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# ELASTIC SOLUTIONS FOR A TRANSVERSELY ISOTROPIC HALF-SPACE SUBJECTED TO A POINT LOAD

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## SUMMARY

We rederive and present the complete closed-form solutions of the displacements and stresses subjected to a point load in a transversely isotropic elastic half-space. The half-space is bounded by a horizontal surface, and the plane of transverse isotropy of the medium is parallel to the horizontal surface. The solutions are obtained by superposing the solutions of two infinite spaces, one acting a point load in its interior and the other being free loading. The Fourier and Hankel transforms in a cylindrical co-ordinate system are employed for deriving the analytical solutions. These solutions are identical with the Mindlin and Boussinesq solutions if the half-space is homogeneous, linear elastic, and isotropic. Also, the Lekhnitskii solution for a transversely isotropic half-space subjected to a vertical point load on its horizontal surface is one of these solutions. Furthermore, an illustrative example is given to show the effect of degree of rock anisotropy on the vertical surface displacement and vertical stress that are induced by a single vertical concentrated force acting on the surface. The results indicate that the displacement and stress accounted for rock anisotropy are quite different for the displacement and stress calculated from isotropic solutions. © 1998 John Wiley & Sons, Ltd.

Key words: closed-form solution; transversely isotropic half-space; Fourier transform; Hankel transform; rock anisotropy

## INTRODUCTION

In general, the magnitude and distribution of the displacements and stresses in rock are predicted by using exact solutions that model rock as a linearly elastic, homogeneous and isotropic continuum. However, for rock masses cut by discontinuities, such as cleavages, foliations, stratifications, schistosities, joints, these analytical solutions should account for anisotropy. Anisotropy rocks are often modelled as orthotropic or transversely isotropic materials from the standpoint of practical considerations in engineering. In this paper, an elastic problem for a transversely isotropic medium is relevant.

A point load solution is the basis of complex loading problems. For an isotropic solid, it has been studied by Kelvin[1] for an infinite space, Boussinesq[2] and Cerruti[3] for a semi-infinite space with a vertical and horizontal point load, respectively. In the case of a single concentrated force acting in the interior of a half-space, Mindlin[4] proposed closed-form solutions for an isotropic medium by using the principle of superposition of 18 nuclei. Mindlin derived the solutions by

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Table I. Existing solutions for transversely isotropic media subjected to a point load

| Author | Space | Type of loading | Solutions |
|---|---|---|---|
| Michell[7] | Half | Vertical | Vertical surface displacement, and partial stresses (inapplicable to boundary value problems) |
| Wolf[8] | Half | Vertical | All displacements and stresses (oversimplified the elastic constants) |
| Koning[9] | Half | Vertical | All displacements and stresses |
| Barden[10] | Half | Vertical | Vertical surface displacement, and stresses on load axis |
| De Urena *et al.*[11] | Half | Vertical | All displacements and stresses |
| Misra *et al.*[12] | Half | Vertical | All displacements, and stresses on load axis (oversimplified the elastic constants) |
| Chowdhury[13] | Full | Vertical | All displacements, and stresses on load axis |
| | Half | Buried, vertical | All displacements, and stresses on load axis |
| Pan[14] | Full | 3-D | All displacements and stresses |
| Kröner[15] | Half | Vertical | All displacements (dimensionally incorrect) |
| Willis[16] | Full | Vertical | All displacements (cumbersome and inaccurate) |
| Lee[17] | Half | Buried, vertical | All stresses (complicated) |
| Lekhnitskii[18] | Half | Vertical | All stresses (incomplete) |
| Elliott[19] | Full | Vertical | All displacements and stresses (incomplete) |
| Shield[20] | Half | Buried, vertical | All displacements and stresses at the surface |
| Eubanks *et al.*[21] | Half | Vertical | (completeness of Lekhnitskii's method) |
| Lodge[22] | | | (transformed anisotropic problem into isotropic one, inapplicable to general boundary value problems) |
| Hata[23] | Half | Vertical | (rederived the Elliott's and Lodge's solution) |
| Chen[24] | Full | Vertical | All displacements and stresses |
| | | Horizontal | All displacements |
| Pan and Chou[25] | Full | 3-D | All displacements and stresses |
| Pan and Chou[26] | Half | Buried, vertical | All displacements, and stresses on load axis |
| | | Buried, horizontal | All displacements, and partial stresses (potential functions assumed are lengthy) |
| Okumura *et al.*[27] | Half | Vertical | All displacements |
| Fabrikant[28] | Full | 3-D | All displacements, and partial stresses |
| | Half | 3-D | All displacements, and partial stresses (solution of the shear stress is wrong) |
| Lin *et al.*[29] | Half | Vertical, horizontal | All displacements and stresses |
| Hanson *et al.*[30] | Half | Buried, 3-D | (only the potential functions listed) |
| Sveklo[31] | Half | Vertical | All displacements |
| Sveklo[32] | Full | Vertical | All displacements |
| | Half | Buried, vertical | All displacements |

following the Kelvin's[1] approach and satisfying the condition of vanishing traction on a plane boundary. However, the calculation of nuclei for a half-space is very difficult.[5] Dean *et al.*[6] recommended another approach for the same problem by using the method of images. Some of the solutions can be extended to anisotropic media, whereas others are difficult.

For the displacements and stresses in transversely isotropic media subjected to a point load, analytical solutions have been presented by several investigators.[7–32] Some of the solutions were directly derived by the approaches for isotropic solutions.[7–14] Nevertheless, others employed complex mathematics techniques, such as Fourier transformations,[15–17] potential functions,[18–30] and complex variables,[31,32] etc. A summary of the existing solutions is given in Table I. Table I indicates the type of analytical space, loads, and the results presented in their solutions. Because of mathematical difficulty or oversimplification for solving the problems, these solutions were limited to three-dimensional problems with partial results of displacements[7,10,20] and stresses,[7,10,12,13,20,26,28] or axially symmetric problems[9,11–13,18,19,31] with neglecting the tangential co-ordinate, $\theta$. Neglecting $\theta$, the solutions cannot be extended to solve a half-space problem subjected to asymmetric loads. Also, it is found that Pan and Chou[26] proposed a more general solution by using the potential functions. In their solution, the buried loads can be vertical or horizontal with respect to the boundary plane. However, only the stress components related to the $z$-direction were given (i.e. $\sigma_{zz}$, $\sigma_{zx}$, $\sigma_{zy}$), and the expressions for the solution are quite lengthy.

A more efficient analysis for a transversely isotropic infinite space was given by Tarn and Wang[33] by employing the Fourier and Hankel transforms. The derivation of their solution is completely systematic and the solution is the same as the Kelvin[1] solution for the medium being isotropic. Following the method, Lu[34] presented analytical solutions for the displacements in an infinite or a semi-infinite soil mass (transverse isotropy) under a long-term consolidation. However, comparing with the Mindlin[4] solution, a part of his solutions for isotropic media is not correct.

Utilizing the approaches proposed by Tarn and Wang,[33] the closed-form solutions of displacements and stresses in a transversely isotropic half-space subjected to a point load are presented in this paper. These solutions indicate that both of the displacements and stresses in a transversely isotropic half-space are affected by the loading types (radial, tangential or normal), and the degree and type of rock anisotropy. An illustrative example is given at the end of this paper to investigate the effect of rock anisotropy on the displacement and stress in a medium subjected to a vertical point load.

## EXACT SOLUTIONS FOR THE DISPLACEMENTS AND STRESSES IN A TRANSVERSELY ISOTROPIC HALF-SPACE

A problem of a point load acting in the interior (including on the surface) of a semi-infinite space is relevant to this paper. The exact solutions for the displacements and stresses in a transversely isotropic half-space are derived by the principle of superposition as shown in Figure 1. Figure 1 depicts that a half-space is composed of two infinite spaces, one subjected to a point load in its interior and the other being free loading, and zero stress conditions on the $z = 0$ plane.[34] Hence, the solutions can be derived from the governing equations for an infinite space (including the general solutions (I) and homogeneous solutions (II)) by satisfying the free traction on the surface of the half-space. The problem of an infinite space acting a point load is first solved below.

### *Displacements and stresses in a full space*

Solving the displacements in an infinite mass subjected to a single concentrated force (Figure 2) proposed by Tarn and Wang,[33] is followed for solving the displacements and stresses in a half-space. Figure 2 depicts that the $r$–$\theta$ plane of a cylindrical co-ordinate system is attached to the planes of elastic symmetry of a transversely isotropic material. The $X$–$Y$ plane of a Cartesian co-ordinate system is parallel to the $r$–$\theta$ plane. Then, the expression of generalized Hooke's law for transversely isotropic solids in a cylindrical co-ordinate system is

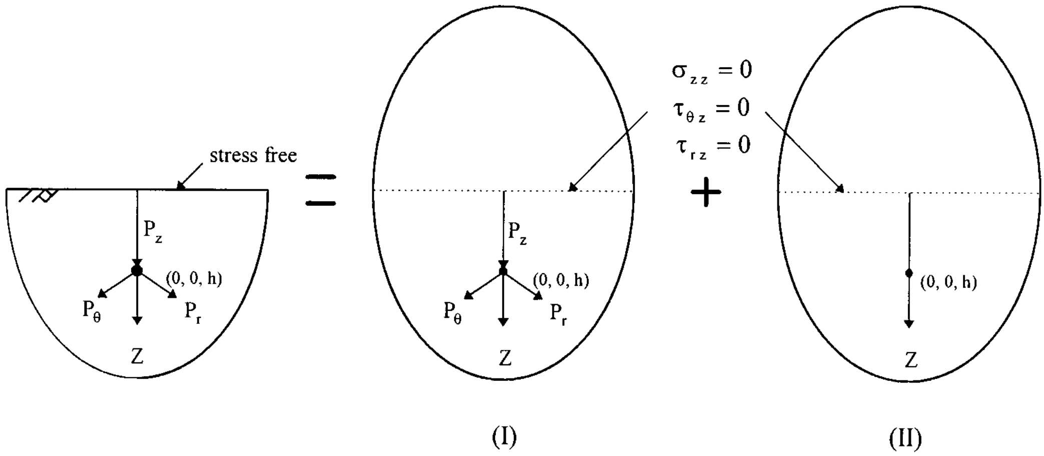


Figure 1. ($P_r$, $P_\theta$, $P_z$) acting in the interior of a semi-infinite space

