ELSEVIER

European Journal of Operational Research 107 (1998) 625–632

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

An approximate approach of global optimization for polynomial programming problems

Han-Lin Li *, Ching-Ter Chang

Institute of Information Management, National Chiao Tung University, Ta Hsieh Rd., Hsinchu 30050, Taiwan, ROC

Received 28 March 1995; accepted 15 August 1996

Abstract

Many methods for solving polynomial programming problems can only find locally optimal solutions. This paper proposes a method for finding the approximately globally optimal solutions of polynomial programs. Representing a bounded continuous variable x_i as the addition of a discrete variable d_i and a small variable ε_i, a polynomial term $x_i x_j$ can be expanded as the sum of $d_i x_j$, $d_j \varepsilon_i$ and $\varepsilon_i \varepsilon_j$. A procedure is then developed to fully linearize $d_i x_j$ and $d_j \varepsilon_i$, and to approximately linearize $\varepsilon_i \varepsilon_j$ with an error below a pre-specified tolerance. This linearization procedure can also be extended to higher order polynomial programs. Several polynomial programming examples in the literature are tested to demonstrate that the proposed method can systematically solve these examples to find the global optimum within a pre-specified error.

Keywords: Global optimization; Linearization; Polynomial program

1. Introduction

This paper develops a method for seeking a global minimum of a polynomial program where the polynomial terms may appear in the objective function and the constraints. None of the convexity restrictions is imposed on these functions and constraints. The mathematical expression of a polynomial programming problem is given below:

(PP Problem)

$$\text{Global Min} \quad \sum_j f_j(X)$$

subject to

$$g_k(X) \le 0, \quad k = 1, \dots, q,$$
$$X = (x_1, \dots, x_n),$$
$$0 \le l_i \le x_i \le \overline{x_i}, \quad i = 1, \dots, n,$$

where $f_j(X)$ and $g_k(X)$ are polynomial functions of X, and l_i and $\overline{x_i}$ are respectively the lower and the upper bound of x_i.

Some approaches for solving the above polynomial programming problems are discussed below.

1.1. Analytical approach

Horst and Tuy [5] proposed outer approximation techniques for solving a PP Problem with Lipschitzian objective function and constraint. Hansen,

* Corresponding author. E-mail:hlli@ccsun2.cc.nctu.edu.tw.

PII S0377-2217(96)00310-4

Jaumard and Lu [3] developed interval analysis based sufficient conditions for convergence, and provided ways to eliminate variables and to reduce the ranges of variables. These analytical approaches, however, are only convergent in the absence of blocking subproblems. As the size of problems increases it becomes more and more likely that successive elimination of variables leads to problems too complicated to allow any further elimination [3]. The analytical approach is promising in finding a global optimum of a PP Problem only if the range of variables of the PP Problem can be easily reduced by analytical techniques.

1.2. Concave minimization approach and binary approach

Particular cases of PP Problems, such as concave programming problems and bilinear programming problems, have attracted much attention. Work on these problems was reviewed by Pardalos and Rosen [11]. Various 0–1 polynomial programs proposed by Hansen, Jaumard and Lu [2] and Li [7,8] perform well in finding globally optimal solutions. These approaches exploit the special structure of the PP Problem and therefore they are not directly applicable to the general polynomial programming problems discussed in this paper.

1.3. Stochastic approach

Many stochastic algorithms for global optimization, such as the multistart with clustering [12,9] used random search to converge asymptotically to a global optimum. The algorithms can also be extended to find global solutions. This approach is quite promising in searching for a global optimum in highly nonlinear programs which are difficult to treat by other methods. However, since this technique requires evaluating a huge amount of starting points, it can only be applied to solve small size problems.

