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Brief Paper

A Multirate Controller Design of Linear Periodic Time Delay Systems*

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Key Words—Sampled-data system; linear periodic system; delay; deadbeat control; suboptimal control; output feedback control.

Abstract—This paper presents a multirate controller design for a linear periodic system with multiple delays at input and output. The approach first converts the periodic time-delay system into a periodic delay-free system, and then stabilizes and optimizes it by a multirate controller with pulse compensation. A significant advantage of this approach is that by using multirate sampling, the controller can provide more substantial design freedoms, so that although the system does not provide complete state information, it remains possible to convert the controller design into the dual of a regular complete state feedback problem. This enables one to derive a simple algorithm for choosing the optimal parameters of the controller and, by use of the optimal pulse compensation, to improve the transient response.

1. *Introduction*

PERIODIC SYSTEMS are an important class of control systems. Many time-varying mechanical and chemical processes exhibit periodical property and are best described by periodic models (Onogi and Matsubara, 1980; Schadlich *et al.*, 1983). Various valuable approaches for controlling such models have been proposed in the past two decades, e.g. optimal periodic filtering and control (Bittani and Bolzern, 1985a; Bittani *et al.*, 1990; Kano and Nishimura, 1985); periodic eigenvalue assignment (Al-Rahmani and Franklin, 1989; Kabamba, 1986); periodic deadbeat control (Grasselli and Lampariello, 1981), etc. Roughly speaking, to control a linear periodic system, a periodic state feedback is sufficient to guarantee the closed-loop asymptotic stability and to obtain the desired performance specification under some constraints.

It is interesting to control a linear periodic time-delay system. However, this problem may encounter some more difficulties than that of an ordinary delay-free system. One difficulty arises from the implementation. This is because in the stabilization of a linear time-delay system, not only the present state, but also the past states or controls are needed. As has been pointed out, (Åström and Wittenmark, 1984) an analog Smith predictor is difficult to implement, because it needs to store and integrate the past controls at every instant, so that for practical implementation, a sampled-data controller is more convenient than an analog controller. Another difficulty is that an optimal algorithm of a linear time-delay system is generally very cumbersome and hard to solve. As a result, convenient suboptimal algorithms are often suggested for controlling a linear time-delay system.

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In this paper, a multirate controller design is proposed for a linear periodic system with multiple delays at input and output. A motivation of this approach came from the paper of Al-Rahmani and Franklin (1990), who have shown that a multirate controller for periodic delay-free systems presents some advantages over that of a single rate controller. For example, the linear quadratic regulation problem subject to the multirate structure can be solved from an algebraic Riccati equation with a dimension equal to the plant while, in general, a single-rate approach needs to solve an algebraic Riccati equation with a dimension higher than the plant (Meyer and Burrus, 1975). Furthermore, it is possible for a multirate controller to sample the state relatively slowly while the response characteristics are still met by the fast rate of control updation. Thus more freedoms can be obtained on the choice of the sampling period.

A significant advantage of the presented multirate controller for a periodic time-delay system is that by taking a large ratio of sampling rate between input and output, the controller can provide more substantial design freedoms, so that although the system does not provide complete state information, it remains possible to convert the controller design into the dual of a regular complete state feedback problem. This enables one to derive a simple algorithm for choosing the optimal parameters of the controller. A distinctive feature of the proposed controller is that a pulse compensation is employed to improve the transient characteristics which might be badly influenced solely by an output feedback because of multiple time delays. In particular, if the initial state is known, then the system can be driven to the zero steady-state in a time no larger than the sum of the maximum input delay and the sampling period.

2. *Preliminary*

2.1. *The plant.* Consider a linear periodic time-delay system

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{f} B_i(t)u(t-h_i), \tag{1a}$$

$$y(t) = \sum_{j=1}^{g} C_j(t)x(t-\bar{h}_j), \tag{1b}$$

where f and g are positive integers, h_i and $\bar{h}_j$ are delay times satisfying $0 \le h_1 < \cdots < h_f$ and $0 \le \bar{h}_1 < \cdots < \bar{h}_g$, $x \in R^n$ is the state vector, $u \in R^{m_1}$ is the control input, $y \in R^{m_2}$ $(m_2 \le n)$ is the measurable output, and the parameters $A(t)$, $B_i(t)$ and $C_j(t)$ are piecewise continuous and satisfying the periodical property that $A(t) = A(t-T)$, $B_i(t) = B_i(t-T)$ and $C_j(t) = C_j(t-T)$ for some positive real T.