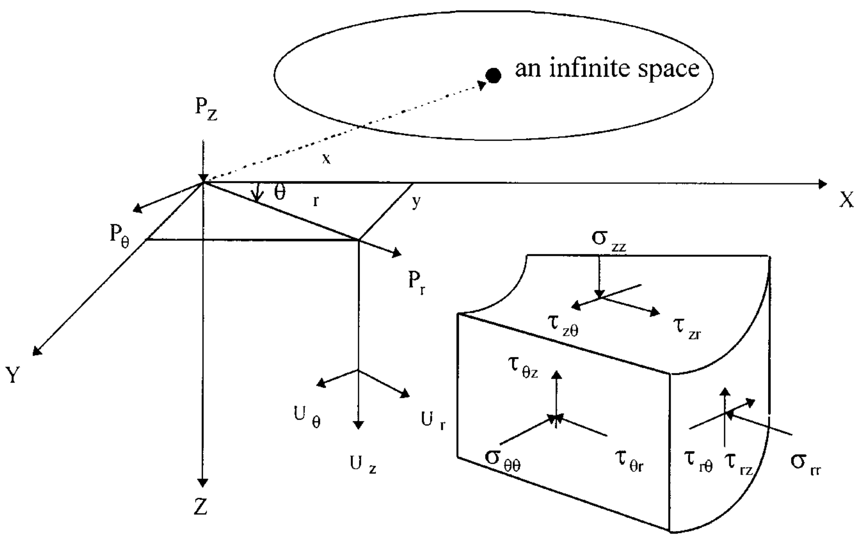


Figure 2. ($P_r$, $P_\theta$, $P_z$) acting in an infinite space

given as follows:

$$\sigma_{rr} = A_{11}\varepsilon_{rr} + (A_{11} - 2A_{66})\varepsilon_{\theta\theta} + A_{13}\varepsilon_{zz} \tag{1}$$

$$\sigma_{\theta\theta} = (A_{11} - 2A_{66})\varepsilon_{rr} + A_{11}\varepsilon_{\theta\theta} + A_{13}\varepsilon_{zz} \tag{2}$$

$$\sigma_{zz} = A_{13}(\varepsilon_{rr} + \varepsilon_{\theta\theta}) + A_{33}\varepsilon_{zz} \tag{3}$$

$$\tau_{r\theta} = A_{66}\gamma_{r\theta} \tag{4}$$

 

$$\tau_{\theta z} = A_{44}\gamma_{\theta z} \tag{5}$$

$$\tau_{rz} = A_{44}\gamma_{rz} \tag{6}$$

where $A_{ij}$ $(i, j = 1\text{–}6)$ are the elastic moduli[35] or elasticity constants[36] of the medium. For a transversely isotropic material, only five independent elastic constants are needed to describe its deformational response. In this paper, the five engineering elastic constants, $E$, $E'$, $v$, $v'$ and $G'$ are adopted and defined as follows:

1. $E$ and $E'$ are Young's moduli in the plane of transverse isotropy and in a direction normal to it, respectively.
2. $v$ and $v'$ are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively.
3. $G'$ is the shear modulus in planes normal to the plane of transverse isotropy. Hence, $A_{ij}$ can be expressed in terms of these elastic constants as

$$A_{11} = \frac{E\left(1 - \frac{E}{E'}\,\upsilon'^2\right)}{(1+\upsilon)\left(1 - \upsilon - \frac{2E}{E'}\,\upsilon'^2\right)}, \qquad A_{13} = \frac{E\upsilon'}{1 - \upsilon - \frac{2E}{E'}\,\upsilon'^2}, \qquad A_{33} = \frac{E'(1-\upsilon)}{1 - \upsilon - \frac{2E}{E'}\,\upsilon'^2},$$

$$A_{44} = G', \qquad A_{66} = \frac{E}{2(1+\upsilon)} \tag{7}$$

For small strain conditions, the expressions of strain–displacement relations in a cylindrical co-ordinate system are

$$\varepsilon_{rr} = -\frac{\partial U_r}{\partial r} \tag{8}$$

$$\varepsilon_{\theta\theta} = -\frac{U_r}{r} - \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} \tag{9}$$

$$\varepsilon_{zz} = -\frac{\partial U_z}{\partial z} \tag{10}$$

$$\gamma_{r\theta} = -\frac{1}{r}\frac{\partial U_r}{\partial \theta} - \frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r} \tag{11}$$

$$\gamma_{\theta z} = -\frac{\partial U_\theta}{\partial z} - \frac{1}{r}\frac{\partial U_z}{\partial \theta} \tag{12}$$

$$\gamma_{rz} = -\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r} \tag{13}$$

where $U_r$, $U_\theta$ and $U_z$ are radial, tangential and vertical displacements, respectively.

 

Also, the partial differential forms of equilibrium equations are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = R \tag{14}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \Theta \tag{15}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = Z \tag{16}$$

where $R$, $\Theta$, $Z$ are the components of the body forces per unit volume on the co-ordinate directions, $r$, $\theta$ and $z$, respectively.

For a dynamic elastic problem, an arbitrary time-harmonic body force in $z$ direction with angular frequency $\omega$ can be expressed as[37,38]

$$Z(r, \theta, z, t) = Z^*(r, \theta, z)\mathrm{e}^{\mathrm{i}\omega t} \tag{17}$$

where $Z^*$ is the complex amplitude of the body force. Following the suggestions,[37,38] a concentrated force in $z$ direction ($P_z$), can be represented as the form of a body force:

$$Z = \frac{P_z}{r}\,\delta(r)\delta(\theta)\delta(z)\mathrm{e}^{\mathrm{i}\omega t} \tag{18}$$

where $\delta(\,)$ is the Dirac delta function. As for a static case concerning about in this paper, $\omega$ in equation (18) will be zero. Hence, a static point load with components ($P_r$, $P_\theta$, $P_z$), acting at the origin of the co-ordinate for an infinite space can be expressed as the form of body forces:

$$R = \frac{P_r}{r}\,\delta(r)\delta(\theta)\delta(z) \tag{19}$$

$$\Theta = \frac{P_\theta}{r}\,\delta(r)\delta(\theta)\delta(z) \tag{20}$$

$$Z = \frac{P_z}{r}\,\delta(r)\delta(\theta)\delta(z) \tag{21}$$

Substituting $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$, $\tau_{r\theta}$, $\tau_{\theta z}$, $\tau_{rz}$ (equations (1)–(6)) and $R$, $\Theta$, $Z$ (equations (19)–(21)) into equations (14)–(16), and adopting the strain–displacement relations (equations (8)–(13)), then equations (14)–(16) can be regrouped as the Navier–Cauchy equations for transversely isotropic materials:

$$A_{11}\left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{U_r}{r^2}\right) + A_{66}\frac{1}{r^2}\frac{\partial^2 U_r}{\partial \theta^2} + A_{44}\frac{\partial^2 U_r}{\partial z^2} + (A_{11} - A_{66})\frac{1}{r}\frac{\partial^2 U_\theta}{\partial r\,\partial \theta}$$
$$- (A_{11} + A_{66})\frac{1}{r^2}\frac{\partial U_\theta}{\partial \theta} + (A_{13} + A_{44})\frac{\partial^2 U_z}{\partial r\,\partial z} = -\frac{P_r}{r}\,\delta(r)\delta(\theta)\delta(z) \tag{22}$$

$$(A_{11}-A_{66})\frac{1}{r}\frac{\partial^2 U_r}{\partial r\,\partial\theta}+(A_{11}+A_{66})\frac{1}{r^2}\frac{\partial U_r}{\partial\theta}+A_{66}\left(\frac{\partial^2 U_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial U_\theta}{\partial r}-\frac{U_\theta}{r^2}\right)$$

$$+A_{11}\frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2}+A_{44}\frac{\partial^2 U_\theta}{\partial z^2}+(A_{13}+A_{44})\frac{1}{r}\frac{\partial^2 U_z}{\partial\theta\,\partial z}=-\frac{P_\theta}{r}\delta(r)\delta(\theta)\delta(z) \tag{23}$$

$$(A_{13}+A_{44})\left(\frac{\partial^2 U_r}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial U_r}{\partial z}\right)+(A_{13}+A_{44})\frac{1}{r}\frac{\partial^2 U_\theta}{\partial\theta\,\partial z}+A_{44}\left(\frac{\partial^2 U_z}{\partial r^2}+\frac{1}{r}\frac{\partial U_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_z}{\partial\theta^2}\right)$$

$$+A_{33}\frac{\partial^2 U_z}{\partial z^2}=-\frac{P_z}{r}\delta(r)\delta(\theta)\delta(z) \tag{24}$$

In order to solve equations (22)–(24), the following mathematic operations are made:

(i) The displacement functions $U_r$, $U_\theta$ and $U_z$ are transformed by a finite Fourier exponential transform with respect to the tangential co-ordinate $\theta$ as

$$\begin{Bmatrix} U_r^* \\ U_\theta^* \\ U_z^* \end{Bmatrix}=\int_0^{2\pi}\begin{Bmatrix} U_r \\ U_\theta \\ U_z \end{Bmatrix}e^{in\theta}\,d\theta \tag{25}$$

(ii) New displacement functions, $\Phi^*$ and $\Psi^*$ are introduced:

$$\Phi^*=U_r^*+iU_\theta^*,\qquad \Psi^*=U_r^*-iU_\theta^* \tag{26}$$

(iii) The displacement functions $\Phi^*$, $\Psi^*$ and $U_z^*$ are transformed by a system of proper Hankel transformations[39,40] with respect to the radial co-ordinate $r$ of order $n-1$,$n+1$ and $n$, respectively, in the following:

$$\begin{Bmatrix} \Phi_{n-1}^{**} \\ \Psi_{n+1}^{**} \\ U_{zn}^{**} \end{Bmatrix}=\int_0^{\infty} r\begin{Bmatrix} \Phi^* J_{n-1}(\xi r) \\ \Psi^* J_{n+1}(\xi r) \\ U_z^* J_n(\xi r) \end{Bmatrix}dr \tag{27}$$

where $J_n(\,)$ is the Bessel function of first kind of order $n$.

