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On the convergence and stability of the standard least squares finite element method for first-order elliptic systems

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Abstract

A general framework of the theoretical analysis for the convergence and stability of the standard least squares finite element approximations to boundary value problems of first-order linear elliptic systems is established in a natural norm. With a suitable density assumption, the standard least squares method is proved to be convergent without requiring extra smoothness of the exact solutions. The method is also shown to be stable with respect to the natural norm. Some representative problems such as the grad-div type problems and the Stokes problem are demonstrated. © 1998 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

The purpose of this paper is to establish a general framework of the analysis for the convergence and stability of the standard least squares finite element method which is applied to boundary value problems of first-order linear elliptic systems. Some examples such as the grad-div type problems and the Stokes problem are of particular interest in this framework.

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In the last ten years, the applications of the use of least squares principles in connection with finite element techniques have been extensively studied for the approximations in many different fields such as fluid dynamics, elasticity, electromagnetism, and semiconductor device physics. The approach offers certain advantages, especially for large-scale computations. For example, it leads to minimization problems rather than saddle point problems by the mixed finite element approach, thus it is not subject to the restriction of the Babuška–Brezzi condition; a single continuous piecewise polynomial space can be used for the approximation of all the unknowns; the resulting algebraic system is symmetric and positive definite; accurate approximations of all the unknowns can be obtained simultaneously.

The least squares finite element approach represents a fairly general methodology that can produce a variety of algorithms. Roughly speaking, according to the boundary treatment, these methods can be classified into the following two categories (see Ref. [1] and references therein for more details): the standard least squares finite element method [2–7] and the weighted least squares finite element method [8,9,1]. Here, *standard* means that the associated least squares functional is defined to be the sum of the squared L^2-norms of the residuals of the differential equations.

In the error analysis of the least squares methods mentioned above for first-order elliptic systems, there is a problem that the error estimates require relatively smooth exact solutions. The error estimates do not guarantee any convergence when the methods are applied to problems with low regularity solutions. Accordingly, in Ref. [10], a new least squares finite element method based on a discrete minus one inner product for first-order systems is proposed. The least squares method developed therein is shown to be convergent and stable in some Sobolev's norm as long as the solution belongs to the space $H^{1+\epsilon}(\Omega)$, for any $\epsilon > 0$. However, this method seems rather tricky to implement in practice.

In the present paper, we shall establish a general framework of the analysis for the convergence and stability of the standard least squares finite element approximations to first-order elliptic systems. By using the standard density argument [11], we prove that, without requiring extra smoothness of the exact solutions, the standard least squares method is convergent in a natural norm associated with the least squares bilinear form. We also show that the method is stable with respect to the natural norm. Furthermore, for many examples as we shall present, the natural norm is equivalent to some appropriate Sobolev's norm. Therefore, for these model problems at least, we have established the convergence and stability in some Sobolev's norm without any extra requirement on the regularity of the exact solutions.

The remainder of the paper is organized as follows. In Section 2, we introduce the standard least squares finite element method for first-order elliptic systems. In Section 3, we establish the main results for the convergence and stability. In Section 4, some representative examples are given. Finally, in Section 5, some concluding remarks are drawn.

2. The least squares finite element method

Throughout this paper, the classical Sobolev space $H^s(\Omega), s \geqslant 0$ integer, with its associated inner product $(\cdot,\cdot)_{s,\Omega}$ and norm $\|\cdot\|_{s,\Omega}$, are employed [11–13]. As usual, $L^2(\Omega) = H^0(\Omega)$. For the product space $[H^s(\Omega)]^m$, the corresponding inner product and norm are also denoted by $(\cdot,\cdot)_{s,\Omega}$ and $\|\cdot\|_{s,\Omega}$, respectively, when there is no chance for confusion.