2.2. *The transformation algorithm.* By using the following transformation algorithm (see Lemma A.1 in the Appendix):

$$\bar{x}(t) = x(t) + \sum_{i=1}^{f} \int_{t-h_i}^{t} \phi(t, s+h_i)B_i(s+h_i)u(s)\,ds, \tag{2a}$$

$$\bar{y}(t) = y(t) + \sum_{j=1}^{g}\sum_{i=1}^{f} C_j(t) \times \int_{t-h_i-\bar{h}_j}^{t} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i)u(s)\,ds, \tag{2b}$$

one can convert the periodic time-delay system (1) into a periodic delay-free system as follows:

$$\dot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t)u(t), \tag{3a}$$

$$\bar{y}(t) = C(t)\bar{x}(t), \tag{3b}$$

where $B(t)$ and $C(t)$ are given by

$$B(t) = \sum_{i=1}^{f} \phi(t, t+h_i)B_i(t+h_i), \tag{4a}$$

$$C(t) = \sum_{j=1}^{g} C_j(t)\phi(t-\bar{h}_j, t), \tag{4b}$$

and $\phi(t, s)$ is the state transition matrix satisfying

$$\frac{d}{dt}\phi(t, s) = A(t)\phi(t, s); \quad \phi(s, s) = I_n. \tag{5}$$

2.3. *Controllability and observability*. Considering the converted periodic delay-free systems (3), the reachability Gramian matrix on $[0\ T]$ is defined as

$$\Omega_b[0\ \ T) = \int_0^T \phi(T, s)B(s)B(s)^\tau\phi(T, s)^\tau\, ds, \tag{6a}$$

where τ denotes the transpose operation of a matrix, and the observability Gramian matrix on $[0\ T)$ is defined as

$$\Omega_c[0\ \ T) = \int_0^T \phi(T, s)B(s)B(s)^\tau\phi(T, s)^\tau\, ds, \tag{6a}$$

It is assumed that $(A(t), B(t))$ is controllable on $[0\ T)$ and $(C(t), A(t))$ is observable on $[0\ T)$ in the sense that $\Omega_b[0\ T)$ and $\Omega_c[0\ T)$ are nonsingular, respectively (Bittani and Bolzern, 1985a; Bittani *et al.*, 1985b, Kabamba, 1986). As will be clear later that the controllability implies that the periodic time-delay system (1a) with any initial state can be driven to the zero steady-state in a time no larger than $h_f + T$, and the observability implies that there exists finite reals $\theta_1, \theta_2, \ldots, \theta_p$ satisfying $0 \le \theta_1 < \theta_2 < \cdots < \theta_p < T$ such that the initial state $x(0)$ can be constructed by $y(-\theta_1)$, $y(-\theta_2), \ldots, y(-\theta_p)$ and $u(-s)$ for all $s \in [0\ h_f + \bar{h}_g + \theta_p)$.

3. *Multirate controller design*

3.1. *Multirate structure*. Based on the converted periodic delay-free system (3), a multirate controller is proposed as follows:

$$u(kT + iT/M + \theta) = P_i(L_1\bar{y}(kT) + L_2\delta_0(kT)), \tag{7}$$

where $0 \in [0\ T/M)$, $i = 0, 1, 2, \ldots, M-1$, M is an integer, $L_1 \in R^{n\times m_2}$, $L_2 \in R^{n\times 1}$, $\delta_0(kT)$ denotes the discrete-time pulse function (i.e. $\delta_0(kT) = 1$ for $k = 0$, and $\delta_0(kT) = 0$ for $k = 1, 2, \ldots$), and $P_i \in R^{m_1\times n}$ denotes the normalized piecewise gains described by

$$P_i = M/T\Omega_i^\tau\Omega^{-1}, \tag{8}$$

in which Ω denotes the generalized reachability Gramian of order M on $[0\ T)$ (Al-Rahmani and Franklin, 1990) given by

$$\Omega = M/T \sum_{i=0}^{M-1} \Omega_i\Omega_i^\tau, \tag{9a}$$

where

$$\Omega_i = \int_{iT/M}^{(i+1)T/M} \phi(T, s)B(s)\, ds. \tag{9b}$$

Since $(A(t), B(t))$ is controllable on $[0\ T)$, the inverse of Ω can be guaranteed if M is sufficiently large (Al-Rahmani and Franklin, 1989).