Then, equations (22)–(24) are rewritten by a system of ordinary differential equations as follows:

$$\left[-(A_{11}+A_{66})\xi^2+2A_{44}\frac{d^2}{dz^2}\right]\Phi_{n-1}^{**}+(A_{11}-A_{66})\xi^2\Psi_{n+1}^{**}+2(A_{13}+A_{44})\xi\frac{d}{dz}U_{zn}^{**}$$

$$=-2(P_r+iP_\theta)J_{n-1}(0)\delta(z) \tag{28}$$

$$(A_{11}-A_{66})\xi^2\Phi_{n-1}^{**}+\left[-(A_{11}+A_{66})\xi^2+2A_{44}\frac{d^2}{dz^2}\right]\Psi_{n+1}^{**}-2(A_{13}+A_{44})\xi\frac{d}{dz}U_{zn}^{**}$$

$$=-2(P_r-iP_\theta)J_{n+1}(0)\delta(z) \tag{29}$$

$$-(A_{13}+A_{44})\xi\frac{\mathrm{d}}{\mathrm{d}z}\Phi^{**}_{n-1}+(A_{13}+A_{44})\xi\frac{\mathrm{d}}{\mathrm{d}z}\Psi^{**}_{n+1}-2\left(A_{44}\xi^2-A_{33}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)U^{**}_{zn}$$
$$=-2P_zJ_n(0)\delta(z) \qquad (30)$$

The homogeneous solutions of equations (28)–(30) are obtained by solving the simultaneous ordinary differential equations[41] as

$$\Phi^{**}_{n-1}(H)=A_1\mathrm{e}^{u_1\xi z}+A_2\mathrm{e}^{u_2\xi z}+A_3\mathrm{e}^{-u_1\xi z}+A_4\mathrm{e}^{-u_2\xi z}+A_5\mathrm{e}^{u_3\xi z}+A_6\mathrm{e}^{-u_3\xi z} \qquad (31)$$

$$\Psi^{**}_{n+1}(H)=B_1\mathrm{e}^{u_1\xi z}+B_2\mathrm{e}^{u_2\xi z}+B_3\mathrm{e}^{-u_1\xi z}+B_4\mathrm{e}^{-u_2\xi z}+B_5\mathrm{e}^{u_3\xi z}+B_6\mathrm{e}^{-u_3\xi z} \qquad (32)$$

$$U^{**}_{zn}(H)=C_1\mathrm{e}^{u_1\xi z}+C_2\mathrm{e}^{u_2\xi z}+C_3\mathrm{e}^{-u_1\xi z}+C_4\mathrm{e}^{-u_2\xi z}+C_5\mathrm{e}^{u_3\xi z}+C_6\mathrm{e}^{-u_3\xi z} \qquad (33)$$

where $A_i$, $B_i$ and $C_i$ ($i=1$–$6$) are the undetermined coefficients and the relations between these three coefficients can be determined by substituting equations (31)–(33) into equations (28)–(30). Then, equations (31)–(33) can be expressed in terms of $B_i$ ($i=1$–$6$) as follows:

$$\Phi^{**}_{n-1}(H)=-B_1\mathrm{e}^{u_1\xi z}-B_2\mathrm{e}^{u_2\xi z}-B_3\mathrm{e}^{-u_1\xi z}-B_4\mathrm{e}^{-u_2\xi z}+B_5\mathrm{e}^{u_3\xi z}+B_6\mathrm{e}^{-u_3\xi z} \qquad (34)$$

$$\Psi^{**}_{n+1}(H)=B_1\mathrm{e}^{u_1\xi z}+B_2\mathrm{e}^{u_2\xi z}+B_3\mathrm{e}^{-u_1\xi z}+B_4\mathrm{e}^{-u_2\xi z}+B_5\mathrm{e}^{u_3\xi z}+B_6\mathrm{e}^{-u_3\xi z} \qquad (35)$$

$$U^{**}_{zn}(H)=-m_1B_1\mathrm{e}^{u_1\xi z}-m_2B_2\mathrm{e}^{u_2\xi z}+m_1B_3\mathrm{e}^{-u_1\xi z}+m_2B_4\mathrm{e}^{-u_2\xi z} \qquad (36)$$

where

$$m_i=\frac{(A_{13}+A_{44})u_i}{A_{33}u_i^2-A_{44}}=\frac{A_{11}-A_{44}u_i^2}{(A_{13}+A_{44})u_i}\quad (i=1,2);\quad u_3=\sqrt{(A_{66}/A_{44})};$$

$u_1$ and $u_2$ are the roots of the following characteristic equation:

$$u^4-su^2+q=0 \qquad (37)$$

where

$$s=\frac{A_{11}A_{33}-A_{13}(A_{13}+2A_{44})}{A_{33}A_{44}},\qquad q=\frac{A_{11}}{A_{33}}$$

Since the strain energy is assumed to be positive definite in the medium, the values of elastic constants are restricted.[42,43] Hence, there is three categories of the characteristic roots, $u_1$ and $u_2$ as follows:

*Case* 1: $u_{1,2}=\pm\sqrt{\{\frac{1}{2}[s\pm\sqrt{(s^2-4q)}]\}}$ are two real distinct roots when $s^2-4q>0$;
*Case* 2: $u_{1,2}=\pm\sqrt{s/2},\ \pm\sqrt{s/2}$ are double equal real roots when $s^2-4q=0$;
*Case* 3: $u_{1,2}=\frac{1}{2}\sqrt{(s+2\sqrt{q})}-i\frac{1}{2}\sqrt{(-s+2\sqrt{q})}=\gamma-\mathrm{i}\delta$, $u_2=\gamma+\mathrm{i}\delta$ are two complex conjugate roots (where $\gamma$ cannot be equal to zero) when $s^2-4q<0$.

Gerrard[44] and Amadei *et al.*[45] demonstrated that for most transversely isotropic rocks, $E/E'$ and $G/G'$ vary between 1 and 3 and the Poisson's ratios $\nu$ and $\nu'$ vary between 0·15 and 0·35. Figure 3 shows the distribution of the three categories of the characteristic roots for transversely isotropic rocks. The figure indicates that approximately two-thirds of transversely isotropic rocks belong to case 1.

 

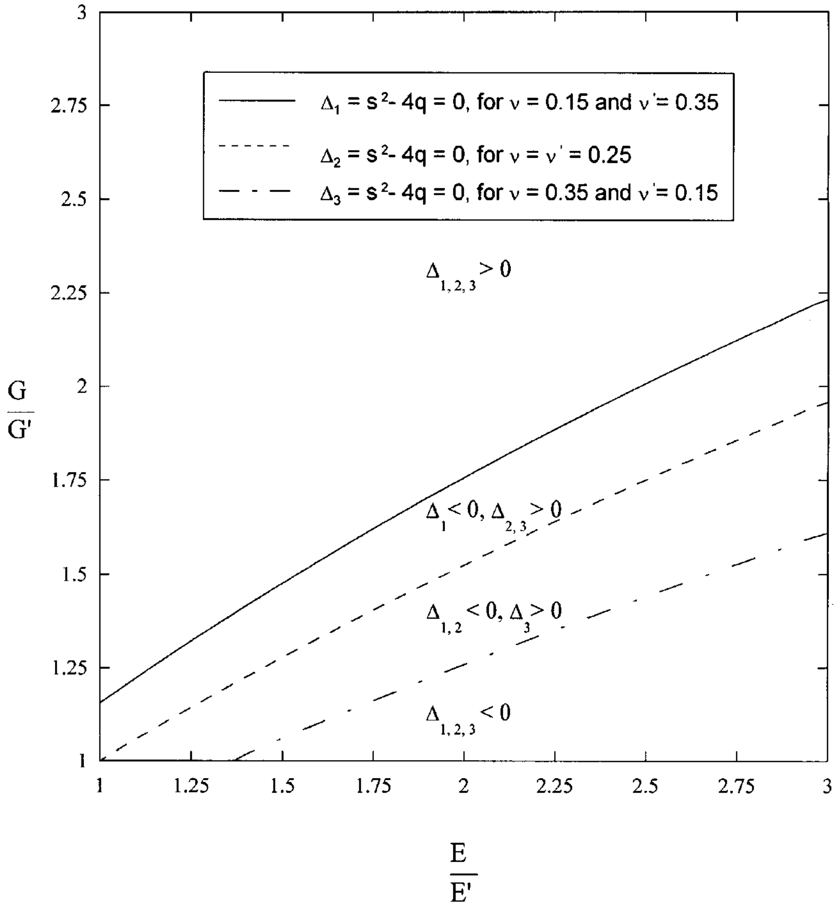


Figure 3. Distribution of the three categories of the characteristic roots for transversely isotropic rocks

In order to derive the particular solutions of equations (28)–(30), we define the three displacement functions as follows:

$$\Phi^{**}_{n-1}(P) = D_1 e^{u_1 \xi z} + D_2 e^{u_2 \xi z} + D_3 e^{-u_1 \xi z} + D_4 e^{-u_2 \xi z} + D_5 e^{u_3 \xi z} + D_6 e^{-u_3 \xi z} \tag{38}$$

$$\Psi^{**}_{n+1}(P) = E_1 e^{u_1 \xi z} + E_2 e^{u_2 \xi z} + E_3 e^{-u_1 \xi z} + E_4 e^{-u_2 \xi z} + E_5 e^{u_3 \xi z} + E_6 e^{-u_3 \xi z} \tag{39}$$

$$U^{**}_{zn}(P) = F_1 e^{u_1 \xi z} + F_2 e^{u_2 \xi z} + F_3 e^{-u_1 \xi z} + F_4 e^{-u_2 \xi z} + F_5 e^{u_3 \xi z} + F_6 e^{-u_3 \xi z} \tag{40}$$

for $z > 0$, and

$$\Phi^{**}_{n-1}(P) = \Psi^{**}_{n+1}(P) = U^{**}_{zn}(P) = 0 \tag{41}$$

for $z < 0$.