As usual, $L_0^2(\Omega)$ will denote the subspace of square integrable functions with zero mean, i.e., $\int_\Omega v \, \mathrm{d}x = 0$ for all $v \in L_0^2(\Omega)$. By $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ we denote the usual Banach spaces of measurable and essentially bounded real-valued functions defined on Ω and $\partial\Omega$ with the norms $\|\cdot\|_{\infty,\Omega}$ and $\|\cdot\|_{\infty,\partial\Omega}$, respectively. Let $\mathscr{D}(\Omega)$ denote the linear space of infinitely differentiable functions with compact support in Ω, and let $\mathscr{D}(\overline{\Omega})$ denote the restrictions of the functions in $\mathscr{D}(\mathbb{R}^d)$ to $\overline{\Omega}$. It is well-known that $\mathscr{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$.

We shall consider the standard least squares finite element approximations to the boundary value problems of first-order linear elliptic systems in the general form:

$$\sum_{i=l}^{d} A_i \frac{\partial U}{\partial x_i} + A_0 U = F \quad \text{in } \Omega, \tag{2.1}$$

$$BU = G \quad \text{on } \partial\Omega, \tag{2.2}$$

where $\Omega \subset \mathbb{R}^d, d \geqslant 2$, is an open bounded connected domain with a smooth boundary $\partial\Omega$, and $U = (u_1, \ldots, u_m)^{\mathrm{T}}, F = (f_1, \ldots, f_m)^{\mathrm{T}}, G = (g_1, \ldots, g_n)^{\mathrm{T}}$. In this paper, we shall always assume that the entries of $m \times m$ matrices $A_i \in [L^\infty(\Omega)]^{m\times m}, 0 \leqslant i \leqslant d$, and the entries of $n \times m$ boundary matrix $B \in [L^\infty(\partial\Omega)]^{n\times m}$ are regular enough on $\overline{\Omega}$ and $\partial\Omega$, respectively, such that problem (2.1) and (2.2) has a unique strong solution $U \in [H^1(\Omega)]^m$ with the given functions $F \in [L^2(\Omega)]^m, G \in [L^2(\partial\Omega)]^n$. For simplicity, we also assume that $G = \mathbf{0}$ on the boundary $\partial\Omega$.

We now introduce the standard least squares finite element method for problem (2.1) and (2.2). Define a function space $\mathscr{V}$ for our problem by

$$\mathscr{V} = \{V \in [H^1(\Omega)]^m; BV = \mathbf{0} \text{ on } \partial\Omega\}, \tag{2.3}$$

and then define a standard least squares energy functional $\mathscr{J} : \mathscr{V} \to \mathbb{R}$ by

$$\mathscr{J}(V) = \left\| \sum_{i=1}^{d} A_i \frac{\partial V}{\partial x_i} + A_0 V - F \right\|_{0,\Omega}^2 . \tag{2.4}$$

Obviously, the exact solution $U \in \mathscr{V}$ of problem (2.1) and (2.2) is the unique zero minimizer of the functional $\mathscr{J}$ on $\mathscr{V}$, that is,

$$\mathscr{J}(U) = 0 = \min\{\mathscr{J}(V); V \in \mathscr{V}\}. \tag{2.5}$$

Applying the variational techniques, we can find that (2.5) is equivalent to

$$\mathscr{B}(U, V) = \mathscr{F}(V) \quad \forall V \in \mathscr{V}, \tag{2.6}$$

where the bilinear form $\mathscr{B}(\cdot,\cdot)$ and the linear form $\mathscr{F}(\cdot)$ are defined, respectively, by

$$\mathscr{B}(V, W) = \int_\Omega \left(\sum_{i=1}^d A_i \frac{\partial V}{\partial x_i} + A_0 V\right)\left(\sum_{i=1}^d A_i \frac{\partial W}{\partial x_i} + A_0 W\right) \mathrm{d}x, \tag{2.7}$$

$$\mathscr{F}(V) = \int_\Omega F\left(\sum_{i=1}^d A_i \frac{\partial V}{\partial x_i} + A_0 V\right) \mathrm{d}x \tag{2.8}$$