1.4. Reformulation – linearization approach

Sherali and Tuncbilek [15] and Adams and Sherali [14] derived a reformulation linearization technique (RLT) which generated polynomial implied constraints, and subsequently linearized the resulting problem by defining new variables. This construct was then used to obtain lower bounds in the context of a proposed branch and bound scheme. Although the RLT process is promising with respect to converging to a global solution, the process is in practice very difficult to implement owing to the following reasons: (i) Several types of implied constraints, or subsets, or surrogates need to be generated in a linearized form. Tightening its representation at the expense of an exponential constraint step by step is a long trial and error process. (ii) The RLT algorithm always needs to generate a huge amount of bounded constraints; many of these constraints are redundant. (iii) These are considerable variants in designing a RLT process, depending on the actual structure of the problem being solved. A user needs to formulate a special RLT scheme corresponding to each of his programs.

This paper develops a new method for solving a PP Problem and to find a global optimum with a prespecified tolerance. The developed method uses a convenient linearization technique to systematically convert a PP Problem into a linear mixed 0–1 problem. The solution of this converted problem can be as close as possible to the global optimum of the original PP Problem. A comparison of this method with other methods reviewed above is given below:

1. The proposed methods can solve general PP Problems. In contrast, the analytical approach [3] can only solve problems in the absence of blocking subproblems. The outer approximation techniques [5] or concave minimization approach [11] can only treat problems with specific objective functions and constraints.
2. Both the proposed method and the multistart method [12,9] can solve a PP Problem to obtain a solution closing in on a global optimum. However, the multistart method requires evaluating a huge amount of starting points.
3. The proposed method can systematically solve the general PP Problem, but the reformulation-linearization approach [15] can only solve particular problems using various RLT processes.

This paper has solved many test problems from the compendiums of Hock and Schittkowski [4], Schittkowski [6], and other sources. The experiment demonstrates that the proposed method stably treats all of the test problems in finding globally optimal solutions within a prespecified tolerance.

2. Preliminaries

Consider a bounded variable x_i, $0 \le l_i \le x_i \le \overline{x_i}$, where l_i and $\overline{x_i}$ are constants. x_i can be represented as follows:

$$x_i = l_i + \omega_i \sum_{j=1}^{J} 2^{j-1} y_{ij} + \varepsilon_i, \tag{2.1}$$

where:

l_i is the lower bound of x_i.
ω_i is the pre-specified positive constant which is the upper bound of ε_i,
y_{ij} is a 0–1 variable.
J is an integer which denotes the number of required 0–1 variables for representing x_i.
ε_i is a small variable, $0 \le \varepsilon_i \le \omega_i$.

For any bounded variable x_i, there is an unique set of y_{ij} ($j = 1, \ldots, J$), ω_i and ε_i such that Eq. (2.1) is satisfied. For example, if x_i is a variable between 10 and 15 and ω_i is chosen as 0.1, x_i can then be represented as

$$x_i = 10 + 0.1 y_{i1} + 0.2 y_{i2} + 0.4 y_{i3} + 0.8 y_{i4} + 1.6 y_{i5} + 3.2 y_{i6} + \varepsilon_i,$$

where $0 \le \varepsilon_i \le 0.1$ and y_{ij}, $j = 1, \ldots, 6$, are 0–1 variables. Suppose $x_i = 13.752$. Then

$$y_{i1} = y_{i3} = y_{i6} = 1,$$
$$y_{i2} = y_{i4} = y_{i5} = 0$$

and $\varepsilon_i = 0.052$.

Referring to Eq. (2.1), a polynomial term $x_1 x_2$ is represented as

$$x_1 x_2 = A_1 x_2 + \varepsilon_1 x_2 = A_1 x_2 + A_2 \varepsilon_1 + \varepsilon_1 \varepsilon_2, \tag{2.2}$$

where

$$A_i = l_i + \omega_i \sum_{j=1}^{J} 2^{j-1} y_{ij} \quad \text{for } i = 1, 2. \tag{2.3}$$

Define e_{12} as a linear approximation of $\varepsilon_1 \varepsilon_2$, expressed as

$$e_{12} = \tfrac{1}{2}(\omega_1 \varepsilon_2 + \omega_2 \varepsilon_1). \tag{2.4}$$