The undetermined coefficients $D_i$, $E_i$ and $F_i$ ($i = 1$–$6$) can be obtained by the method of variation of parameters.[41] The general solutions are the sum of the homogeneous and the particular solutions. The constants $B_i$ ($i = 1$–$6$) can be determined by the conditions that the effect of the point load must vanish at infinity. Therefore, the final resulting expressions of general solutions for $\Phi^{**}_{n-1}$, $\Psi^{**}_{n+1}$ and $U^{**}_{zn}$ are

$$\Phi^{**}_{n-1}(G) = \frac{P_r + \mathrm{i}P_\theta}{4\xi}\left[\frac{k}{m_1}\,\mathrm{e}^{-u_1\xi|z|} - \frac{k}{m_2}\,\mathrm{e}^{-u_2\xi|z|} + \frac{1}{u_3A_{44}}\,\mathrm{e}^{-u_3\xi|z|}\right]J_{n-1}(0)$$
$$+ \frac{P_r - \mathrm{i}P_\theta}{4\xi}\left[-\frac{k}{m_1}\,\mathrm{e}^{-u_1\xi|z|} + \frac{k}{m_2}\,\mathrm{e}^{-u_2\xi|z|} + \frac{1}{u_3A_{44}}\,\mathrm{e}^{-u_3\xi|z|}\right]J_{n+1}(0)$$
$$+ \frac{P_z}{2\xi}\left[\pm k\mathrm{e}^{-u_1\xi|z|} \mp k\mathrm{e}^{-u_2\xi|z|}\right]J_n(0) \qquad (42)$$

$$\Psi^{**}_{n+1}(G) = \frac{P_r + \mathrm{i}P_\theta}{4\xi}\left[-\frac{k}{m_1}\,\mathrm{e}^{-u_1\xi|z|} + \frac{k}{m_2}\,\mathrm{e}^{-u_2\xi|z|} + \frac{1}{u_3A_{44}}\,\mathrm{e}^{-u_3\xi|z|}\right]J_{n-1}(0)$$
$$+ \frac{P_r - \mathrm{i}P_\theta}{4\xi}\left[\frac{k}{m_1}\,\mathrm{e}^{-u_1\xi|z|} - \frac{k}{m_2}\,\mathrm{e}^{-u_2\xi|z|} + \frac{1}{u_3A_{44}}\,\mathrm{e}^{-u_3\xi|z|}\right]J_{n+1}(0)$$
$$+ \frac{P_z}{2\xi}\left[\mp k\mathrm{e}^{-u_1\xi|z|} \pm k\mathrm{e}^{-u_2\xi|z|}\right]J_n(0) \qquad (43)$$

$$U^{**}_{zn}(G) = \frac{P_r + \mathrm{i}P_\theta}{4\xi}\left[\mp k\mathrm{e}^{-u_1\xi|z|} \pm k\mathrm{e}^{-u_2\xi|z|}\right]J_{n-1}(0)$$
$$+ \frac{P_r - \mathrm{i}P_\theta}{4\xi}\left[\pm k\mathrm{e}^{-u_1\xi|z|} \mp k\mathrm{e}^{-u_2\xi|z|}\right]J_{n+1}(0)$$
$$+ \frac{P_z}{2\xi}\left[-k\mathrm{e}^{-u_1\xi|z|} + k\mathrm{e}^{-u_2\xi|z|}\right]J_n(0) \qquad (44)$$

where the upper sign is for $z > 0$ (the sign of $z$ is downward positive), the lower sign is for $z < 0$, and $k = (A_{13} + A_{44})/A_{33}A_{44}(u_1^2 - u_2^2)$.

The desired solutions of the displacements $U_r$, $U_\theta$ and $U_z$ can be obtained by taking the inverse Hankel transform[46] with respect to $\xi$, and inverse Fourier transform with respect to $n$, respectively, in the following:

$$\begin{Bmatrix} \Phi^* \\ \Psi^* \\ U_z^* \end{Bmatrix} = \int_0^\infty \xi \begin{Bmatrix} \Phi^{**}_{n-1}J_{n-1}(r\xi) \\ \Psi^{**}_{n+1}J_{n+1}(r\xi) \\ U^{**}_{zn}J_n(r\xi) \end{Bmatrix} \mathrm{d}\xi \qquad (45)$$

$$\begin{Bmatrix} U_r \\ U_\theta \\ U_z \end{Bmatrix} = \frac{1}{2\pi}\sum_{n=-\infty}^{n=\infty} \begin{Bmatrix} U_r^* \\ U_\theta^* \\ U_z^* \end{Bmatrix} \mathrm{e}^{-\mathrm{i}n\theta} \qquad (46)$$

The expression of complete components for the displacements in a transversely isotropic medium with three root types mentioned above is lengthy. Since two-thirds of transversely isotropic rocks may have two real distinct roots for equation (37), only the exact solutions for case 1 in an infinite space denoted by $U'_r$, $U'_\theta$, $U'_z$ are presented below:

$$U'_r = \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi}\left[k\left(\frac{z_1 R_1^*}{m_1 r^2 R_1} - \frac{z_2 R_2^*}{m_2 r^2 R_2}\right) + \frac{1}{u_3 A_{44}}\frac{R_3^*}{r^2}\right] \mp \frac{P_z}{4\pi} k\left(\frac{R_1^*}{rR_1} - \frac{R_2^*}{rR_2}\right) \tag{47}$$

$$U'_\theta = \frac{(P_r \sin\theta - P_\theta \cos\theta)}{4\pi}\left[-k\left(\frac{R_1^*}{m_1 r^2} - \frac{R_2^*}{m_2 r^2}\right) - \frac{1}{u_3 A_{44}}\frac{z_3 R_3^*}{r^2 R_3}\right] \tag{48}$$

$$U'_z = \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi} k\left[\mp\left(\frac{R_1^*}{rR_1} - \frac{R_2^*}{rR_2}\right)\right] - \frac{P_z}{4\pi} k\left(\frac{m_1}{R_1} - \frac{m_2}{R_2}\right) \tag{49}$$

where $z_i = u_i|z|$, $R_i = \sqrt{r^2 + z_i^2}$, $R_i^* = R_i - z_i$ $(i = 1, 2, 3)$.

From equations (47)–(49), (8)–(13) and (1)–(6), the stresses in an infinite space ($z > 0$, case 1) are denoted by $\sigma'_{rr}$, $\sigma'_{\theta\theta}$, $\sigma'_{zz}$, $\tau'_{r\theta}$, $\tau'_{\theta z}$, $\tau'_{rz}$, and expressed as

$$\begin{aligned}\sigma'_{rr} = \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi}&\left\{\frac{k}{m_1}\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{r}{R_1^3}\right) - 2A_{66}\left(\frac{R_1^{*2}}{r^3 R_1}\right)\right]\right.\\
&\left.-\frac{k}{m_2}\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{r}{R_2^3}\right) - 2A_{66}\left(\frac{R_2^{*2}}{r^3 R_2}\right)\right] + 2u_3\left(\frac{R_3^{*2}}{r^3 R_3}\right)\right\}\\
&+\frac{P_z}{4\pi} k\left\{\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{z_1}{R_1^3}\right) - 2A_{66}\left(\frac{R_1^*}{r^2 R_1}\right)\right]\right.\\
&\left.-\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{z_2}{R_2^3}\right) - 2A_{66}\left(\frac{R_2^*}{r^2 R_2}\right)\right]\right\}\end{aligned} \tag{50}$$

$$\begin{aligned}\sigma'_{\theta\theta} = \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi}&\left\{\frac{k}{m_1}\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{r}{R_1^3}\right) - 2A_{66}\left(\frac{2z_1 R_1^*}{r^3 R_1} - \frac{z_1^2}{rR_1^3}\right)\right]\right.\\
&\left.-\frac{k}{m_2}\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{r}{R_2^3}\right) - 2A_{66}\left(\frac{2z_2 R_2^*}{r^3 R_2} - \frac{z_2^2}{rR_2^3}\right)\right] + 2u_3\left(\frac{1}{rR_3} - \frac{2R_3^*}{r^3}\right)\right\}\\
&+\frac{P_z}{4\pi} k\left\{\left[(A_{11} - 2A_{66} - u_1 m_1 A_{13})\left(\frac{z_1}{R_1^3}\right) + 2A_{66}\left(\frac{R_1^*}{r^2 R_1}\right)\right]\right.\\
&\left.-\left[(A_{11} - 2A_{66} - u_2 m_2 A_{13})\left(\frac{z_2}{R_2^3}\right) + 2A_{66}\left(\frac{R_2^*}{r^2 R_2}\right)\right]\right\}\end{aligned} \tag{51}$$

