ELSEVIER

Information Processing Letters 66 (1998) 215–219

Information Processing Letters

On the arrangement graph

Wei-Kuo Chiang [a,*], Rong-Jaye Chen [b]

[a] *Computer Telephony Applications Department, Computer & Communication Research Laboratories, Industrial Technology Research Institute, K200 CCL Bldg. 51, 195-11 Sec. 4, Chung Hsing Road, Chutung, Hsinchu, Taiwan 310*

[b] *Department of Computer Science and Information Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan 300*

Received 27 August 1996
Communicated by F. Dehne

Abstract

The arrangement graph was proposed as a generalization of the star graph topology. In this paper we investigate the topological properties of the (n, k)-arrangement graph $A_{n,k}$. It has been shown that the $(n, n-2)$-arrangement graph $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n. In addition, the exact value of average distance of $A_{n,k}$ has been derived.

Keywords: Interconnection networks; Graph isomorphism; Average distance

1. Introduction

Recently, a widely studied interconnection network topology called the star graph was proposed by Akers et al. [2]. It has been known as an attractive alternative to the hypercube [1]. The star graph is node and edge symmetric, and strongly hierarchical as is the case of the hypercube. The n-star graph S_n is regular of degree $n-1$, the number of nodes $n!$, and diameter $\lfloor 3(n-1)/2 \rfloor$. For a similar number of nodes, the star graph has a lower node degree, a shorter diameter, and a smaller average distance than the comparable hypercube.

In addition, Jow et al. presented another interconnection scheme based on the Cayley graph of the alternating group, called the alternating group graph [8]. The n-alternating group graph AG_n is regular of degree $2(n-2)$, the number of nodes $n!/2$, and diameter $\lfloor 3(n-2)/2 \rfloor$. The alternating group graph is also node and edge symmetric.

A common drawback of S_k and AG_n is the restriction on the number of nodes: $n!$ for S_n and $n!/2$ for AG_n. The set of values of $n!$ (or $n!/2$) is spread widely over the set of integers; so, one may be faced with the choice of either too few or too many available nodes.

Even before [8] was published, Day and Tripathi [5] proposed a generalized star graph, called the arrangement graph, as an attractive interconnection scheme for massively parallel systems. An arrangement graph is specified by two parameters n and k, satisfying $1 \leqslant k \leqslant n-1$. The (n, k)-arrangement graph $A_{n,k}$ is regular of degree $k(n-k)$, the number of nodes $n!/(n-k)!$, and diameter $\lfloor 3k/2 \rfloor$. The $(n, n-1)$-arrangement graph $A_{n,n-1}$ is isomorphic to the n-star graph S_n [6]. The arrangement graph provides more flexibility than the star graph in terms of choosing the major design parameters: degree, diameter, and number of nodes. The arrangement graph has been shown

* Corresponding author. Email: wkchiang@dcs.ccl.itri.org.tw.

0020-0190/98/$19.00 © 1998 Published by Elsevier Science B.V. All rights reserved.
PII: S0020-0190(98)00052-0

to be node and edge symmetric, strongly hierarchical, maximally fault tolerant, and strongly resilient [5].

In this paper we further look into the topological properties of the arrangement graph. We first show that the $(n, n-2)$-arrangement graph $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n. Then, we derive the exact value for the average distance of $A_{n,k}$. Due to the isomorphism between $A_{n,n-2}$ and AG_n, the average distance derived here for the arrangement graph as well as those discussed in the literature [4–7] can be applied directly to the alternating group graph. Therefore, we solve two of four open problems for the alternating group graph listed in [8]:
(1) enumeration of the node disjoint paths; and
(2) exact value of the average distance.

2. Graph definitions and basic properties

In this section we introduce the definitions and notations of the arrangement graph and the alternating group graph, and address their basic topological properties.

2.1. Graph definitions

For simplicity, let $\langle n\rangle = \{1, 2, \ldots, n\}$ and $\langle k\rangle = \{1, 2, \ldots, k\}$.