for all $V, W \in \mathscr{V}$. Therefore, the standard least squares finite element method for problem (2.1) and (2.2) is to determine $U_h \in \mathscr{V}_h$ such that

$$\mathscr{B}(U_h, V_h) = \mathscr{F}(V_h) \quad \forall V_h \in \mathscr{V}_h, \tag{2.9}$$

where the finite element space $\mathscr{V}_h \subset \mathscr{V}$ is assumed to satisfy the following approximation property. For any $V \in \mathscr{V} \cap [H^{p+1}(\Omega)]^m, p \geqslant 0$ integer, there exists $V_h \in \mathscr{V}_h$ such that

$$\|V - V_h\|_{1,\Omega} \leqslant Ch^p \|V\|_{p+1,\Omega}, \tag{2.10}$$

where the positive constant C is independent of V and the mesh parameter h. Approximation property (2.10) is satisfied for usual finite element spaces provided the associated family of triangulations $\{\mathscr{T}_h\}$ of $\overline{\Omega}$ is regular [11].

Throughout this paper, in any estimate or inequality the quantity C will denote a generic positive constant always independent of h and need not necessarily be the same constant in different places.

3. Convergence and stability

It is clear that $\mathscr{B}(\cdot,\cdot)$ defines an inner product on $\mathscr{V} \times \mathscr{V}$ since the positive-definiteness is ensured from the fact that problem (2.1) and (2.2) possesses the unique solution $U = \mathbf{0}$ for $F = \mathbf{0}$ and $G = \mathbf{0}$. Denote the associated natural norm by

$$\|V\|_b = \{\mathscr{B}(V, V)\}^{1/2} \quad \forall V \in \mathscr{V}. \tag{3.1}$$

Although we do not know whether the $\|\cdot\|_b$-norm is equivalent to the $\|\cdot\|_{1,\Omega}$-norm or not, evidently there exists a positive constant C such that

$$\|V\|_b \leqslant C\|V\|_{1,\Omega} \quad \forall V \in \mathscr{V} \tag{3.2}$$

since $\sum_{i=1}^d A_i(\partial/\partial x_i) + A_0$ in (2.1) is a first-order differential operator.

We first state some fundamental properties of the standard least squares finite element scheme (2.9).

Theorem 3.1. *Let $U \in [H^1(\Omega)]^m$ be the exact solution of (2.1) and (2.2) with the given functions $F \in [L^2(\Omega)]^m$ and $G = \mathbf{0}$.*

(i) Problem (2.9) has a unique solution $U_h \in \mathcal{V}_h$ which satisfies the following stability estimate:

$$\|U_h\|_b \leqslant \|F\|_{0,\Omega}. \tag{3.3}$$

(ii) The matrix of the linear system associated with problem (2.9) is symmetric and positive definite.

(iii) The following orthogonality relation holds:

$$\mathscr{B}(U - U_h, V_h) = 0 \quad \forall V_h \in \mathcal{V}_h. \tag{3.4}$$

(iv) The approximate solution U_h is a best approximation of U in the $\|\cdot\|_b$-norm, that is,

$$\|U - U_h\|_b = \inf_{V_h \in \mathcal{V}_h} \|U - V_h\|_b. \tag{3.5}$$

Proof. To prove the unique solvability, it suffices to prove the uniqueness of solution since $\mathcal{V}_h$ is a finite-dimensional space. Let U_h be a solution of (2.9), then we have

$$\begin{aligned}
\|U_h\|_b^2 &= \mathscr{B}(U_h, U_h) = \mathscr{F}(U_h) \\
&\leqslant \|F\|_{0,\Omega} \left\| \sum_{i=1}^{d} A_i \frac{\partial U_h}{\partial x_i} + A_0 U_h \right\|_{0,\Omega} \\
&\leqslant \|F\|_{0,\Omega} \|U_h\|_b,
\end{aligned}$$

which implies (3.3). Consequently, the solution U_h of (2.9) is unique.