$$\begin{aligned}\sigma'_{zz} = \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi}& k\left[\frac{(A_{13} - u_1 m_1 A_{33})}{m_1}\left(\frac{r}{R_1^3}\right) - \frac{(A_{13} - u_2 m_2 A_{33})}{m_2}\left(\frac{r}{R_2^3}\right)\right]\\
&+\frac{P_z}{4\pi} k\left[(A_{13} - u_1 m_1 A_{33})\left(\frac{z_1}{R_1^3}\right) - (A_{13} - u_2 m_2 A_{33})\left(\frac{z_2}{R_2^3}\right)\right]\end{aligned} \tag{52}$$

$$\tau'_{r\theta} = \frac{(P_r \sin\theta - P_\theta \cos\theta)}{4\pi} \left[ \frac{2kA_{66}}{m_1} \left( -\frac{1}{rR_1} + \frac{2z_1 R_1^*}{r^3 R_1} \right) - \frac{2kA_{66}}{m_2} \left( -\frac{1}{rR_2} + \frac{2z_2 R_2^*}{r^3 R_2} \right) \right.$$
$$\left. - u_3 \left( -\frac{R_3^*}{r^3} + \frac{3z_3 R_3^*}{r^3 R_3} - \frac{z_3^2}{rR_3^3} \right) \right] \tag{53}$$

$$\tau'_{\theta z} = -\frac{(P_r \sin\theta - P_\theta \cos\theta)}{4\pi} \left[ \frac{k(u_1 + m_1)A_{44}}{m_1} \left( \frac{R_1^*}{r^2 R_1} \right) - \frac{k(u_2 + m_2)A_{44}}{m_2} \left( \frac{R_2^*}{r^2 R_2} \right) - \left( \frac{R_3^*}{r^2 R_3} - \frac{z_3}{R_3^3} \right) \right] \tag{54}$$

$$\tau'_{rz} = \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi} \left[ \frac{k(u_1 + m_1)A_{44}}{m_1} \left( -\frac{R_1^*}{r^2 R_1} + \frac{z_1}{R_1^3} \right) \right.$$
$$\left. - \frac{k(u_2 + m_2)A_{44}}{m_2} \left( -\frac{R_2^*}{r^2 R_2} + \frac{z_2}{R_2^3} \right) + \frac{R_3^*}{r^2 R_3} \right]$$
$$- \frac{P_z}{4\pi} kA_{44} \left[ (u_1 + m_1) \left( \frac{r}{R_1^3} \right) - (u_2 + m_2) \left( \frac{r}{R_2^3} \right) \right] \tag{55}$$

For the medium with double equal real roots (case 2), the exact solutions for the displacements and stresses can be obtained from equations (47)–(49) and (50)–(55) by approaching $u_2$ to $u_1$, and using the L'Hôpital rule, respectively. When $u_1 (= u_2) = 1$, these solutions are in agreement with the Kelvin[1] solution for an isotropic material. Regarding the medium with complex conjugate roots (case 3), the closed-form solutions can be easily obtained by replacing the distinct root $u_1$ by the complex root $\gamma - i\delta$, and $u_2$ by $\gamma + i\delta$ into equations (47)–(49) and (50)–(55), respectively.

*Displacements and stresses in a half-space*

As mentioned above, the solutions of displacement functions, $\Phi_{n-1}^{**}$, $\Psi_{n+1}^{**}$ and $U_{zn}^{**}$ for the half-space problem can be directly obtained from the superposition of general solutions (equations (42)–(44)) by shifting $|z|$ to $|z-h|$ and being denoted by $\Phi_{n-1}'^{**}(G)$, $\Psi_{n+1}'^{**}(G)$, $U_{zn}'^{**}(G)$, and homogeneous solutions (equations (34)–(36)) in which $B_i$ ($i = 1$–$6$) are denoted by $B_i'$ ($i = 1$–$6$) and $z$ is replaced by $(z-h)$ as shown in Figure 1 are

$$\Phi_{n-1}^{**} = \Phi_{n-1}'^{**}(G) - B_1' e^{u_1 \xi (z-h)} - B_2' e^{u_2 \xi (z-h)} - B_3' e^{-u_1 \xi (z-h)} - B_4' e^{-u_2 \xi (z-h)}$$
$$+ B_5' e^{u_3 \xi (z-h)} + B_6' e^{-u_3 \xi (z-h)} \tag{56}$$

$$\Psi_{n+1}^{**} = \Psi_{n+1}'^{**}(G) + B_1' e^{u_1 \xi (z-h)} + B_2' e^{u_2 \xi (z-h)} + B_3' e^{-u_1 \xi (z-h)} + B_4' e^{-u_2 \xi (z-h)}$$
$$+ B_5' e^{u_3 \xi (z-h)} + B_6' e^{-u_3 \xi (z-h)} \tag{57}$$

$$U_{zn}^{**} = U_{zn}'^{**}(G) - m_1 B_1' e^{u_1 \xi (z-h)} - m_2 B_2' e^{u_2 \xi (z-h)} + m_1 B_3' e^{-u_1 \xi (z-h)} + m_2 B_4' e^{-u_2 \xi (z-h)} \tag{58}$$

For an elastic semi-infinite space with free traction on the bounding plane, the boundary conditions can be expressed in terms of displacements as follows:

$$\sigma_{zz} = -A_{13} \left( \frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} \right) - A_{33} \frac{\partial U_z}{\partial z} = 0 \tag{59}$$

$$\tau_{\theta z} = -A_{44}\left(\frac{\partial U_\theta}{\partial z} + \frac{1}{r}\frac{\partial U_z}{\partial \theta}\right) = 0 \tag{60}$$

$$\tau_{rz} = -A_{44}\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right) = 0 \tag{61}$$

The stress functions $\sigma_{zz}$, $\tau_{\theta z}$, and $\tau_{rz}$ in equations (59)–(61) are transformed by a finite Fourier exponential transform with respect to $\theta$ as

$$\begin{Bmatrix} \sigma^*_{zz} \\ \tau^*_{\theta z} \\ \tau^*_{rz} \end{Bmatrix} = \int_0^{2\pi} \begin{Bmatrix} \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \end{Bmatrix} \mathrm{e}^{\mathrm{i}n\theta}\, \mathrm{d}\theta \tag{62}$$

Using the displacement functions $\Phi^*$ and $\Psi^*$ in equation (26) and introducing two stress functions, $\alpha^*$ and $\beta^*$, equations (59)–(61) can be transformed as

$$\sigma^*_{zz} = -\frac{A_{13}}{2}\left[\left(\frac{\partial}{\partial r} - \frac{n-1}{r}\right)\Phi^* + \left(\frac{\partial}{\partial r} + \frac{n+1}{r}\right)\Psi^*\right] - A_{33}\frac{\partial U^*_z}{\partial z} = 0 \tag{63}$$

$$\alpha^* = \tau^*_{rz} + \mathrm{i}\tau^*_{\theta z} = -A_{44}\left[\frac{\partial}{\partial z}\Phi^* + \left(\frac{\partial}{\partial r} + \frac{n}{r}\right)U^*_z\right] = 0 \tag{64}$$

$$\beta^* = z^*_{rz} - \mathrm{i}\tau^*_{\theta z} = -A_{44}\left[\frac{\partial}{\partial z}\Psi^* + \left(\frac{\partial}{\partial r} - \frac{n}{r}\right)U^*_z\right] = 0 \tag{65}$$

Hankel transformations of $\sigma^*_{zz}$, $\alpha^*$ and $\beta^*$ with respect to $r$ of order $n$, $n-1$ and $n+1$, respectively, are given as

$$\begin{Bmatrix} \sigma^{**}_{zzn} \\ \alpha^{**}_{n-1} \\ \beta^{**}_{n+1} \end{Bmatrix} = \int_0^{\infty} r \begin{Bmatrix} \sigma^*_{zz} J_n(\xi r) \\ \alpha^* J_{n-1}(\xi r) \\ \beta^* J_{n+1}(\xi r) \end{Bmatrix} \mathrm{d}r \tag{66}$$

then, equations (63)–(65) are rewritten as follows:

$$\sigma^{**}_{zzn} = -\frac{A_{13}}{2}\left(-\xi\Phi^{**}_{n-1} + \xi\Psi^{**}_{n+1}\right) - A_{33}\frac{\mathrm{d}}{\mathrm{d}z}U^{**}_{zn} = 0 \tag{67}$$

$$\alpha^{**}_{n-1} = -A_{44}\left[\frac{\mathrm{d}}{\mathrm{d}z}\Phi^{**}_{n-1} + \xi U^{**}_{zn}\right] = 0 \tag{68}$$

$$\beta^{**}_{n+1} = -A_{44}\left[\frac{\mathrm{d}}{\mathrm{d}z}\Psi^{**}_{n+1} - \xi U^{**}_{zn}\right] = 0 \tag{69}$$