Now, by the periodical property of $A(t)$, $B(t)$ and $C(t)$, the state of the converted periodic delay-free system (3) satisfies

$$\bar{x}((k+1)T) = \phi(T, 0)\bar{x}(kT) + \int_0^T \phi(T, s)B(s)u(kT+s)\, ds. \tag{10}$$

By substituting the multirate control (7) into (10), one can obtain

$$\bar{x}((k+1)T) = (\bar{A} + L_1\bar{C})\bar{x}(kT) + L_2\delta_0(kT), \tag{11}$$

where $\bar{A} = \phi(T, 0)$ and $\bar{C} = C(kT) = C(0)$. If $\bar{A} + L_1\bar{C}$ is asymptotically stable (i.e. all eigenvalues lie inside the unit complex circle), then $\bar{x}(kT) \to 0$ as $k \to \infty$, so that by (7) and (2), $x(t) \to 0$ and $u(t) \to 0$ as $t \to \infty$.

3.2. *Computation of* $\bar{y}$(kT). The converted output $\bar{y}(kT)$ can be calculated from a discrete-time model. To do so, let $\bar{C}_j = C_j(kT) = C_j(0)$ for $j = 1, \ldots, g$, N be a positive integer satisfying $(N-1)T < h_f + \bar{h}_g \le NT$, and

$$W(s) = \sum_{i=1}^{f}\sum_{j=1}^{g} W_{ij}(s), \tag{12}$$

where

$$W_{ij}(s) = \begin{cases} \bar{C}_j\phi(-\bar{h}_j, h_i - s)B_i(h_i - s) & \text{when } 0 \le s < h_i + \bar{h}_j, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

From (2b), one can rearrange $\bar{y}(kT)$ as

$$\begin{aligned}\bar{y}(kT) &= y(kT) + \int_0^{NT} W(s)u(kT-s)\, ds \\ &= y(kT) + \sum_{j=0}^{N-1}\int_0^T W(jT+s)u(kT-jT-s)\, ds.\end{aligned} \tag{14}$$

So after $k \ge N$, $\bar{y}(kT)$ can be calculated by

$$\begin{aligned}&\bar{y}(kT) \\ &\quad = y(kT) + \sum_{j=1}^{N} \gamma_j(L_1\bar{y}((k-j)T) + L_2\delta_0((k-j)T)),\end{aligned} \tag{15}$$

where

$$\gamma_j = \sum_{z=0}^{M-1}\int_{zT/M}^{(z+1)T/M} W((j-1)T+s)P_{m-z-1}\, ds. \tag{16}$$

Since $\bar{y}(kT)$ may not obey the discrete-time model (15) for $k = 0, 1, 2, \ldots, N-1$, a precise calculation of $\bar{y}(kT)$ can be done as follows:

$$\bar{y}(kT) = \begin{cases} y(KT) + \int_0^{NT} W(s)u(kT-s)\, ds, & \text{for } k < N \\ y(kT) + \sum_{j=1}^{N} \gamma_j(L_1\bar{y}((k-j)T) + L_2\delta_0((k-j)T) & \text{for } k \ge N. \end{cases} \tag{17}$$

4. *An optimal approach*

Define the performance index

$$J = \sum_{k=1}^{\infty} E(\bar{x}^\tau(kT)Q\bar{x}(kT)), \tag{18}$$

where $E(\#)$ denotes the expectation of a random vector #, and $Q \in R^{n\times n}$ is a positive-definite matrix (denoted by $Q > 0$). By the closed-loop sampled-data system (11), the quadratic performance index (18) can be expressed by Kwakernaak and Sivan (1972)

$$J = \mathrm{Tr}\, VE(\bar{x}(T)\bar{x}^\tau(T)), \tag{19}$$

where $V \in R^{n\times n}$ is a positive-definite matrix solved from the following Lyapunov equation:

$$(\bar{A} + L_1\bar{C})^\tau V(\bar{A} + L_1\bar{C}) - V + Q = 0. \tag{20}$$

By canceling redundant output variables, it does not lose the generality to assume that $\bar{C}$ is of full rank. The following theorem shows that the optimal parameters L_1 and L_2 of the multirate controller (7) to minimize the quadratic performance index (18) is unique and only dependent on the expectation and the covariance of the converted initial state $\bar{x}(0)$.