Assertion (ii) follows from the fact that the bilinear form $\mathscr{B}(\cdot,\cdot)$ is symmetric and positive definite. (iii) is obtained by subtracting Eq. (2.9) from Eq. (2.6). Using (3.4) and the Cauchy–Schwarz inequality,

$$\begin{aligned}
\|U - U_h\|_b^2 &= \mathscr{B}(U - U_h, U - U_h) \\
&= \mathscr{B}(U - U_h, U - V_h) \quad \forall V_h \in \mathcal{V}_h \\
&\leqslant \|U - U_h\|_b \|U - V_h\|_b \quad \forall V_h \in \mathcal{V}_h,
\end{aligned}$$

we prove (iv). □

Estimate (3.3) indicates that the standard least squares method is stable with respect to the $\|\cdot\|_b$-norm, that is, when we change the given data function F slightly in the L^2-norm, the least squares solution U_h changes only slightly in the $\|\cdot\|_b$-norm. Moreover, by using the standard density argument [11], we can obtain the following results for the convergence.

Theorem 3.2. *Assume that there exists a subspace* $\mathscr{S} \subset \mathscr{V} \cap [H^{q+1}(\Omega)]^m$, *for some integer* $q \geqslant 1$, *which is dense in the space* $\mathscr{V}$ *with respect to the* $\|\cdot\|_{1,\Omega}$-*norm. Then the standard least squares finite element method (2.9) is convergent with respect to the* $\|\cdot\|_b$-*norm without requiring any extra regularity assumption on the exact solution* U, *i.e.,*

$$\lim_{h\to 0}\|U - U_h\|_b = \lim_{h\to 0}\left\|\sum_{i=1}^{d} A_i\frac{\partial U_h}{\partial x_i} + A_0U_h - F\right\|_{0,\Omega} = 0. \tag{3.6}$$

Moreover, if the exact solution $U \in \mathscr{V} \cap [H^{p+1}(\Omega)]^m$, *then we have the following error estimate:*

$$\|U - U_h\|_b \leqslant Ch^p\|U\|_{p+1,\Omega}, \tag{3.7}$$

where C *is a positive constant independent of* h.

Proof. Since the subspace $\mathscr{S} \subset \mathscr{V} \cap [H^{q+1}(\Omega)]^m$, is dense in $\mathscr{V}$ with respect to the $\|\cdot\|_{1,\Omega}$-norm, for any $\epsilon > 0$, there exists $\tilde{U} \in \mathscr{S}$ independent of h such that

$$\|U - \tilde{U}\|_{1,\Omega} < \frac{\epsilon}{2C},$$

where C is the same constant as in (3.2), which implies

$$\|U - \tilde{U}\|_b < \frac{\epsilon}{2}. \tag{3.8}$$

For this fixed smooth function $\tilde{U} \in \mathscr{S} \subset [H^{q+1}(\Omega)]^m, q \geqslant 1$, by the approximation property (2.10), we can find $\tilde{U}_h \in \mathscr{V}_h$ so that,

$$\|\tilde{U} - \tilde{U}_h\|_{1,\Omega} \leqslant Ch^q\|\tilde{U}\|_{q+1,\Omega}$$

which implies, for sufficiently small h,

$$\|\tilde{U} - \tilde{U}_h\|_b \leqslant C\|\tilde{U} - \tilde{U}_h\|_{1,\Omega} < \frac{\epsilon}{2}. \tag{3.9}$$

Combining inequalities (3.8) and (3.9) with (3.5), we immediately obtain

$$0 \leqslant \|U - U_h\|_b \leqslant \|U - \tilde{U}_h\|_b \leqslant \|U - \tilde{U}\|_b + \|\tilde{U} - \tilde{U}_h\|_b < \epsilon$$

which implies (3.6). We now assume that $U \in \mathscr{V} \cap [H^{p+1}(\Omega)]^m$. By (3.5), (3.2) and the approximation property (2.10) of the finite element space $\mathscr{V}_h$, we obtain (3.7). This completes the proof. □

4. Examples

Unless otherwise specified, we assume in this section that the dimension d is two or three. We begin with two preliminary lemmas.