Definition 1 [5]. The (n, k)-arrangement graph

$$A_{n,k} = (V_1, E_1), \quad 1 \leqslant k \leqslant n-1,$$

is defined as follows:

$$V_1 = \{p_1 p_2 \ldots p_k \mid p_i \in \langle n\rangle \text{ and } p_i \neq p_j \text{ for } i \neq j\},$$

and

$$E_1 = \{(p, q) \mid p \text{ and } q \text{ in } V_1 \text{ and for some } i \text{ in } \langle k\rangle, \; p_i \neq q_i \text{ and } p_j = q_j \text{ for } j \neq i\}.$$

That is, the nodes of $A_{n,k}$ are the arrangements of k elements out of the n symbols $\langle n\rangle$, and the edges of $A_{n,k}$ connect arrangements which differ in exactly one of their k positions. An edge of $A_{n,k}$ connecting two arrangements which differ only in position i is called an i-edge. An example of $A_{n,k}$ for $n = 4$ and $k = 2$ is given in Fig. 1.

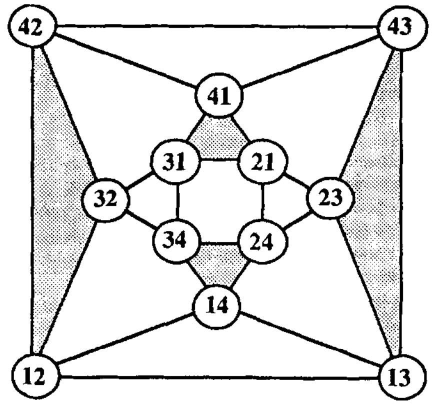

Fig. 1. A (4, 2)-arrangement graph.

To define the alternating group graph, we describe some terminologies and notations for ease of exposition. In a permutation $p = p_1 p_2 \ldots p_i \ldots p_j \ldots p_n$, the pair (i, j), $i < j$, is said to constitute an inversion if $p_i > p_j$. A permutation p is called even or odd depending on the number of inversions in p being even or odd. The alternating group A_n is defined as the set of all even permutations of n elements [3].

Let

$$g_i^+ = (1\ 2\ i), \qquad g_i^- = (1\ i\ 2),$$
$$\Omega = \{g_i^+ \mid 3 \leqslant i \leqslant n\} \cup \{g_i^- \mid 3 \leqslant i \leqslant n\},$$

where Ω is known to be a generator set for A_n [8]. Let T_{ij} denote the transposition that swaps the elements at positions i and j, then $g_i^+ = T_{12} \cdot T_{2i}$ and $g_i^- = T_{2i} \cdot T_{12}$ for $3 \leqslant i \leqslant n$.

Definition 2 [8]. The n-alternating group graph $AG_n = (V_2, E_2)$ is defined as follows:

$$V_2 = A_n, \quad \text{and}$$
$$E_2 = \{(p, q) \mid p, q \in V_2, \text{ and } q = p \circ g \text{ for some } g \in \Omega\},$$

where "$\circ$" is the composition operator. The nodes of AG_n are even permutations of the n symbols $\langle n\rangle$. An edge of AG_n connecting two even permutations p and q is called an i-edge if $q = p \circ g_i^a$ and $a = +$ or $-$. Note that $q = p \circ g_i^+$ if and only if $p = q \circ g_i^-$. An example of AG_n for $n = 4$ is given in Fig. 2.

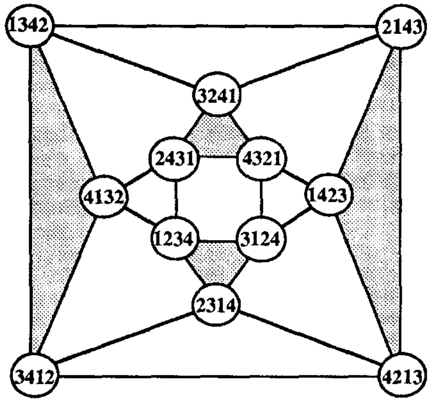

Fig. 2. A 4-alternating group graph AG_4.