Theorem 1. Assume $(\bar{C}, \bar{A})$ is observable, and the expectation and the covariance of the initial state $\bar{x}(0)$ are given by

$$E[\bar{x}(0)] = E[x(0)] + \sum_{i=1}^{f} \int_{-h_i}^{0} \phi(0, s + h_i)B_i(s + h_i)u(s)\,ds = \Psi_1, \quad (21a)$$

and

$$\mathrm{Cov}(\bar{x}(0)) = \mathrm{Cov}(x(0)) = E[(x(0) - E(x(0)))(x(0) - E(X(0)))^{\tau}] = \Psi_2 > 0, \quad (21b)$$

then the optimal parameters L_1 and L_2 of the multirate controller (7) to minimize the performance index (18) can be obtained as

$$L_1 = -\bar{A}\lambda\bar{C}^{\tau}(\bar{C}\lambda\bar{C}^{\tau})^{-1}, \quad (22a)$$

and

$$L_2 = -\bar{A}\Psi_1 - L_1\bar{y}(0), \quad (22b)$$

where $\lambda \in R^{n\times n}$ is the positive-definite matrix solved from the following algebraic Riccati equation:

$$\bar{A}\lambda\bar{A}^{\tau} - \bar{A}\lambda\bar{C}^{\tau}(\bar{C}\lambda\bar{C}^{\tau})^{-1}\bar{C}\lambda\bar{A}^{\tau} - \lambda + \bar{A}\Psi_2\bar{A}^{\tau} = 0. \quad (22c)$$

Proof. From the closed-loop sampled-data system (11) one has

$$\bar{x}(T) = \bar{A}x(0) + L_1\bar{y}(0) + L_2 = \bar{A}\{\bar{x}(0) - E(\bar{x}(0))\} + \{L_1\bar{y}(0) + L_2 + \bar{A}E(\bar{x}(0))\}. \quad (23)$$

By substituting (23) into (19), the index can be expressed by

$$J = \mathrm{Tr}\, V\{\bar{A}E[(\bar{x}(0) - E(\bar{x}(0)))(\bar{x}(0) - E(\bar{x}(0)))^{\tau}]\bar{A}^{\tau} + (L_1\bar{y}(0) + L_2 + \bar{A}E(\bar{x}(0)))(L_1\bar{y}(0) + L_2 + \bar{A}E(\bar{x}(0)))^{\tau}\}. \quad (24)$$

From (24), it is obvious that

$$L_2 = -\bar{A}E(\bar{x}(0)) - L_1\bar{y}(0), \quad (25)$$

is the gain to minimize J for every specified L_1. Hence (22b) is obtained. On the other hand, by substituting (22b) into (24), the performance index is simplified as

$$J = \mathrm{Tr}\{V\bar{A}\,\mathrm{Cov}(\bar{x}(0))\bar{A}^{\tau}\} = \mathrm{Tr}\{V\bar{A}\,\mathrm{Cov}(x(0))\bar{A}^{\tau}\} = \mathrm{Tr}\{V\bar{A}\Psi_2\bar{A}^{\tau}\}. \quad (26)$$

To minimize the performance index (26) subject to the Lyapunov equation (20), an augmented cost can be introduced as follows (Bryson and Ho, 1975):

$$J = \mathrm{Tr}\{V\bar{A}\Psi_2\bar{A}^{\tau} + \lambda((\bar{A} + L_1\bar{C})^{\tau}V(\bar{A} + L_1\bar{C}) - V + Q)\}, \quad (27)$$

where $\lambda \in R^{n\times n}$ is the associated Lagrange multiplier. By taking $dJ/d\lambda = 0$, one obtains (20), and by taking $dJ/dV = 0$, one obtains

$$(\bar{A} + L_1\bar{C})\lambda(\bar{A} + L_1\bar{C})^{\tau} - \lambda + \bar{A}\Psi_2\bar{A}^{\tau} = 0. \quad (28)$$

Also, by taking $dJ/dL_1 = 0$, one obtains

$$V(\bar{A} + L_1\bar{C})\lambda\bar{C}^{\tau} = 0. \quad (29)$$

Since Q and Ψ_2 are positive-definite, the solution V and λ of the Lyapunov equations (20) and (28) are positive-definite, thus (29) leads to (22a). By substituting (22a) into (28), one also obtains (22c). Hence the necessity of (22) is proved. Besides, by (28) and (20), one has

$$J = \mathrm{Tr}\, V\bar{A}\Psi_2\bar{A}^{\tau} = -\mathrm{Tr}\, V\{(\bar{A} + L_1\bar{C})\lambda(\bar{A} + L_1\bar{C})^{\tau} - \lambda\} = -\mathrm{Tr}\,\lambda\{(\bar{A} + L_1\bar{C})^{\tau}V(\bar{A} + L_1\bar{C}) - V\} = \mathrm{Tr}\,\lambda Q. \quad (30)$$

It is known (Payne and Silverman, 1973; Caines and Mayne, 1970) that for the discrete-time Algebraic Riccati equation (22c) and (22a), there exists a unique stable solution to minimize the index (30) therefore the theorem is derived. □