Lemma 4.1 (The Poincaré–Wirtinger inequality [13]). *Let Ω be an open bounded connected subset of $\mathbb{R}^d, d \geqslant 2$, with a C^1 boundary $\partial\Omega$. Then there exists a constant $C = C(d, \Omega) > 0$ such that for every $v \in H^1(\Omega)$, we have*

$$\|v - v_\Omega\|_{0,\Omega} \leqslant C\|\nabla v\|_{0,\Omega} \tag{4.1}$$

where

$$v_\Omega := \frac{1}{\text{meas}(\Omega)} \int_\Omega v \, \mathrm{d}x \tag{4.2}$$

is the average of v over Ω.

Defining the function space

$$\tilde{\mathscr{D}}(\overline{\Omega}) = \mathscr{D}(\overline{\Omega}) \cap L_0^2(\Omega), \tag{4.3}$$

we obtain, from Lemma 4.1, the following result.

Lemma 4.2. *The space $\tilde{\mathscr{D}}(\overline{\Omega})$ is dense in $H^1(\Omega) \cap L_0^2(\Omega)$.*

Proof. Since $\mathscr{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$, for every $v \in H^1(\Omega) \cap L_0^2(\Omega)$, there exists a sequence $\{v_n\}$ in $\mathscr{D}(\overline{\Omega})$ such that

$$\|v_n - v\|_{1,\Omega}^2 = \|v_n - v\|_{0,\Omega}^2 + \|\nabla(v_n - v)\|_{0,\Omega}^2 \to 0 \quad \text{as } n \to \infty.$$

Define the following sequence $\{c_n\}$ of real numbers,

$$c_n := \frac{1}{\text{meas}(\Omega)} \int_\Omega v_n \, \mathrm{d}x,$$

then

$$c_n = \frac{1}{\text{meas}(\Omega)} \int_\Omega (v_n - v) \, \mathrm{d}x$$

since $\int_\Omega v \, \mathrm{d}x = 0$. Applying the Poincaré–Wirtinger inequality to $v_n - v$ for all n, we have

$$\begin{aligned} 0 \leqslant \|(v_n - c_n) - v\|_{1,\Omega}^2 &= \|(v_n - c_n) - v\|_{0,\Omega}^2 + \|\nabla(v_n - c_n) - \nabla v\|_{0,\Omega}^2 \\ &= \|(v_n - c_n) - v\|_{0,\Omega}^2 + \|\nabla(v_n - v)\|_{0,\Omega}^2 \\ &\leqslant C\|\nabla(v_n - v)\|_{0,\Omega}^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This completes the proof. □

Let $H_0^1(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$, then

$$H_0^1(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \partial\Omega\}. \tag{4.4}$$

We now introduce our first example.

Example 4.1 (*The velocity–vorticity–pressure Stokes equations*). Let $\Omega \subset \mathbb{R}^d, d = 2$, and let $\mathbf{f} = (f_1, f_2)^{\mathrm{T}} \in [L^2(\Omega)]^2$ be a given function representing the body force. The Stokes equations with homogeneous velocity boundary conditions can be posed as:

$$-\Delta\mathbf{u} + \operatorname{grad}(p) = \mathbf{f} \quad \text{in } \Omega, \tag{4.5}$$
$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{4.6}$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{4.7}$$
$$(p, 1)_{0,\Omega} = 0, \tag{4.8}$$