2.2. Basic properties

There are $(n-1)!/(n-k)!$ nodes in $A_{n,k}$ which have element i in position j, for any fixed i and j ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$). These nodes are interconnected in a manner identical to an $A_{n-1,k-1}$ graph. For a fixed position j, an $A_{n,k}$ can be partitioned into n node-disjoint copies of $A_{n-1,k-1}$. This partitioning of $A_{n,k}$ into n copies of $A_{n-1,k-1}$ can be done in k different ways corresponding to the k possible values of j ($1 \leqslant j \leqslant k$) and can be carried out recursively. Fig. 1 shows that $A_{4,2}$ can be viewed as an interconnection of four $A_{3,1}$'s by fixing the symbol in position 2.

Due to the node symmetry of the (n,k)-arrangement graph, the problem of routing between two arbitrary nodes in $A_{n,k}$ is reduced to the problem of routing between an arbitrary node and the identity node I_k. In order to solve the problem of the routing, a cycle representation for the label of each node in $A_{n,k}$ was introduced in [5]. The cycle representation of a node p with c cycles including e external cycles can be denoted as

$$C(p) = C_1, C_2, \ldots, C_e, C'_{e+1}, C'_{e+2}, \ldots, C'_c,$$

where $C_1, C_2, \ldots, C_e$ are external cycles and $C'_{e+1}, C'_{e+2}, \ldots, C'_c$ are internal cycles. Let m denote the total number of elements in these cycles. It has been shown in [5] that the distance $d(p)$ from node p to the identity node I_k in $A_{n,k}$ is given by:

$$d(p) = c + m - 2e. \qquad (1)$$

In addition, it has been presented in [5] that the connectivity $\kappa(A_{n,k})$ of $A_{n,k}$ is $k(n-k)$ and the fault-diameter $D_f(A_{n,k})$ of $A_{n,k}$ is at most $\lfloor 3k/2 \rfloor + 4$. In [6], it has been known that the arrangement graph $A_{n,k}$, for $1 \leqslant k \leqslant n-2$, contains cycles of any arbitrary length L, $3 \leqslant L \leqslant |A_{n,k}|$, where $|A_{n,k}|$ is the number of nodes in $A_{n,k}$. Furthermore, it has been shown in [7] that an $(n-k+1) \times (n-k+2) \times \cdots \times (n-1) \times n$ grid can be embedded into $A_{n,k}$ with unit expansion and dilation three.

3. Isomorphism between $A_{n,n-2}$ and AG_n

In this section, we show the alternating group graph is a spatial arrangement graph. Before proving it, we give a formal definition of isomorphism between two graphs in the following.

Definition 3. Two graphs G_1 and G_2 are *isomorphic* if there is a one-to-one function f from $V(G_1)$ onto $V(G_2)$ such that $(p,q) \in E(G_1)$ if and only if $(f(p), f(q)) \in E(G_2)$, where $f(p)$ is the image of p.

Theorem 4. *The $(n, n-2)$-arrangement graph $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n.*

Proof. To prove that AG_n and $A_{n,n-2}$ are isomorphic, we define a one-to-one function f_1 from the nodes of AG_n to those of $A_{n,n-2}$ by:

$$f_i(p_1 p_2 p_3 \ldots p_{n-1} p_n) = p_3 \ldots p_{n-1} p_n.$$

Note that $f_1^{-1}(q_3 \ldots q_{n-1} q_n) = q_1 q_2 q_3 \ldots q_{n-1} q_n$ or $q_2 q_1 q_3 \ldots q_{n-1} q_n$ depending on which one is an even permutation.

Now, we prove f_1 preserves adjacency. Let p and q be two nodes linked with an i-edge in AG_n. Then $f_1(p)$ and $f_1(q)$ are linked with an $(i-2)$-edge in $A_{n,n-2}$. Conversely, let s and t be two nodes linked with a $(j-2)$-edge in $A_{n,n-2}$, i.e.,

$$s = s_3 \ldots s_{j-1} s_j s_{j+1} \ldots s_n \quad \text{and}$$
$$t = s_3 \ldots s_{j-1} t_j s_{j+1} \ldots s_n.$$