For solving equations (56)–(58), the undetermined coefficients $B'_i$ ($i = 1$–$6$) can be obtained by the assumptions that displacements $U_r$, $U_\theta$ and $U_z$ must be finite when $z$ is approaching to infinity. Hence, $B'_1 = B'_2 = B'_5 = 0$. The remaining coefficients $B'_3$, $B'_4$ and $B'_6$ can also be obtained from the transformed boundary conditions (equations (67)–(69)) as

$$B'_3 = \frac{P_r + \mathrm{i}P_\theta}{4\xi}\,[T_1 \mathrm{e}^{-2u_1\xi h} - T_2 \mathrm{e}^{-(u_1+u_2)\xi h}]J_{n-1}(0)$$
$$- \frac{P_r - \mathrm{i}P_\theta}{4\xi}\,[T_1 \mathrm{e}^{-2u_1\xi h} - T_2 \mathrm{e}^{-(u_1+u_2)\xi h}]J_{n+1}(0)$$
$$- \frac{P_z}{2\xi}\,[m_1 T_1 \mathrm{e}^{-2u_1\xi h} - m_2 T_2 \mathrm{e}^{-(u_1+u_2)\xi h}]J_n(0) \qquad (70)$$

$$B'_4 = -\frac{P_r + \mathrm{i}P_\theta}{4\xi}\,[T_3 \mathrm{e}^{-(u_1+u_2)\xi h} - T_4 \mathrm{e}^{-2u_2\xi h}]J_{n-1}(0)$$
$$+ \frac{P_r - \mathrm{i}P_\theta}{4\xi}\,[T_3 \mathrm{e}^{-(u_1+u_2)\xi h} - T_4 \mathrm{e}^{-2u_2\xi h}]J_{n+1}(0)$$
$$+ \frac{P_z}{2\xi}\,[m_1 T_3 \mathrm{e}^{-(u_1+u_2)\xi h} - m_2 T_4 \mathrm{e}^{-2u_2\xi h}]J_n(0) \qquad (71)$$

$$B'_6 = \left[\frac{P_r + \mathrm{i}P_\theta}{4\xi}\,J_{n-1}(0) + \frac{P_r - \mathrm{i}P_\theta}{4\xi}\,J_{n+1}(0)\right]\frac{1}{u_3 A_{44}}\,\mathrm{e}^{-2u_3\xi h} \qquad (72)$$

where

$$T_1 = \frac{k}{m_1}\frac{u_1 + u_2}{u_2 - u_1}, \quad T_2 = \frac{k}{m_2}\frac{2u_1(u_2 + m_2)}{(u_2 - u_1)(u_1 + m_1)}, \quad T_3 = \frac{k}{m_1}\frac{2u_2(u_1 + m_1)}{(u_2 - u_1)(u_2 + m_2)}, \quad T_4 = \frac{k}{m_2}\frac{u_1 + u_2}{u_2 - u_1}$$

.

Finally, the displacements in a transversely isotropic half-space with a point load ($P_r$, $P_\theta$, $P_z$) acting at $z = h$ are obtained by inverse Hankel transforms[46] and inverse Fourier transforms as

$$U_r = U'_r + \frac{(P_r\cos\theta + P_\theta\sin\theta)}{4\pi}\left[-T_1\left(\frac{z_a R_a^*}{r^2 R_a}\right) + T_2\left(\frac{z_b R_b^*}{r^2 R_b}\right) + T_3\left(\frac{z_c R_c^*}{r^2 R_c}\right)\right.$$
$$\left. - T_4\left(\frac{z_d R_d^*}{r^2 R_d}\right) + \frac{1}{u_3 A_{44}}\left(\frac{R_e^*}{r^2}\right)\right] - \frac{P_z}{4\pi}\left\{m_1\left[T_1\left(\frac{R_a^*}{rR_a}\right) - T_3\left(\frac{R_c^*}{rR_c}\right)\right]\right.$$
$$\left. - m_2\left[T_2\left(\frac{R_b^*}{rR_b}\right) - T_4\left(\frac{R_d^*}{rR_d}\right)\right]\right\} \qquad (73)$$

$$U_\theta = U'_\theta + \frac{(P_r\sin\theta - P_\theta\cos\theta)}{4\pi}\left[T_1\left(\frac{R_a^*}{r^2}\right) - T_2\left(\frac{R_b^*}{r^2}\right) - T_3\left(\frac{R_c^*}{r^2}\right) + T_4\left(\frac{R_d^*}{r^2}\right) - \frac{1}{u_3 A_{44}}\left(\frac{z_e R_e^*}{r^2 R_e}\right)\right] \qquad (74)$$

$$U_z = U'_z + \frac{(P_r\cos\theta + P_\theta\sin\theta)}{4\pi}\left\{m_1\left[T_1\left(\frac{R_a^*}{rR_a}\right) - T_2\left(\frac{R_b^*}{rR_b}\right)\right] - m_2\left[T_3\left(\frac{R_c^*}{rR_c}\right) - T_4\left(\frac{R_d^*}{rR_d}\right)\right]\right\}$$
$$- \frac{P_z}{4\pi}\left\{m_1\left[T_1 m_1\left(\frac{1}{R_a}\right) - T_2 m_2\left(\frac{1}{R_b}\right)\right] - m_2\left[T_3 m_1\left(\frac{1}{R_c}\right) - T_4 m_2\left(\frac{1}{R_d}\right)\right]\right\} \qquad (75)$$

where $U'_r$, $U'_\theta$ and $U'_z$ are the displacement components of an infinite space given in equations (47)–(49), and $z_a = u_1(z+h)$, $z_b = u_1 z + u_2 h$, $z_c = u_1 h + u_2 z$, $z_d = u_2(z+h)$, $z_e = u_3(z+h)$, $R_i = \sqrt{r^2 + z_i^2}$, $R_i^* = R_i - z_i$ ($i$ = a, b, c, d, e).

From equations (73)–(75), (8)–(13) and (1)–(6), the stresses in a semi-infinite space with two real distinct roots (case 1) can be expressed as

$$\begin{aligned}
\sigma_{rr} = \sigma'_{rr} &- \frac{(P_r\cos\theta + P_\theta\sin\theta)}{4\pi}\left\{T_1\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{r}{R_a^3}\right) - 2A_{66}\left(\frac{R_a^{*2}}{r^3 R_a}\right)\right]\right.\\
&- T_2\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{r}{R_b^3}\right) - 2A_{66}\left(\frac{R_b^{*2}}{r^3 R_b}\right)\right]\\
&- T_3\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{r}{R_c^3}\right) - 2A_{66}\left(\frac{R_c^{*2}}{r^3 R_c}\right)\right]\\
&\left.+ T_4\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{r}{R_d^3}\right) - 2A_{66}\left(\frac{R_d^{*2}}{r^3 R_d}\right)\right] - 2u_3\left(\frac{R_e^{*2}}{r^3 R_e}\right)\right\}\\
&+ \frac{P_z}{4\pi}\left\{T_1 m_1\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{z_a}{R_a^3}\right) - 2A_{66}\left(\frac{R_a^*}{r^2 R_a}\right)\right]\right.\\
&- T_2 m_2\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{z_b}{R_b^3}\right) - 2A_{66}\left(\frac{R_b^*}{r^2 R_b}\right)\right]\\
&- T_3 m_1\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{z_c}{R_c^3}\right) - 2A_{66}\left(\frac{R_c^*}{r^2 R_c}\right)\right]\\
&\left.+ T_4 m_2\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{z_d}{R_d^3}\right) - 2A_{66}\left(\frac{R_d^*}{r^2 R_d}\right)\right]\right\}
\end{aligned} \tag{76}$$

$$\begin{aligned}
\sigma_{\theta\theta} = \sigma'_{\theta\theta} &- \frac{(P_r\cos\theta + P_\theta\sin\theta)}{4\pi}\left\{T_1\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{r}{R_a^3}\right) - 2A_{66}\left(\frac{2z_a R_a^*}{r^3 R_a} - \frac{z_a^2}{r R_a^3}\right)\right]\right.\\
&- T_2\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{r}{R_b^3}\right) - 2A_{66}\left(\frac{2z_b R_b^*}{r^3 R_b} - \frac{z_b^2}{r R_b^3}\right)\right]\\
&- T_3\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{r}{R_c^3}\right) - 2A_{66}\left(\frac{2z_c R_c^*}{r^3 R_c} - \frac{z_c^2}{r R_c^3}\right)\right]\\
&\left.+ T_4\left[(A_{11} - u_2 m_2 A_{13})\left(\frac{r}{R_d^3}\right) - 2A_{66}\left(\frac{2z_d R_d^*}{r^3 R_d} - \frac{z_d^2}{r R_d^3}\right)\right] - 2u_3\left(\frac{1}{r R_e} - \frac{2R_e^*}{r^3}\right)\right\}\\
&+ \frac{P_z}{4\pi}\left\{T_1 m_1\left[(A_{11} - 2A_{66} - u_1 m_1 A_{13})\left(\frac{z_a}{R_a^3}\right) + 2A_{66}\left(\frac{R_a^*}{r^2 R_a}\right)\right]\right.\\
&- T_2 m_2\left[(A_{11} - 2A_{66} - u_1 m_1 A_{13})\left(\frac{z_b}{R_b^3}\right) + 2A_{66}\left(\frac{R_b^*}{r^2 R_b}\right)\right]
\end{aligned}$$

$$- T_3 m_1 \left[ (A_{11} - 2A_{66} - u_2 m_2 A_{13}) \left( \frac{z_c}{R_c^3} \right) + 2A_{66} \left( \frac{R_c^*}{r^2 R_c} \right) \right]$$

$$+ T_4 m_2 \left[ (A_{11} - 2A_{66} - u_2 m_2 A_{13}) \left( \frac{z_d}{R_d^3} \right) + 2A_{66} \left( \frac{R_d^*}{r^2 R_d} \right) \right] \Bigg\} \tag{77}$$

$$\sigma_{zz} = \sigma'_{zz} - \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi} \Bigg\{ (A_{13} - u_1 m_1 A_{33}) \left[ T_1 \left( \frac{r}{R_a^3} \right) - T_2 \left( \frac{r}{R_b^3} \right) \right]$$

$$- (A_{13} - u_2 m_2 A_{33}) \left[ T_3 \left( \frac{r}{R_c^3} \right) - T_4 \left( \frac{r}{R_d^3} \right) \right] \Bigg\}$$