where $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ is the velocity and p is the pressure, all of which are assumed to be nondimensionalized. Introducing the auxiliary variable $\omega = \operatorname{curl}(\mathbf{u}) := \partial u_2/\partial x - \partial u_1/\partial y$, which is known as the vorticity, we can transform (4.5)–(4.8) into a first-order system [14] as follows:

$$\operatorname{curl}(\omega) + \operatorname{grad}(p) = \mathbf{f} \quad \text{in } \Omega, \tag{4.9}$$
$$\operatorname{curl}(\mathbf{u}) - \omega = 0 \quad \text{in } \Omega, \tag{4.10}$$
$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{4.11}$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{4.12}$$
$$(p, 1)_{0,\Omega} = 0, \tag{4.13}$$

where $\operatorname{curl}(\omega) := (\omega_y, -\omega_x)^{\mathrm{T}}$ is another curl operator.

Applying the least squares finite element scheme (2.9) to the first-order system (4.9)–(4.13) with

$$\mathscr{V} = \left\{(\mathbf{v}, \varphi, q)^{\mathrm{T}} \in [H_0^1(\Omega)]^2 \times H^1(\Omega) \times H^1(\Omega); q \in L_0^2(\Omega)\right\}$$

and taking

$$\mathscr{S} = \mathscr{D}(\Omega) \times \mathscr{D}(\Omega) \times \mathscr{D}(\overline{\Omega}) \times \tilde{\mathscr{D}}(\overline{\Omega}),$$

which is contained in $\mathscr{V} \cap [H^2(\Omega)]^4$, we have the following convergence results that follow from (4.4), Lemma 4.2, and Theorem 3.2,

$$\|\operatorname{curl}(\omega - \omega_h) + \operatorname{grad}(p - p_h)\|_{0,\Omega} + \|\operatorname{curl}(\mathbf{u} - \mathbf{u}_h) - (\omega - \omega_h)\|_{0,\Omega} + \|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \to 0 \quad \text{as} \quad h \to 0, \tag{4.14}$$

provided the exact solution $U = (\mathbf{u}, \omega, p)^{\mathrm{T}} \in \mathscr{V}$; and

$$\|\operatorname{curl}(\omega - \omega_h) + \operatorname{grad}(p - p_h)\|_{0,\Omega} + \|\operatorname{curl}(\mathbf{u} - \mathbf{u}_h) - (\omega - \omega_h)\|_{0,\Omega} + \|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \leqslant Ch^p \|U\|_{p+1,\Omega}, \tag{4.15}$$

if $U = (\mathbf{u}, \omega, p)^{\mathrm{T}} \in \mathscr{V} \cap [H^{p+1}(\Omega)]^4$, where $U_h = (\mathbf{u}_h, \omega_h, p_h)^{\mathrm{T}} \in \mathscr{V}_h$ is the least squares finite element solution.

For the case of sufficiently smooth exact solutions, the error estimates can be found in, for example, Refs. [15–17]. More specifically, in Ref. [17], we have proved that there exists a positive constant C such that

$$\|\mathbf{v}\|_{1,\Omega} + \|\varphi\|_{0,\Omega} + \|q\|_{0,\Omega} \leqslant C\big(\|\mathrm{curl}(\varphi) + \mathrm{grad}(q)\|_{0,\Omega} + \|\mathrm{curl}(\mathbf{v}) - \varphi\|_{0,\Omega}. \\ + \|\mathrm{div}(\mathbf{v})\|_{0,\Omega}\big) \quad \forall V = (\mathbf{v}, \varphi, q)^{\mathrm{T}} \in \mathscr{V}. \tag{4.16}$$

Combining (4.16) with (4.14) and (4.15), we have, respectively,

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\omega - \omega_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \to 0 \quad \text{as} \quad h \to 0, \tag{4.17}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\omega - \omega_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leqslant Ch^p \|U\|_{p+1,\Omega}. \tag{4.18}$$

In this example, we focus our attention on the two-dimensional velocity–vorticity–pressure Stokes equations with velocity boundary conditions. All of the results developed here can be extended to the three-dimensional case directly [15,17].