Without loss of generality, assume s is an even permutation, then

$$f_1^{-1}(s) = s_1 s_2 s_3 \ldots s_{j-1} s_j s_{j+1} \ldots s_n \quad \text{and}$$
$$f_1^{-1}(t) = s_2 s_j s_3 \ldots s_{j-1} t_j s_{j+1} \ldots s_n,$$

where $s_1 < s_2$ and $t_j = s_1$, and thus $f_1^{-1}(t) = f_1^{-1}(s) \circ g_j^+$. That is, $f_1^{-1}(t)$ and $f_1^{-1}(s)$ are linked with a j-edge in AG_n. □

The example for $A_{4,2}$ in Fig. 1 is the resulting graph when f_1 is applied to AG_4 in Fig. 2.

The next corollary solves one of four open problems listed in [8]. We get the result by using Theorem 4 and the fact that for any two nodes of $A_{n,k}$, there exists $k(n-k)$ node-disjoint paths of length at most four plus the distance between the two nodes [4].

Corollary 5. *Given any distinct nodes s and t in an n-alternating group graph AG_n, there are $2(n-2)$ node-disjoint paths of length at most four plus the distance between the two nodes. That is, the connectivity $\kappa(AG_n)$ of AG_n is $2(n-2)$ and the fault-diameter $D_f(AG_n)$ of AG_n is at most $\lfloor 3(n-2)/2 \rfloor + 4$.*

The following corollary describes the routing distance in AG_n. Although the result was discussed in [8], here we get it directly by applying Eq. (1) with Theorem 4.

Corollary 6. *The distance $d(p)$ from node p to the identity node I_n in AG_n is:*

$$d(p) = \begin{cases} n+k-1, & \text{if } p_1 = 1 \text{ and } p_2 = 2, \\ n+k-l-2, & \text{if } p_1 \neq 1 \text{ and } p_2 = 2, \\ n+k-l-2, & \text{if } p_1 = 1 \text{ and } p_2 \neq 2, \\ n+k-l-3, & \text{if } 1 \text{ and } 2 \in C_i, \\ n+k-l-4, & \text{if } 1 \in C_i \text{ and } 2 \in C_j, \end{cases}$$

where k and l denote the numbers of cycles and invariants in $C(p)$.

Proof. The distance from $p = p_1p_2p_3 \ldots p_n$ to $I_n = 123\ldots n$ in AG_n is equivalent to the distance from $p'' = p_3 \ldots p_n$ to $I_n'' = 3\ldots n$ in $A_{n,n-2}$. Let c'', m'', and e'' denote the number of cycles, the total number of elements in these cycles, and the number of external cycles in $C(p'')$ with respect to I_n''; that is, $d(p) = d(p'') = c'' + m'' - 2e''$. Note that symbols 1 and 2 with respect to I_n'' are foreign symbols.

(a) If $p_1 = 1$ and $p_2 = 2$, then $c'' = k$, $m'' = n - l$, $e'' = 0$, and $d(p'') = n + k - l$.
(b) If $p_1 \neq 1$ and $p_2 = 2$, then $c'' = k$, $m'' = n - l$, $e'' = 1$, and $d(p'') = n + k - l - 2$.
(c) If $p_1 = 1$ and $p_2 \neq 2$, then $c'' = k$, $m'' = n - l$, $e'' = 1$, and $d(p'') = n + k - l - 2$.
(d) For $1, 2 \in C_i$, if either $p_1 = 2$, or $p_2 = 1$, $c'' = k$, $m'' = n - l - 1$, $e'' = 1$; otherwise, $c'' = k + 1$, $m'' = n - l$, $e'' = 2$. Therefore, $d(p'') = n + k - l - 3$.
(e) If $1 \in C_i$ and $2 \in C_j$, $c'' = k$, $m'' = n - l$, and $e'' = 2$; so, $d(p'') = n + k - l - 4$. □

4. Average distance

In this section, we derive the exact values for the average distances of the (n, k)-arrangement graph $A_{n,k}$ and the n-alternating group graph AG_n. The average distance of a symmetric interconnection network is determined by the summation of distances of all nodes from a given node over the total number of nodes. Average distance is a better indicator of the average message delay in an interconnection network than its diameter. Since $A_{n,k}$ is node-symmetric, its average distance among all pairs of nodes p and q (possibly $p = q$) equals the average distance from the identity node I_k to all nodes.