Remark 1. In a general case, $(\bar{C}, \bar{A})$ may not be observable even if $(C(t), A(t))$ is observable on $[0\ T)$. In this condition, one can use a new observation as

$$y_{\mathrm{new}}(t) = \sum_{v=1}^{p} \bar{C}_v y(t - \theta_v) = \sum_{v=1}^{p}\sum_{j=1}^{g} \bar{C}_v C_j(t - \theta_v)x(t - \bar{h}_j - \theta_v), \quad (31)$$

where $\bar{C}_v \in R^{m_3\times m_2}$ (m_3 is a selected positive integer) and $0 \le \theta_1 < \theta_2 < \cdots < \theta_p < T$. Now, if the output (1b) is replaced by (31), the output transformation (2b) is replaced by

$$\bar{y}_{\mathrm{new}}(t) = y_{\mathrm{new}}(t) + \sum_{v=1}^{p}\sum_{j=1}^{g}\sum_{i=1}^{f} \bar{C}_v C_j(t - \theta_v) \times \int_{t-h_i-\bar{h}_j-\theta_v}^{t} \phi(t - \bar{h}_j - \theta_v, s + h_i)B_i(s + h_i)u(s)\,ds, \quad (32)$$

and the converted output matrix (4b) is replaced by

$$C(t)_{\mathrm{new}} = \sum_{v=1}^{p}\sum_{j=1}^{g} \bar{C}_v C_j(t - \theta_v)\phi(t - \bar{h}_j - \theta_v, t) = \sum_{v=1}^{p} \bar{C}_v C(t - \theta_v)\phi(t - \theta_v, t), \quad (33a)$$

hence one has

$$\bar{C}_{\mathrm{new}} = C(0)_{\mathrm{new}} = \sum_{v=1}^{p} \bar{C}_v C(-\theta_v)\phi(-\theta_v, 0). \quad (33b)$$

In view of the full rankness of $\Omega_c[0\ T)$, one can choose $\bar{C}_v$ and θ_v such that $(\bar{C}_{\mathrm{new}}, \bar{A})$ is observable.

Remark 2. One can estimate $\bar{x}(0)$ from an available observation. To do so, by substituting (31) into (32), one obtains

$$\bar{y}_{\mathrm{new}}(0) = \sum_{v=1}^{p} \bar{C}_v y(-\theta_v) + \sum_{v=1}^{p}\sum_{j=1}^{g}\sum_{i=1}^{f} \bar{C}_v C_j(-\theta_v) \times \int_{-h_i-\bar{h}_j-\theta_v}^{0} \phi(-\bar{h}_j - \theta_v, s + h_i)B_i(s + h_i)u(s)\,ds = \bar{C}_{\mathrm{new}}\bar{x}(0). \quad (34)$$

Now, one defines the least square error estimate of $\bar{x}(0)$ subject to the observation (31) as the vector $\psi \in R^n$ which minimizes the index $J_0 = \psi^{\tau}\psi$ subject to the equality that $\bar{y}_{\mathrm{new}}(0) = \bar{C}_{\mathrm{new}}\psi$. By minimizing the following augmented cost

$$J_0 = \psi^{\tau}\psi + l^{\tau}(\bar{C}_{\mathrm{new}}\psi - \bar{y}_{\mathrm{new}}(0)), \quad (35)$$

where $l \in R^{m_3}$ is the Lagrange multiplier, one obtains the estimate as follows (no loss of the generality to assume $\mathrm{rank}[\bar{C}_{\mathrm{new}}] = m_3 \le n$):

$$\psi = \bar{C}_{\mathrm{new}}^{\tau}(\bar{C}_{\mathrm{new}}\bar{C}_{\mathrm{new}}^{\tau})^{-1}\bar{y}_{\mathrm{new}}(0). \quad (36)$$

In particular, one has $\psi = \bar{x}(0)$ when $\mathrm{rank}[\bar{C}_{\mathrm{new}}] = m_3 = n$, so that $\bar{x}(0)$ can be reconstructed from the past inputs and outputs.

Remark 3. If Ψ_1 is substituted by the exact value $\bar{x}(0)$, then by (22b) and (11), $\bar{x}(kT) = 0$ for all $k \ge 1$. Thus by (2) and (7), $x(t) = 0$ and $u(t) = 0$ for all $t \ge h_f + T$. Therefore the periodic time-delay system (1a) can be driven to the zero steady-state in a time no larger than $h_f + T$. In general, $\bar{x}(0)$ is unknown, for practical applications, Ψ_1 can be substituted by an estimated vector (e.g. the least square error estimate subject to an available observation), $\Psi_2 = \mathrm{Cov}[\bar{x}(0)]$ can be substituted by a chosen positive-definite matrix to reflect the estimation error.