So far, we have been mainly interested in the case of $\mathscr{V} \subset [H^1(\Omega)]^m$; but all of the results developed in Section 3 can be easily applied to the function space $\mathscr{V}$ with less regularity. Bearing this in mind, we introduce the following function spaces:

$$\mathbf{H}(\mathrm{div};\Omega) := \{V \in [L^2(\Omega)]^d; \mathrm{div}\, V \in L^2(\Omega)\}, \tag{4.19}$$

$$\mathbf{H}_0(\mathrm{div};\Omega) := \text{the closure of } [\mathscr{D}(\Omega)]^d \text{ in } \mathbf{H}(\mathrm{div};\Omega), \tag{4.20}$$

where the space $\mathbf{H}(\mathrm{div};\Omega)$ is equipped with the following inner product and norm,

$$(V, W)_{\mathbf{H}(\mathrm{div};\Omega)} := (V, W)_{0,\Omega} + (\mathrm{div}\, V, \mathrm{div}\, W)_{0,\Omega} \quad \forall V, W \in \mathbf{H}(\mathrm{div};\Omega), \tag{4.21}$$

$$\|V\|_{\mathbf{H}(\mathrm{div};\Omega)} := \big(\|V\|_{0,\Omega}^2 + \|\mathrm{div}\, V\|_{0,\Omega}^2\big)^{1/2} \quad \forall V \in \mathbf{H}(\mathrm{div};\Omega), \tag{4.22}$$

which make it a Hilbert space [12].

The following lemmas will be required later and their proofs can be found in [12].

Lemma 4.3. *The space* $[\mathscr{D}(\overline{\Omega})]^d$ *is dense in* $\mathbf{H}(\mathrm{div};\Omega)$.

Lemma 4.4. *Let* $\mathbf{n}$ *be the outward unit normal vector to* $\partial\Omega$, *then we have*

$$\mathbf{H}_0(\mathrm{div};\Omega) = \{V \in \mathbf{H}(\mathrm{div};\Omega); V \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}. \tag{4.23}$$

Example 4.2 *(Grad-div type problems)*. Let the smooth boundary $\partial\Omega$ of domain $\Omega \subset \mathbb{R}^d, d = 2$ or 3, be partitioned into two disjoint open parts, Γ_D and Γ_N, such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. We consider the following boundary value problem with mixed type boundary conditions:

$$-\operatorname{div}(A \operatorname{grad} u) = f \quad \text{in } \Omega, \tag{4.24}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{4.25}$$

$$(A \operatorname{grad} u) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N, \tag{4.26}$$

where $A = (a_{ij}(\mathbf{x}))_{d\times d}, a_{ij} \in L^\infty(\overline{\Omega})$, and we assume that A is symmetric and uniformly positive definite on $\overline{\Omega}$. It is understood that if $\Gamma_N = \partial\Omega$, we further require the compatibility condition, $\int_\Omega f \, dx = 0$, and impose an additional constraint on u such as $\int_\Omega u \, dx = 0$, for the well-posedness.

Introducing the auxiliary variables, $\mathbf{p} = -A \operatorname{grad} u$ on $\overline{\Omega}$, we can reformulate problem (4.24)–(4.26) to the following equivalent first-order form:

$$\mathbf{p} + A \operatorname{grad} u = \mathbf{0} \quad \text{in } \Omega, \tag{4.27}$$

$$\operatorname{div} \mathbf{p} = f \quad \text{in } \Omega, \tag{4.28}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{4.29}$$

$$\mathbf{p} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \tag{4.30}$$

Following Section 3 with minor modifications, we can apply the standard least squares method (2.9) over the extended first-order system (4.27)–(4.30) with