The error of approximating $\varepsilon_1 \varepsilon_2$ is computed as

$$0 \le e_{12} - \varepsilon_1 \varepsilon_2 = \tfrac{1}{2}(\omega_1 \varepsilon_2 + \omega_2 \varepsilon_1) - \varepsilon_1 \varepsilon_2 \le \tfrac{1}{4}\omega_1 \omega_2. \tag{2.5}$$

The maximal difference between e_{12} and $\varepsilon_1 \varepsilon_2$ is $\frac{1}{4}\omega_1 \omega_2$, which occurs at $\varepsilon_1 = \frac{1}{2}\omega_1$ and $\varepsilon_2 = \frac{1}{2}\omega_2$. Substituting $\varepsilon_1 \varepsilon_2$ by e_{12}, expression (2.2) can be approximately linearized as

$$x_1 x_2 = A_1 x_2 + A_2 \varepsilon_1 + \tfrac{1}{2}(\omega_1 \varepsilon_2 + \omega_1 \varepsilon_1). \tag{2.6}$$

The following section will show that the polynomial terms $A_1 x_2$ and $A_2 \varepsilon_1$ can be fully linearized. The maximum error of approximately linearizing $x_1 x_2$ in Eq. (2.6) is therefore less than $\frac{1}{4}\omega_1 \omega_2$. By specifying smaller ω_1 and ω_2 values, a more accurate approximation can be obtained. Choosing a smaller ω_i, however, requires using more binary variables to represent a bounded variable x_i in Eq. (2.1).

3. Linear strategies

A polynomial term $x_1 x_2$ can be approximated as follows, referring to Eqs. (2.2), (2.3), (2.4), (2.5) and (2.6):

$$x_1 x_2 = l_1 x_2 + \omega_1 \sum_{i=1}^{J} 2^{i-1} y_{1i} x_2 + l_2 \varepsilon_1 + \omega_2 \sum_{j=1}^{J} 2^{j-1} y_{2j} \varepsilon_1 + \tfrac{1}{2}(\omega_1 \varepsilon_2 + \omega_2 \varepsilon_1), \tag{3.1}$$

where l_1, l_2, ω_1, ω_2, I and J are constants, ε_1 and ε_2 are continuous variables, y_{1i} and y_{2j} are binary variables, and $0 \le \varepsilon_k \le \omega_k$, for $k = 1, 2$. The terms $y_{1i} x_2$ and $y_{2j} \varepsilon_1$ in Eq. (3.1) can be fully linearized based on Proposition 1 discussed below.

Proposition 1. *Given a polynomial term $y_1 y_2, \ldots, y_n x$, in which $y_1 y_2, \ldots, y_n$ are binary variables and $0 \le x \le \bar{x}$, with $\bar{x}$ a constant, $y_1 y_2, \ldots, y_n x$ can be fully linearized as*

$$y_1 y_2, \ldots, y_n x = q,$$

where the following inequalities are satisfied:
1. $x + (y_1 + y_2 + \cdots + y_n - n)\bar{x} \le q \le x$;
2. $0 \le q \le \bar{x} y_i, \quad i = 1, \ldots, n.$

This proposition can be checked as follows: If one of the $y_1 y_2, \ldots, y_n$ equals 0, then $q = 0$; and if all of the $y_1 y_2, \ldots, y_n$ equal 1, then $q = x$.

Proposition 2. *Based on Proposition 1, a linear approximation of $x_1 x_2$, denoted as $[x_1 x_2]$, can be expressed as*

$$[x_1 x_2] = l_1 x_2 + w_1 \sum_{i=1}^{I} 2^{i-1} q_{1i} + l_2 \varepsilon_1 + w_2 \sum_{j=1}^{J} 2^{j-1} q_{2j} + \tfrac{1}{2}(w_1 \varepsilon_2 + w_2 \varepsilon_1),$$

where q_{1i} and q_{2j} are bounded by the following inequalities:

$$0 \le q_{1i} \le \bar{x}_1 y_{1i}, \quad i = 1, \ldots, I,$$

$$\bar{x}_2 (y_{1i} - 1)\bar{x} \le q_{1i} \le x_2, \quad i = 1, \ldots, I,$$

$$0 \le q_{2j} \le w_1 y_{2j}, \quad j = 1, \ldots, J,$$

$$\varepsilon_1 + (y_{2j} - 1) w_1 \le q_{2j} \le \varepsilon_1, \quad j = 1, \ldots, J.$$

It is clear that $[x_1 x_2] \ge x_1 x_2$, and the maximal error of this approximation is

$$\max\{[x_1 x_2] - x_1 x_2\} = \tfrac{1}{4}\omega_1 \omega_2.$$

The same linearization process can be applied to a higher order polynomial term, discussed in the proposition below.