$$+ \frac{P_z}{4\pi} \Bigg\{ (A_{13} - u_1 m_1 A_{33}) \left[ T_1 m_1 \left( \frac{z_a}{R_a^3} \right) - T_2 m_2 \left( \frac{z_b}{R_b^3} \right) \right]$$

$$- (A_{13} - u_2 m_2 A_{33}) \left[ T_3 m_1 \left( \frac{z_c}{R_c^3} \right) - T_4 m_2 \left( \frac{z_d}{R_d^3} \right) \right] \Bigg\} \tag{78}$$

$$\tau_{r\theta} = \tau'_{r\theta} - \frac{(P_r \sin\theta - P_\theta \cos\theta)}{4\pi} \Bigg[ 2T_1 A_{66} \left( -\frac{1}{rR_a} + \frac{2z_a R_a^*}{r^3 R_a} \right) - 2T_2 A_{66} \left( -\frac{1}{rR_b} + \frac{2z_b R_b^*}{r^3 R_b} \right)$$

$$- 2T_3 A_{66} \left( -\frac{1}{rR_c} + \frac{2z_c R_c^*}{r^3 R_c} \right) + 2T_4 A_{66} \left( -\frac{1}{rR_d} + \frac{2z_d R_d^*}{r^3 R_d} \right)$$

$$+ u_3 \left( -\frac{R_e^*}{r^3} + \frac{3z_e R_e^*}{r^3 R_e} - \frac{z_e^2}{rR_e^3} \right) \Bigg] \tag{79}$$

$$\tau_{\theta z} = \tau'_{\theta z} + \frac{(P_r \sin\theta - P_\theta \cos\theta)}{4\pi} \Bigg\{ (u_1 + m_1) A_{44} \left[ T_1 \left( \frac{R_a^*}{r^2 R_a} \right) - T_2 \left( \frac{R_b^*}{r^2 R_b} \right) \right]$$

$$- (u_2 + m_2) A_{44} \left[ T_3 \left( \frac{R_c^*}{r^2 R_c} \right) - T_4 \left( \frac{R_d^*}{r^2 R_d} \right) \right] + \left( \frac{R_e^*}{r^2 R_e} - \frac{z_e}{R_e^3} \right) \Bigg\} \tag{80}$$

$$\tau_{rz} = \tau'_{rz} + \frac{(P_r \cos\theta + P_\theta \sin\theta)}{4\pi} \Bigg\{ (u_1 + m_1) A_{44} \left[ T_1 \left( -\frac{R_a^*}{r^2 R_a} + \frac{z_a}{R_a^3} \right) - T_2 \left( -\frac{R_b^*}{r^2 R_b} + \frac{z_b}{R_b^3} \right) \right]$$

$$- (u_2 + m_2) A_{44} \left[ T_3 \left( -\frac{R_c^*}{r^2 R_c} + \frac{z_c}{R_c^3} \right) - T_4 \left( -\frac{R_d^*}{r^2 R_d} + \frac{z_d}{R_d^3} \right) \right] - \frac{R_e^*}{r^2 R_e} \Bigg\}$$

$$+ \frac{P_z}{4\pi} A_{44} \Bigg\{ (u_1 + m_1) \left[ T_1 m_1 \left( \frac{r}{R_a^3} \right) - T_2 m_2 \left( \frac{r}{R_b^3} \right) \right]$$

$$- (u_2 + m_2) \left[ T_3 m_1 \left( \frac{r}{R_c^3} \right) - T_4 m_2 \left( \frac{r}{R_d^3} \right) \right] \Bigg\} \tag{81}$$

where $\sigma'_{rr}$, $\sigma'_{\theta\theta}$, $\sigma'_{zz}$, $\tau'_{r\theta}$, $\tau'_{\theta z}$ and $\tau'_{rz}$ are the stress components of an infinite space given in equations (50)–(55).

 

The solutions for the displacements and stresses in a medium bounded by a horizontal surface with double equal real roots (case 2) or complex conjugate roots (case 3) could be easily derived by the same approaches for solving the problem of an infinite space.

Equations (73)–(75) and (76)–(81) indicate that both of the displacements and stresses in a transversely isotropic half-space induced by a point load are affected by several factors. The factors include the loading types (radial, tangential or normal), and the degree and type of rock anisotropy. Considering only a vertical point load ($P_r = P_\theta = 0$) acting at $z = h$ in the interior of a half-space, these solutions are identical with the Mindlin[4] solution when the medium is isotropic. If $h = 0$, in other words, a point load is applied at the surface, these solutions are in agreement with the Lekhnitskii[18] solution that was based on the assumption of axisymmetry for a transversely isotropic half-space. Also, the Boussinesq[2] solution for an isotropic medium is a special case of these analytical solutions.

## ILLUSTRATIVE EXAMPLE

The closed-form solutions, equations (73)–(75) and (76)–(81) can be utilized to calculate the displacements and stresses in a transversely isotropic half-space induced by a point load. A FORTRAN program based on the solutions was written for conducting a parametric study.

A vertical point load acting on the bounded surface is considered as an example (Figures 4 and 5) for verifying the presented formulations. Several types of isotropic and transversely isotropic rocks are considered to constitute the foundation materials. Their elastic properties are listed in Table II with $E/E'$ and $G/G'$ ranging between 1 and 3 and $v/v'$ varying between 0·75–1·5. The values adopted in Table II of $E$ and $v$ are 50 GPa and 0·25, respectively. The chosen domains of variation are based on the suggestions of Gerrard[44] and Amadei *et al.*[45]

A parametric study is conducted for looking at the effect of the ratio $E/E'$, $v/v'$ and $G/G'$ on the displacements and stresses in the foundation. However, only parts of the results, including the vertical displacement ($U_z$) on the surface and vertical stress ($\sigma_{zz}$) in the foundation are presented in the following.

Firstly, the influence of the degree and type of rock anisotropy on the vertical surface displacement is investigated. Figure 4 presents the effect of ratio $E/E'$, $v/v'$ and $G/G'$ on the normalized vertical surface displacement. This figure indicates that the normalized vertical surface displacement is less than the value of 0·025 when the radial distance is large than 0·5 m for all the constituted foundation materials. It means that the elastic settlement in these cases is little. However, the magnitude of surface displacement is influenced by rock anisotropy. Figure 4 shows that the vertical displacement increases with the increase of $E/E'$ with $v/v' = G/G' = 1$, and $G/G'$ with $E/E' = v/v' = 1$. It reflects that the vertical surface displacement increases with the increase of deformability in the direction parallel to the applied load. However, the variation of $v/v'$ on the vertical displacement is little for all the cases.

Secondly, the effect of rock anisotropy on the vertical stress in the medium is studied. In order to investigate the variation of $\sigma_{zz}$ point by point in the $r$–$z$ plane, the relation of two non-dimensional factors, $r/z$ and $z^2\sigma_{zz}/P_z$ is presented in Figure 5. The figure indicates that the vertical stress decreases with the increase of $E/E'$ ($v/v' = G/G' = 1$), and is little affected by the value of $v/v'$ ($E/E' = G/G' = 1$). For the variation of $G/G'$ ($E/E' = v/v' = 1$), it can be seen that the increase of the ratio, the non-dimensional stress could be larger than one unit. Thus, when a point load acting on the surface of a transversely isotropic medium, it should be noted that the excessive compressive-stress may appear in the medium.

 