$$\mathscr{V} = \left\{ (v, \mathbf{q})^{\mathrm{T}} \in H^1(\Omega) \times \mathbf{H}(\operatorname{div};\Omega); v = 0 \text{ on } \Gamma_D \text{ and } \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N \right\},$$

provided the associated finite element space $\mathscr{V}_h \subset \mathscr{V}$ also possesses the following approximation property [2,7]. For any $V = (v, \mathbf{q})^{\mathrm{T}} \in \mathscr{V} \cap [H^{p+1}(\Omega)]^{d+1}$, $p \geqslant 0$ integer, there exists $V_h = (v_h, \mathbf{q}_h)^{\mathrm{T}} \in \mathscr{V}_h$ such that

$$\|v - v_h\|_{1,\Omega} + \|\mathbf{q} - \mathbf{q}_h\|_{\mathbf{H}(\operatorname{div};\Omega)} \leqslant Ch^p \left\{ \|v\|^2_{p+1,\Omega} + \|\mathbf{q}\|^2_{p+1,\Omega} \right\}^{1/2}, \tag{4.31}$$

where the positive constant C is independent of V and the mesh parameter h.

Similar to Theorem 3.2, we can prove that, without any extra regularity assumption on the exact solution $U = (u, \mathbf{p})^{\mathrm{T}}$,

$$\begin{aligned} &\|(\mathbf{p} - \mathbf{p}_h) + A \operatorname{grad}(u - u_h)\|_{0,\Omega} + \|\operatorname{div}(\mathbf{p} - \mathbf{p}_h)\|_{0,\Omega} \\ &\quad = \|\mathbf{p}_h + A \operatorname{grad} u_h\|_{0,\Omega} + \|\operatorname{div} \mathbf{p}_h - f\|_{0,\Omega} \to 0 \quad \text{as} \quad h \to 0, \end{aligned} \tag{4.32}$$

when $\Gamma_D = \partial\Omega$ or $\Gamma_N = \partial\Omega$, where $U_h = (u_h, \mathbf{p}_h)^{\mathrm{T}} \in \mathscr{V}_h$ is the least squares finite element solution, and we choose

$$\mathscr{S} = \mathscr{D}(\Omega) \times [\mathscr{D}(\overline{\Omega})]^d \quad \text{if } \Gamma_D = \partial\Omega$$

and

$$\mathscr{S} = \tilde{\mathscr{D}}(\overline{\Omega}) \times [\mathscr{D}(\Omega)]^d \quad \text{if } \Gamma_N = \partial\Omega$$

both of which are contained in $\mathscr{V} \cap [H^2(\Omega)]^d$ and dense in $\mathscr{V}$ with respect to the $H^1(\Omega) \times \mathbf{H}(\text{div};\Omega)$ norm (cf. (4.4), Lemma 4.3, and Lemma 4.2; (4.20), Lemma 4.4, respectively).

For sufficiently smooth exact solutions, the error estimates can be found in, for example Refs. [2,3,7].

5. Concluding remarks

In this paper, we establish a general framework of the theoretical analysis for the convergence and stability of the standard least squares finite element method for first-order problems. With a suitable density assumption, the method is proved to be convergent in a natural norm without requiring extra regularity of the exact solutions, and with respect to the same norm, the method is also stable.

By the example shown in the previous section, we observe that choosing a suitable dense subset $\mathscr{S}$ of $\mathscr{V}$ plays a crucial role in the analysis of the convergence. A useful tool for choosing $\mathscr{S}$ is based on the following property of Banach spaces [12], p. 26:

A subspace $\mathscr{S}$ of a Banach space $\mathscr{V}$ is dense in $\mathscr{V}$ if and only if

every element of $\mathscr{V}'$ that vanishes on $\mathscr{S}$ also vanishes on $\mathscr{V}$,

where $\mathscr{V}'$ denotes the dual space of $\mathscr{V}$. Actually, Lemma 4.3 and Lemma 4.4 are two applications of the property, whose proofs can be found in Ref. [12].

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