Proposition 3. *A linear approximation of $x_1 x_2 x_3$, denoted as $[x_1 x_2 x_3]$, can be expressed as*

$$[x_1 x_2 x_3] = (l_3 + \varepsilon_3)[x_1 x_2] + w_3 \sum_{k=1}^{K} 2^{k-1} q_{3k},$$

where each of the q_{3k} $(k = 1, \ldots, K)$ satisfies the following inequalities:
1. $0 \le q_{3k} \le [x_1 x_2]$;
2. $[x_1 x_2] + (y_{3k} - 1)\bar{x}_1 \bar{x}_2 \le q_{3k} \le \bar{x}_1 \bar{x}_2 y_{3k}$;

where $[x_1 x_2]$ is the linear approximation of $x_1 x_2$, expressed in Proposition 2.

Since $[x_1 x_2 x_3] \ge x_1 x_2 x_3$, the maximum error of this approximation becomes

$$\max\{[x_1 x_2 x_3] - x_1 x_2 x_3\} = \tfrac{1}{4}\omega_1 \omega_2 \bar{x}_3.$$

This proposition is checked as follows: $[x_1 x_2 x_3]$ is expressed as

$$[x_1 x_2 x_3] = [x_1 x_2]\left(l_3 + w_3 \sum_{k=1}^{K} 2^{k-1} y_{3k} + w_3 \right).$$

Replacing $[x_1 x_2]\sum_{k=1}^{K} 2^{k-1} y_{3k}$ by $\sum_{k=1}^{K} 2^{k-1} q_{3k}$, $[x_1 x_2 x_3]$ is then converted into the form expressed in Proposition 3.

4. Numerical examples

Example 1. Solve the following global optimization problem adopted from Sherali and Tuncbilek [15]:

Global Min

$$\mathrm{PP}(X) = x_1 x_2 x_3 + x_1^2 - 2 x_1 x_2 - 3 x_1 x_3 + 5 x_2 x_3 - x_3^2 + 5 x_2 + x_3$$

subject to

$$4 x_1 + 3 x_2 + x_3 \le 20,$$

$$x_1 + 2 x_2 + x_3 \ge 1,$$

$$2 \le x_1 \le 5,$$

$$0 \le x_2 \le 10,$$

$$4 \le x_3 \le 8.$$

First, to determine the values of ω_1, ω_2 and ω_3, denote $[*]$ as the approximation of a polynomial term $*$, where $[*]$ is obtained by the proposed linearization method mentioned before. Since $[*] \ge *$, the maximal error of linearizing PP(X), denoted as ε, is computed as

$$\varepsilon = [x_1 x_2 x_3] - x_1 x_2 x_3 + [x_1^2] - x_1^2 + 5([x_2 x_3] - x_2 x_3).$$

The relationships among ε, ω_1, ω_2, $\bar{x}_i$ and l_i are stated below:

$$\varepsilon \le \tfrac{1}{4}(\omega_2 \omega_3 \bar{x}_1 + \omega_1^2 + 5\omega_2 \omega_3),$$

$$\frac{\bar{x}_i - l_i}{\omega_i} \le \sum_{i=1}^{I} 2^{i-1} y_i, \quad i = 1,2,3.$$

Suppose ε is chosen as $\varepsilon \le 0.03$. Then ω_1, ω_2, and ω_3 can be set as $\omega_1 \ge 0.125$, $\omega_2 \ge 0.125$, and $\omega_3 \ge 0.137$.