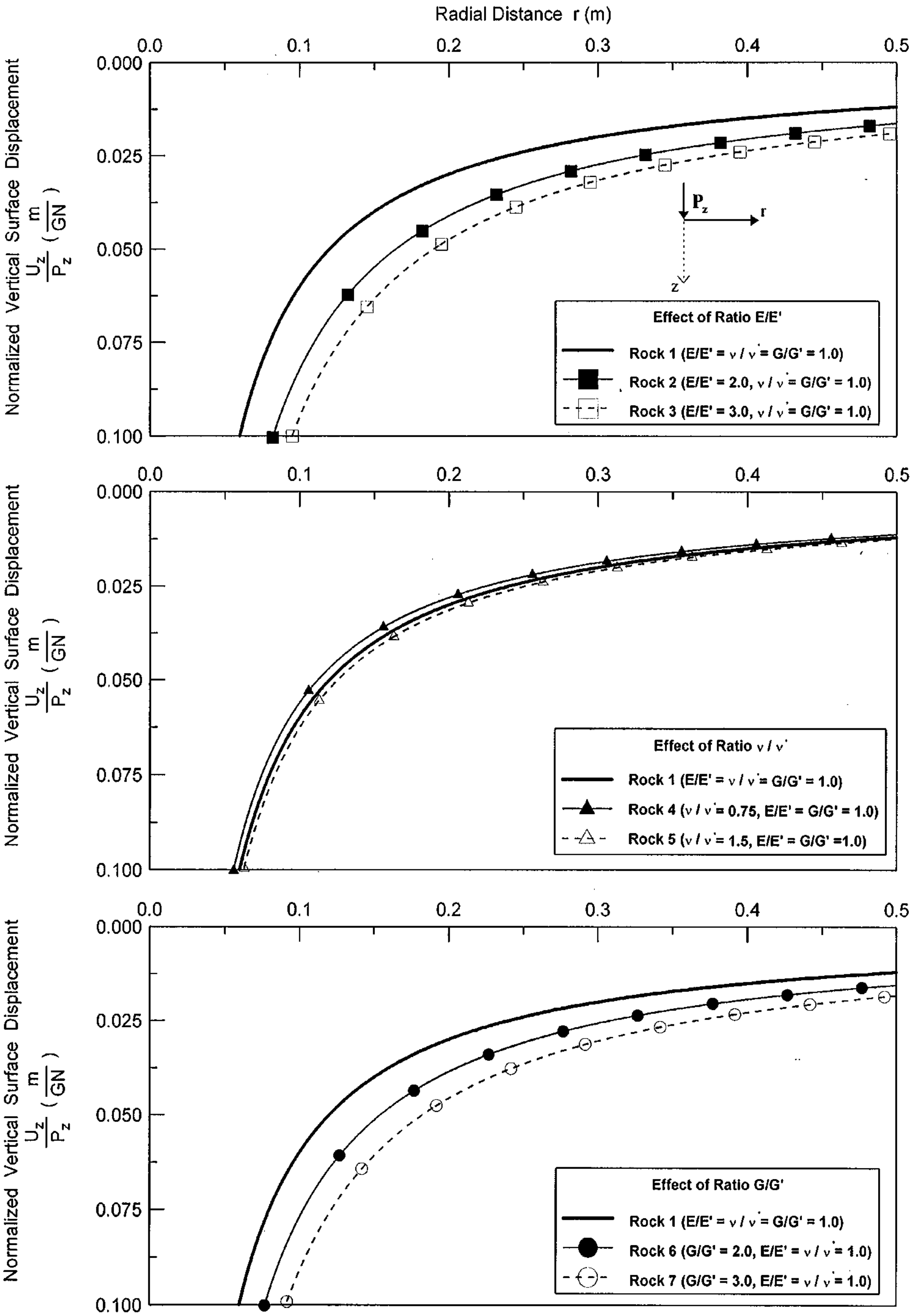


Figure 4. Effect of ratios of $E/E'$, $\nu/\nu'$ and $G/G'$ on normalized vertical surface displacement

 

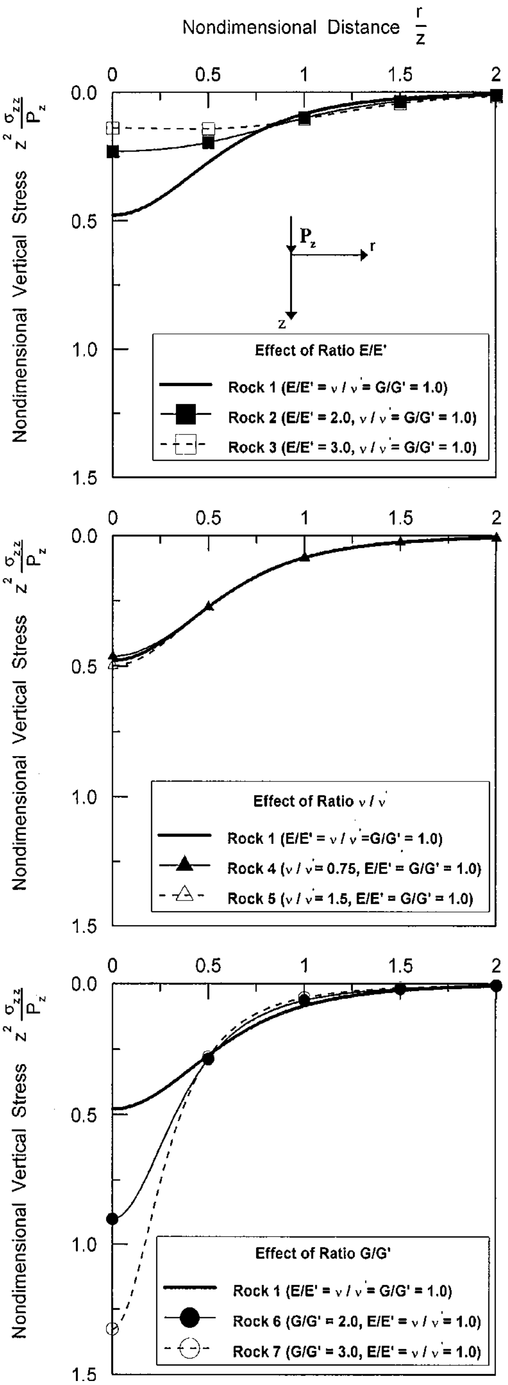


Figure 5. Effect of ratios of $E/E'$, $\nu/\nu'$ and $G/G'$ on non-dimensional vertical stress

Table II. Elastic properties and root types for different rocks

| Rock type | $E/E'$ | $\nu/\nu'$ | $G/G'$ | Root type |
|---|---|---|---|---|
| Rock 1. Isotropic | 1·0 | 1·0 | 1·0 | Equal |
| Rock 2. Transversely isotropic | 2·0 | 1·0 | 1·0 | Complex |
| Rock 3. Transversely isotropic | 3·0 | 1·0 | 1·0 | Complex |
| Rock 4. Transversely isotropic | 1·0 | 0·75 | 1·0 | Complex |
| Rock 5. Transversely isotropic | 1·0 | 1·5 | 1·0 | Distinct |
| Rock 6. Transversely isotropic | 1·0 | 1·0 | 2·0 | Distinct |
| Rock 7. Transversely isotropic | 1·0 | 1·0 | 3·0 | Distinct |

The above example was utilized to examine the solutions and investigate the effect of rock anisotropy on the displacement and stress distributions in the medium. The results show that the displacement and stress in the medium subjected to a point load (on the surface or in the interior) are easy and correct to calculate by the presented solutions. Also, the results indicate that the displacement and stress accounted for rock anisotropy are quite different for the displacement and stress calculated from isotropic solutions.

## CONCLUSIONS

Closed-form solutions for the displacements and stresses in a transversely isotropic half-space subjected to a point load are proposed. The point load can be applied on the horizontal surface or in the interior of the half-space. The solutions are the same as the Lekhnitskii[18] solution when the load applied at the surface. Also, the Mindlin[4] and the Boussinesq[2] solution for an isotropic material belong to the special cases of these exact solutions. Since the Fourier and Hankel transformations are adopted for solving the problem, the calculation of displacements and stresses by these solutions are more efficient. By an illustrative example to study the effect of rock anisotropy on the vertical surface displacement and vertical stress, it can be found that the displacement and stress calculated from isotropic solutions are quite different from these anisotropic solutions.

In practice, these equations can be applied to calculate the elastic displacements and stresses around a single end-bearing pile.[47] For floating pile groups, the principle of superposition could be utilized to analyse any compound pile group. These solutions can be extended to solve the three-dimensional displacements and stresses in a transversely isotropic half-space subjected to asymmetric loading types. These solutions also can be employed for preparing influence charts of a series of displacements and stresses for a transversely isotropic half-space subjected to irregular surface loads. The results will be presented in the forthcoming papers.

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# APPENDIX

*Notation*

| | |
|---|---|
| $A_{ij}\,(i,j=1\text{–}6)$ | elastic moduli or elasticity constants |
| $E, E', \nu, \nu', G'$ | elastic constants of a transversely isotropic rock |
| $h$ | in half-spaces, a distance of the surface, as seen in Figure 1 |
| i | complex number ($=\sqrt{-1}$) |
| $J_n(\ )$ | Bessel function of first kind of order $n$ |
| $k$ | coefficient (see equations (42)–(44)) |
| $m_1, m_2$ | coefficients (see equations (34)–(36)) |
| $n$ | integer used in Fourier transforms |
| $P_r, P_\theta, P_z$ | components of a point load in a cylindrical co-ordinate system |
| $q, s$ | coefficients (see equation (37)) |
| $r, \theta, z$ | cylindrical co-ordinates |
| $R, \Theta, Z$ | body force components in a cylindrical co-ordinate system |
| $T_1, T_2, T_3, T_4$ | coefficients (see equations (70)–(72)) |
| $u_1, u_2, u_3$ | roots of the characteristic equation |
| $U_r, U_\theta, U_z$ | displacement components of a semi-infinite space |
| $U_r^*, U_\theta^*, U_z^*$ | Fourier transforms of $U_r, U_\theta, U_z$ |
| $U'_r, U'_\theta, U'_z$ | displacement components of an infinite space |
| $U_{zn}^{**}$ | Hankel transform of $U_z^*$ |
| $U'^{**}_{zn}$ | transformed general solutions of an infinite space (see equation (58)) |
| $X, Y, Z$ | Cartesian co-ordinates |
| $Z^*$ | complex amplitude of the body force |

*Greek letters*

| | |
|---|---|
| $\Phi^*, \Psi^*$ | new displacement functions (see equation (26)) |
| $\Phi^{**}_{n-1}, \Psi^{**}_{n+1}$ | Hankel transforms of $\Phi^*$ and $\Psi^*$, respectively |
| $\Phi'^{**}_{n-1}, \Psi'^{**}_{n+1}$ | transformed general solutions of an infinite space (see equations (56), (57)) |
| $\alpha^*, \beta^*$ | Hankel transforms of the stress functions (see equations (64), (65)) |
| $\alpha^{**}_{n-1}, \beta^{**}_{n+1}$ | Fourier transforms of $\alpha^*$ and $\beta^*$ of order $n-1$ and $n+1$, respectively |
| $\gamma, \delta$ | real and imaginary part of the complex conjugate roots, respectively |
| $\delta(\ )$ | Dirac delta function |
| $\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$ | normal strain components |
| $\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{rz}$ | shear strain components |
| $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ | normal stress components of a semi-infinite space |
| $\sigma'_{rr}, \sigma'_{\theta\theta}, \sigma'_{zz}$ | normal stress components of an infinite space |
| $\sigma^*_{zz}$ | Hankel transforms of $\sigma_{zz}$ |
| $\sigma^{**}_{zzn}$ | Fourier transforms of $\sigma^*_{zz}$ of order $n$ |
| $\tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$ | shear stress components of a semi-infinite space |
| $\tau'_{r\theta}, \tau'_{\theta z}, \tau'_{rz}$ | shear stress components of an infinite space |
| $\omega$ | angular frequency |
| $\xi$ | Hankel transform parameter |

 

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