5. *Example*

Consider the following periodic time-delay system:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} \sin(2\pi t) \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \cos(2\pi t) \end{pmatrix} u(t-1), \quad (37a)$$

$$y(t) = [\cos(2\pi t) \quad 0]x(t) + [0 \quad \sin(2\pi t)](x(t-1)). \quad (37b)$$

Assume $u(-\theta) = 0$, $0 \le \theta < 2$, $E(x(0)) = [1 \ 1]^{\tau}$ and $\mathrm{Cov}(x(0)) = \epsilon I_2$, ϵ is a positive real. Using the transformation (2), this system is converted into the following periodic delay-free system:

$$\dot{\bar{x}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x}(t) + \begin{pmatrix} \sin(2\pi t) - \sin(1)\cos(2\pi t) \\ \cos(1)\cos(2\pi t) \end{pmatrix} u(t), \quad (38a)$$

$$\bar{y}(t) = \{[\cos(2\pi t) \quad 0] + \sin(2\pi t)[\sin(1) \quad \cos(1)]\}\bar{x}(t). \quad (38b)$$

By selecting $T = 1$ and $M = 10$, one has

$$\bar{A} = \begin{pmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{pmatrix}, \quad \bar{C} = [1 \quad 0]. \quad (39)$$

From (8) and (9), the normalized piecewise gains P_i are given by

$$P_0 = [2.6324 \quad 3.7952], \quad P_5 = [-1.3741 \quad -2.7266],$$
$$P_1 = [1.5140 \quad 1.1151], \quad P_6 = [-2.1336 \quad -2.1882],$$
$$P_2 = [0.5167 \quad -1.2664], \quad P_7 = [-2.8002 \quad -1.7031],$$
$$P_3 = [-0.1948 \quad -2.6847], \quad P_8[-2.9225 \quad -1.1940],$$
$$P_4 = [-0.7502 \quad -3.0512], \quad P_9 = [-2.0862 \quad -0.3665]. \quad (40)$$

Solving the equations (22), one obtains the following optimal gains:

$$L_1 = \begin{pmatrix} -0.7872 \\ 0.6829 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -1.3818 \\ 0.3012 \end{pmatrix} - L_1\bar{y}(0). \quad (41)$$

Besides, by (16), one obtains

$$\gamma_1 = [0.1075 \quad -0.3212], \quad \gamma_2 = [0 \quad 0]. \quad (42)$$

Thus the optimal multirate controller is taken as the following form:

$$u(kT + iT/M + \theta) = P_i(L_1\bar{y}(kT) + L_2\delta_0(kT)), \quad (43a)$$

where $i = 0, 1, \ldots, 9$, and $\bar{y}(kT)$ is given by

$$\bar{y}(kT) = \begin{cases} y(0) & \text{for } k = 0 \\ y(kT) + \gamma_1(L_1\bar{y}((k-1)T) + L_2\delta_0((k-1)T)) & \text{otherwise.} \end{cases} \quad (43b)$$

For comparison, this controller is simulated for six typical initial states (see the output plots (a)–(f) in Fig. 1 and the input plots (a)–(f) in Fig. 2). Notice that this controller is