Express x_1, x_2 and x_3 as follows:

$$x_1 = 2 + 0.125 \sum_{i=1}^{5} 2^{i-1} y_{1i} + \varepsilon_1,$$

$$x_2 = 0.125 \sum_{j=1}^{7} 2^{j-1} y_{2j} + \varepsilon_2,$$

$$x_3 = 4 + 0.137 \sum_{k=1}^{5} 2^{k-1} y_{3k} + \varepsilon_3.$$

The term $x_1 x_2 x_3$ can be linearized by Proposition 3, while all other polynomial terms can be linearized by Proposition 2. Solving the linearized program using LINDO [10], the found optimal solution is

$$x^* = (x_1^*, x_2^*, x_3^*) = (3, 0, 8).$$

In fact, this is the global optimum. The same solution was found by Sherali and Tuncbilek [15] through a complicated heuristic algorithm for solving a mixed 0–1 problem.

Example 2. Consider the optimal design problem of a pressure vessel in Sandgren [13] depicted in Fig. 1. Sandgren solved this problem by a Quadratic Integer Programming Method, and Fu et al. [1] solved it by an Integer Penalty method. Recently, Li and Chou [9] solved this problem by the Multistart Method. The problem is formulated below:

$$\text{Min} \quad f(X) = 0.6224 x_3 x_4 + 1.7781 x_2 x_3^2 + 3.1661 x_1^2 x_4 + 19.84 x_1^2 x_3$$

subject to

$$-x_1 + 0.0193 x_3 \le 0,$$

$$-x_2 + 0.00954 x_3 \le 0,$$

$$-\pi x_3^2 x_4 - \tfrac{4}{3}\pi x_3^3 + 750.1728 \le 0,$$

$$-240 + x_4 \le 0,$$

$$1.000 \le x_1 \le 1.375,$$

$$0.625 \le x_2 \le 1.000,$$

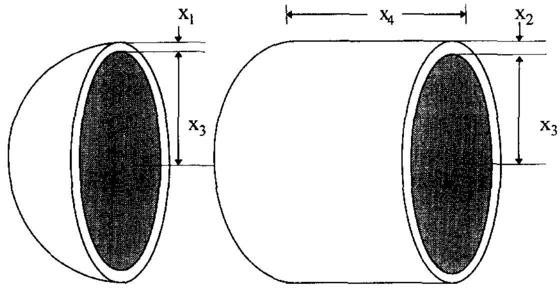

Fig. 1. Tube and end section of pressure vessel (from Sandgren [12]).

where x_1 and x_2 are discrete variables with discreteness 0.0625, and

$$47.5 \le x_3 \le 52.5,$$

$$90.00 \le x_4 \le 112.00,$$

where x_3 and x_4 are continuous variables. x_1 is the spherical head thickness, x_2 is the shell thickness, x_3 is radius and x_4 is the length of the shell.

Since x_1 and x_2 are discrete variables, they can be completely expressed by binary variables without error, as shown below:

$$x_1 = 1 + 0.0625 y_{11} + 0.01250 y_{12} + 0.2500 y_{13},$$

$$x_2 = 0.625 + 0.0625 y_{21} + 0.1250 y_{22} + 0.2500 y_{23}.$$

Suppose the tolerable error of approximating $f(X)$ (i.e. the objective function) is set as 0.05. Then we can choose $\omega_3 = 0.34$ and $\omega_4 = 0.375$. The variables x_3 and x_4 are then rewritten as

$$x_3 = 47.50 + 0.34 y_{31} + 0.68 y_{32} + 1.36 y_{33} + 2.72 y_{34} + \varepsilon_3,$$

$$x_4 = 90.00 + 0.375 y_{41} + 0.75 y_{42} + 1.5 y_{43} + 3 y_{44} + 6 y_{45} + 12 y_{46} + \varepsilon_4.$$

Table 1
A comparison of optimum solutions for Example 2

Items	Optimal solution by Sandgren's method	Optimal solution by Fu et. al's method	Optimal solution by the proposed method
x_1	1.125	1.125	1.000
x_2	0.625	0.625	0.625
x_3	48.95	48.38	51.25
x_4	106.72	111.745	90.991
$f(X)$	7982.5	8048.6	7127.3

