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A SIMPLE AND DIRECT DERIVATION FOR THE NUMBER OF NONCROSSING PARTITIONS

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Abstract. Kreweras considered the problem of counting noncrossing partitions of the set $\{1, 2, \cdots, n\}$, whose elements are arranged into a cycle in its natural order, into p parts of given sizes $n_1, n_2, \cdots, n_p$ (but not specifying which part gets which size). He gave a beautiful and surprising result whose proof resorts to a recurrence relation. In this paper we give a direct, entirely bijective, proof starting from the same initial idea as Kreweras' proof.

1. Introduction

A *noncrossing partition* of the set $[n] = \{1, 2, \cdots, n\}$, whose elements are arranged into a cycle in its natural order, is a partition π of the set $[n]$ with the property that there do not exist four numbers $a < b < c < d$ such that a and c are in one part but b and d are in another part. The study of noncrossing partitions goes back at least to Becker [1], where they are called "planar rhyme schemes." The systematic study of noncrossing partitions began with Kreweras [7] and Poupard [10]. For some further work on noncrossing partitions, see [2], [3], [5], [6], [9], [10], [11], [12], [13], [14] and the references given there. Let $f(n_1, n_2, \cdots, n_p)$ denote the number of noncrossing partitions of $[n]$ into p parts of given sizes $n_1, n_2, \cdots, n_p$ (but not specifying which part gets which size); and let p_k denote the number of parts with size k. Kreweras [7] gave the beautiful and surprising result (also see [4]):

Theorem 1. $f(n_1, n_2, \cdots, n_p) = n(n-1)\cdots(n-p+2)/\prod_{k\geq 1} p_k!$

Namely, $f(n_1, n_2, \cdots, n_p)$ depends on $n_1, n_2, \cdots, n_p$ only through p_k. An immediate consequence is that if the n_i's are distinct, then $f(n_1, n_2, \cdots, n_p) = n(n-1)\cdots(n-p+2)$, independently of $n_1, n_2, \cdots, n_p$. Kreweras' proof resorts to a combinatorial equality derived in another paper [8]. In this paper we give a direct, entirely bijective, proof starting from the same initial idea as Kreweras' proof.

2. A simple proof of the theorem

We give a vector representation of a noncrossing partition.

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Lemma 2. *Suppose the parts are distinguishable. Then there is a 1-1 onto mapping between the set $\mathcal{N}$ of noncrossing partitions of $[n]$ into p parts with given sizes $n_1, n_2, \cdots, n_p$ and the set $\mathcal{V}$ of vectors $(k_1, k_2, \cdots, k_{p-1})$ where $1 \le k_i \le n$ and the k_i's are distinct for $1 \le i \le p-1$.*

Proof. Suppose $\pi \in \mathcal{N}$ and s is the maximum element of π_p. For $1 \le i \le p-1$, choose k_i as the first element of π_i when we traverse the cycle from s clockwise. (Actually, any $s \in \pi_p$ would give the same k_i's.) Let $g(\pi) = (k_1, k_2, \cdots, k_{p-1})$. Then g is clearly a mapping from $\mathcal{N}$ to $\mathcal{V}$.

Conversely, suppose $(k_1, k_2, \cdots, k_{p-1}) \in \mathcal{V}$. We shall construct a unique noncrossing partition π as follows. Initially, all elements $1, 2, \cdots, n$ in the cycle are unmarked. We perform the following two steps.

Step 1. Find the first unmarked k_i such that the number of unmarked elements from k_i to the next unmarked $k_{i'}$ (including k_i but not $k_{i'}$) is at least n_i. Choose the first n_i such elements in clockwise order to form π_i, and mark them off.

Step 2. Go back to Step 1 until all k_i are marked. The remaining elements form π_p.

Note that in Step 1, such a k_i always exists since the number of unmarked elements is equal to n_p plus the sum of those n_j's for which k_j is unmarked. Note also that the construction ensures that the partition π is noncrossing. Hence $h(k_1, k_2, \cdots, k_{p-1}) = \pi$ is a mapping from $\mathcal{V}$ to $\mathcal{N}$.

For any $\pi' \in \mathcal{N}$, let $(k_1, k_2, \cdots, k_{p-1}) = g(\pi')$. Construct π from $(k_1, \cdots, k_{p-1})$ according to Steps 1 and 2. We prove $\pi = \pi'$. Suppose in the construction of π, the first iteration of Step 1 identifies k_i. Then $\pi_i = \{k_i, k_i+1, \cdots, k_i+n_i-1\}$. Note that π'_i also starts with k_i. Furthermore, for $j \ne i$, k_j does not lie between k_i and k_i+n_i-1 or Step 1 would not identify k_i. Thus no element of π'_j for all $j \ne i$ and $j \ne p$ can lie between k_i and k_i+n_i-1. Finally, no element of π'_p can lie between k_i and k_i+n_i-1, for otherwise all elements of π'_p would lie between k_i and k_i+n_i-1 and, starting from s, the first element of π'_i would not be k_i. Therefore $\pi'_i = \pi_i$. By deleting π'_i and π_i from π' and π respectively, a similar argument holds for the part chosen in the second iteration of Step 1, and so on for the third, fourth, ..., iteration.

On the other hand, for $(k'_1, k'_2, \cdots, k'_{p-1}) \in \mathcal{V}$, let $\pi = h(k'_1, k'_2, \cdots, k'_{p-1})$. We prove $g(\pi) = (k'_1, k'_2, \cdots, k'_{p-1})$. Consider the step in the construction of π when π_i is chosen to be marked. There is no unmarked element lying between the first and the last elements of π_i in the clockwise order of the cycle. Hence when we traverse the cycle from *any* unmarked element, in particular, from s, the first element of π_i we encounter must be k'_i. This shows that k'_i is the same k_i in the definition of $g(\pi)$.

Therefore h is the inverse of g. Thus g and h are 1-1 and onto. □

Proof of Theorem 1. First suppose that the parts are distinguishable. Then, by Lemma 2, $|\mathcal{N}| = |\mathcal{V}| = n(n-1)\cdots(n-p+2)$. However, when $n_i = n_j$, then interchanging the elements of π_i and π_j (including π_p) does not lead to a different partition, since parts can be identified only through their sizes. Thus we must divide by $\prod_{k\ge 1} p_k!$. □

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