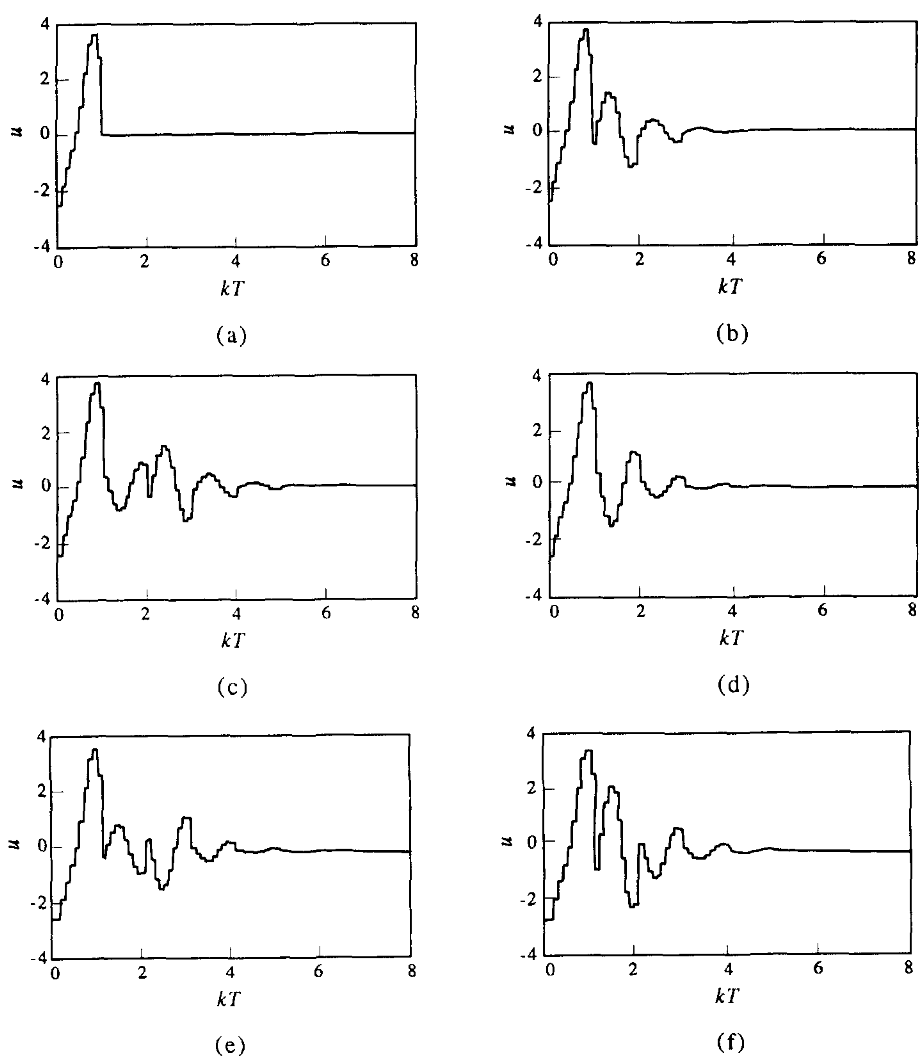

FIG. 1. The output responses of the periodic time-delay system (37) with the multirate controller (43). The initial state $x^{\tau}(0)$ are assumed by (a): [1 1] (b): [1 0], (c): [2 1], (d): [1 2], (e): [0 1] and (f): [0 0].

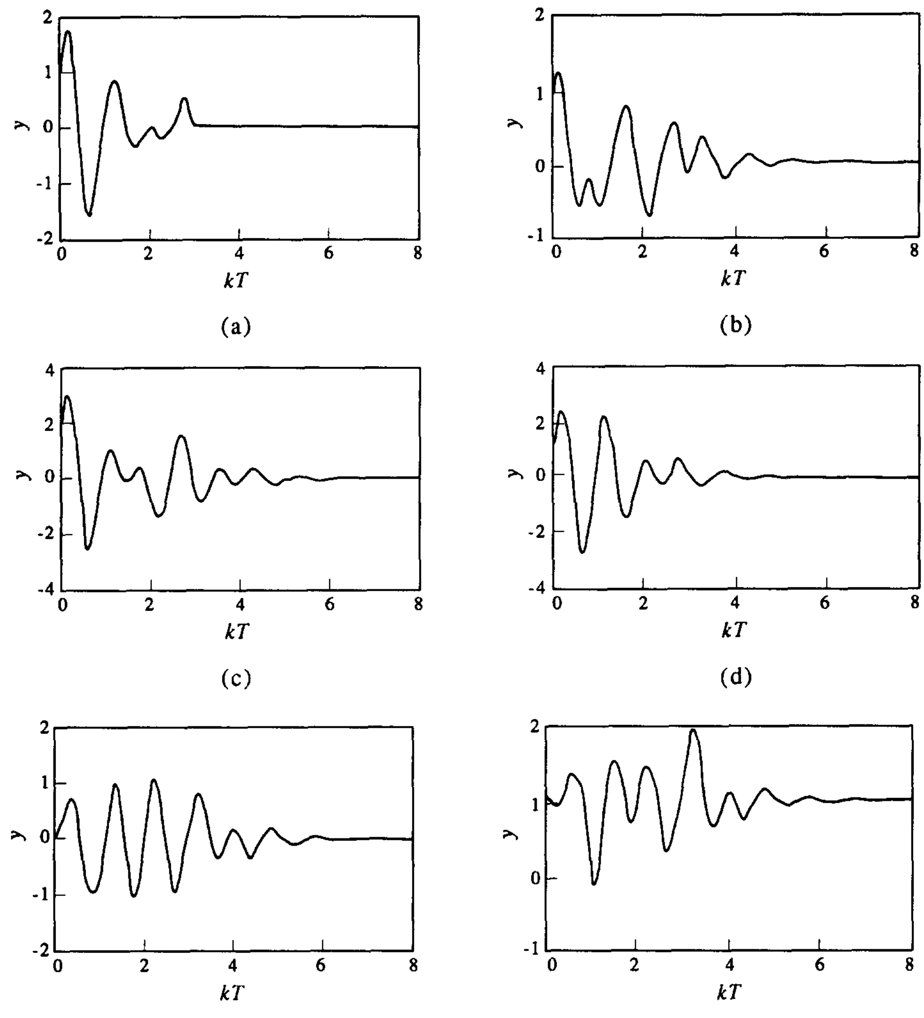

FIG. 2. The input responses of the periodic time-delay system (37) with the multirate controller (43). The same initial states as in Fig. 1 are assumed.