The maximal error of $f(X)$ after linearization is computed as

$$\frac{0.6224}{4}\omega_3\omega_4+\frac{1.7781}{4}\omega_3^2\le 0.05.$$

Solving this program by LINDO [10], the optimal solution is

$$x^*=(1.000, 0.625, 51.252, 90.991),$$

with objective function value $f(X)=7127.3$. The same problem was solved by Sandgren [13] and Fu et al. [1]. The best solutions they can find, after testing many starting points, are listed in Table 1. Table 1 shows that the proposed method can find a solution better than the ones obtained by Sandgren and Fu et al.

Example 3. To demonstrate the superiority of the proposed method in finding the global optimum of polynomial programs, several test examples of polynomial programming problems listed in Hock and Schittkowski [4] and Schittkowski [6] have been evaluated. These examples have been solved by six well-known optimization codes shown in Table 2. Since none of these six codes could solve all test problems successfully, and since none of them can guarantee having found the global solution, Hock and Schittkowski solved each test example by all of these six codes. They executed each code with very low stopping tolerances (10^{-7}) and a huge amount of starting points, thus computing one solution as precise as possible. Only the best result obtained by six codes is reported as the solution of the test example.

The proposed method resolves these test examples by setting the various tolerance values. The experiment shows that the proposed method successfully finds solutions for all test examples. Parts of the results are listed in Table 3. The proposed method resolves each of these problems and finds the same or even a better solution than the best solution found by the other six codes. Take Problem 338 in [6] for instance. The optimization problem is listed below:

Table 2
Optimization programs for evaluating optimal solutions (Hock and Schittkowski [4]; Schittkowski [6])

Code	Author	Method
VFO2AD	Powell	Quadratic approximation
OPRQP	Bartholomew-Biggs	Quadratic approximation
GRGA	Abadie	Generalized reduced gradient
VFO1A	Fletcher	Multiplier
FUNMIN	Kraft	Multiplier
FMIN	Kraft, Lootsma	Penalty

(Problem 338 in [6])

Min $F(X)=-(X_1^2+X_2^2+X_3^2)$

subject to

$$0.5X_1+X_2+X_3-1=0,$$
$$X_1^2+\tfrac{2}{3}X_2^2+\tfrac{1}{4}X_3^2-4=0,$$

where X_1, X_2, X_3 are unbounded.

The best solution of Problem 338 found by the six codes in Table 2 is

$$(X_1, X_2, X_3)=(0.3669, 2.244, -1.427),$$
$$F(x)=-7.2057.$$

in which the stopping tolerance is specified as 10^{-7} and a huge amount of starting points is tested.

We resolve this problem by specifying the tolerance as $\omega_1=\omega_2=\omega_3=0.001$ to obtain the solution as

$$(X_1, X_2, X_3)=(-0.363659, -1.66324, 2.84507),$$
$$F(x)=-10.993.$$

Solving this problem again by respecifying the tolerance as $\omega_1=\omega_2=\omega_3=0.0001$, we find the same solution. Denote $F(x^*)$ as the value of objective function for global optimum, the maximal error of our solution with respect to $F(x^*)$, is estimated below referring to Proposition 2:

$$F(x)-F(x^*)=3\left(\frac{0.0001^2}{4}\right)\cong 10^{-8}.$$

The obtained solution therefore is very close to the global optimum.

The comparison between these six codes and the proposed method is discussed below:

1. The quality of the solution for these six codes depends on the choice of starting point. In contrast, there is no requirement for the proposed method to specify an initial point.
2. None of these six codes can guarantee having found the global solution, but the proposed method can ensure finding the global solution within a pre-specified error.

Table 3
A comparison of optimum solutions found by the proposed method and six other codes listed in Table 2

	Best solution by the other six codes		Optimal solution by the proposed method				
	X	Objective value $F^*(X)$	X	Objective value $F^*(X)$	0–1 variables	No. of constraints	Note
Problem 21 in [4]	(2, 0)	−99.96	(2, 0)	−99.96	12	91	*
Problem 36 in [4]	(20, 11, 15)	−3300	(20, 11, 15)	−3300	34	1346	*
Problem 37 in [4]	(24, 12, 12)	−3456	(24, 12, 12)	−3456	34	1347	*
Problem 44 in [4]	(0.3, 0.4)	−15	(0.3, 0.4)	−15	40	326	*
Problem 338 in [6]	(0.336, 2.244, −1.427)	−7.2057	(−0.363659, −1.66324, 2.84507)	−10.993	42	134	**
Problem 340 in [6]	(0.6, 0.3, 0.3)	−0.054	(0.6, 0.3, 0.3)	−0.054	12	183	*

Note: * = finds the same solution.
** = finds a better solution.

3. These six codes can only be applied to solving problems with continuous variables. But the proposed method can solve problems containing both continuous and discrete variables.

5. Conclusions

This paper proposes a practical and useful linearization technique to approximate polynomial terms in a polynomial program. Using the technique, a program with a polynomial objective function and constraints can be solved to reach a global optimum within a pre-specified tolerance. Many examples in the literature are tested, which demonstrate that the proposed method is very promising with regard to solving a general polynomial program to obtain an approximately global optimum.

Theoretically, the proposed method can solve a polynomial program to find a solution which is as close as possible to a global optimum. A major difficulty of implementing the proposed method is that if the range of a variable x_i is large and the upper bound of the tolerant error of x_i (i.e., ω_i) is small, then it requires to add many binary variables to represent x_i. This will increase the computational burden in the solution process. One possible way of overcoming this difficulty is to divide the interval of a variable before solving the problem. This remains for further study.

References

[1] Fu, J.F., et al., A mixed integer-discrete-continuous programming method and its application to engineering design optimization, *Engineering Optimization* 17 (1991) 263–280.

[2] Hansen, P., Jaumard, B., and Lu, S.H., A framework for algorithms in globally optimal design, *ASME, Journal of Mechanisms, Transmissions, and Automation in Design* 111 (1989) 353–360.

[3] Hansen, P., Jaumard, B., and Lu, S.J., An Analytical approach to global optimization, *Mathematical Programming* 52 (1991) 227–254.

[4] Hock, W., and Schittkowski, K., *Test Examples for Nonlinear Programming Codes*, Lecture Notes in Economics and Mathematical Systems No. 18, Springer-Verlag, Heidelberg, 1981.

[5] Horst, R., and Tuy, H., *Global Optimization: Deterministic Approaches*, Springer-Verlag, Berlin, 1990.

[6] Schittkowski, K., *More Test Examples for Nonlinear Programming Codes*, Lecture Notes in Economics and Mathematical Systems No. 282, Springer-Verlag, Heidelberg, 1987.

[7] Li, H.L., An approximate method for local optima for nonlinear mixed integer programming problems, *Computers & Operations Research* 19/5 (1992) 435–444.

[8] Li, H.L., A global approach for general 0–1 fractional programming, *European Journal of Operational Research* 73 (1994) 590–596.

[9] Li, H.L., and Chou, C.T., A global approach for nonlinear mixed discrete programming in design optimization, *Engineering Optimization* 22 (1994) 109–122.

[10] Linus, S., LINDO Release 5.3, LINDO System, Inc., 1994.

[11] Pardalos, P.M., and Rosen, J.B., *Constrained Global Optimization: Algorithms and Application*, Springer-Verlag, Berlin, 1987.

[12] Rinnooy Kan, A.H.G., and Timmer, G., Towards global optimization methods (I and II), *Mathematical Programming* 39 (1987) 27–78.

[13] Sandgren, E., Nonlinear integer and discrete programming in mechanical design optimization, *Journal of Mechanical Design* 112 (1990) 223–229.

[14] Sherali, H.D., and Adams, W.P., A hierarchy of relaxations between the continuous and convex hull representations for zero–one programming problems, *SIAM Journal on Discrete Mathematics* 3 (1990) 411–430.

[15] Sherali, H., and Tuncbilek, C., A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique, *Journal of Global Optimization* 2 (1992) 101–112.