optimal from a statistical viewpoint for all possible initial states. In particular, if $E(x(0))$ is obtained from a precise reconstruction (i.e. $\mathrm{Cov}(x(0)) \approx 0$), the response almost approaches the deadbeat case (a).

6. *Conclusions*

In this paper, a multirate output feedback controller is presented for a linear periodic system with multiple delays at input and output. A simple algorithm is also derived for choosing the optimal parameters of the controller. Such an algorithm tends to yield a two-stage control. At the first stage, the optimal pulse compensation drives the system to approach the zero steady-state in a time no larger than the sum of the maximum input delay and the sampling period (if the estimation error of the converted initial state is small). Then at the second stage, the multirate output feedback control serves as a closed-loop suboptimal control to guarantee the asymptotic stability.

It is interesting to compare the presented multirate output feedback control scheme with the multirate state feedback control scheme of periodic delay-free systems suggested by Al-Rahmani and Franklin (1990). Since they have derived the control scheme based on the minimization of a continuous-time quadratic performance index, better inter-sampling behavior can be obtained. However, in general, complete state measurements are costly. Moreover, to a periodic system with multiple delays at input and output, the transient response during the period of time delay may be influenced solely by an output feedback. In this condition, a pulse compensation will be valuable for improving the transient response.

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Appendix: Converting a periodic time-delay system into a periodic delay-free system

Lemma A.1. By using the transformation (2), the periodic time-delay system (1) can be converted into the periodic delay-free system (3).

Proof. It is shown (Artstein, 1982) that by using the transformation (2a), the periodic time-delay system (1) can be converted into (3a). Thus by (1b), (2a) and (4a), one may rearrange $y(t)$ as

$$\begin{aligned}
y(t) &= \sum_{j=1}^{g} C_j(t)\Big\{\bar{x}(t-\bar{h}_j) - \sum_{i=1}^{f} \\
&\quad \times \int_{t-h_i-\bar{h}_j}^{t-\bar{h}_j} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i)u(s)\,ds\Big\} \\
&= \sum_{j=1}^{g} C_j(t)\Big\{\phi(t-\bar{h}_j, t)\bar{x}(t) \\
&\quad - \int_{t-\bar{h}_j}^{t} \phi(t-\bar{h}_j, s)B(s)u(s)\,ds \\
&\quad - \sum_{i=1}^{f}\int_{t-h_i-\bar{h}_j}^{t-\bar{h}_j} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i)u(s)\,ds\Big\} \\
&= \sum_{j=1}^{g} C_j(t)\Big\{\phi(t-\bar{h}_j, t)\bar{x}(t) \\
&\quad - \int_{t-\bar{h}_j}^{t} \phi(t-\bar{h}_j, s)\Big(\sum_{i=1}^{f} \phi(s, s+h_i)B_i(s+h_i)\Big)u(s)\,ds \\
&\quad - \sum_{i=1}^{f}\int_{t-h_i-\bar{h}_j}^{t-\bar{h}_j} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i)u(s)\,ds\Big\} \\
&= \sum_{j=1}^{g} C_j(t)\phi(t-\bar{h}_j, t)\bar{x}(t) - \sum_{j=1}^{g}\sum_{i=1}^{f} C_j(t) \\
&\quad \times \Big\{\int_{t-\bar{h}_j}^{t} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i))u(s)\,ds \\
&\quad + \int_{t-h_i-\bar{h}_j}^{t-\bar{h}_j} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i)u(s)\,ds\Big\} \\
&= \sum_{j=1}^{g} C_j(t)\phi(t-\bar{h}_j, t)\bar{x}(t) \\
&\quad - \sum_{j=1}^{g}\sum_{i=1}^{f} C_j(t)\int_{t-h_i-\bar{h}_j}^{t} \phi(t-\bar{h}_j, s+h_i)B_i(s+h_i)u(s)\,ds.
\end{aligned} \tag{A.1}$$

Hence by taking $\bar{y}(t)$ and $C(t)$ as (2b) and (4b), respectively, the converted output (3b) can be obtained. □