Let $\overline{D}(A_{n,k})$ denote the average distance of $A_{n,k}$. The value of this measure for the (n, k)-arrangement graph is

$$\overline{D}(A_{n,k}) = \sum_{p \in A_{n,k}} d(p)/N(n,k),$$

where $N(n,k) = n!/(n-k)!$.

It is known that the average number of cycles including invariants in a permutation of n symbols is H_n, where $H_n = \sum_{i=1}^{n} 1/i$ denotes the nth Harmonic number [9]. Here, we show the average number of cycles including invariants in a permutation of k elements out of n symbols is H_k.

Lemma 7. *The average number of cycles including invariants in a permutation of k elements out of n symbols is H_k.*

Proof. We consider a permutation p formed by choosing arbitrarily k elements out of the n symbols $\langle n \rangle$. Let p' be the permutation obtained from replacing each foreign element of p with its corresponding nonforeign element. Then the total number of cycles of $C(p)$ is equal to that of $C(p')$. The number of cycles in all permutations of the k symbols $\langle k \rangle$ is $k! \times H_k$. Therefore, the total number of cycles in all permutations of k elements out of the n symbols $\langle n\rangle$ is

$$\binom{n}{k} \times k! \times H_k.$$

Dividing it by the total number of permutations of k elements out of the n symbols $\langle n\rangle$, we derive the average number of cycles in a permutation as H_k. □

Given a node p, let u be the total number of invariants, c^* be the total number of cycles including invariants, and m^* be the total number of misplaced symbols, i.e., $m^* = k - u$. In Eq. (1), c denotes the total number of cycles of $C(p)$ excluding invariants, and m denotes the total number of elements in these cycles including the corresponding nonforeign elements for the external cycles. It then follows that for c and m as defined above, $c = c^* - u$ and $m = m^* + e$. Similar to the argument of Theorem 2 in [1], we rewrite Eq. (1) as

$$d(p) = k + c^* - e - 2u. \tag{2}$$

The following theorem gives the exact value for the average distance of $A_{n,k}$, which is obtained by computing the total value of Eq. (2) for each node in $A_{n,k}$ and then dividing by $n!/(n-k)!$.

Theorem 8. *The average distance $\overline{D}(A_{n,k})$ of the (n,k)-arrangement graph is given by:*

$$H_k + \frac{k(k-2)}{n}. \tag{3}$$

Proof. The first term in the summation of Eq. (2) over all nodes of $A_{n,k}$ is $k \times n!/(n-k)!$, and the second term is $H_k \times n!/(n-k)!$. The third term is the total number of foreign symbols in all nodes of $A_{n,k}$, i.e., $k \times (n-k) \times (n-1)!/(n-k)!$. The final term is the total number of symbols in the correct position for all nodes multiplied by -2, i.e., $-2 \times k \times (n-1)!/(n-k)!$. Therefore,

$$\begin{aligned}\overline{D}(A_{n,k}) &= k + H_k - k \times (n-k)/n - 2k/n \\ &= H_k + k \times (k-2)/n. \quad \square\end{aligned}$$

The next corollary solves another open problem listed in [8]. That is, the exact value for the average distance of AG_n is found while the result for the average distance of AG_n proposed in [8] is just an upper bound.

Corollary 9. *The average distance $\overline{D}(AG_n)$ of the n-alternating group graphs is*

$$n + H_n + \frac{7}{n} - \frac{1}{n-1} - 6.$$

Proof. By applying Eq. (3) with Theorem 4,

$$\overline{D}(AG_n) = \overline{D}(A_{n,n-2}). \quad \square$$

References

[1] S.B. Akers, D. Horel, B. Krishnamurthy, The star graph: An attractive alternative to the n-cube, in: Proc. Internat. Conf. on Parallel Processing, 1987, pp. 393–400.

[2] S.B. Akers, B. Krishnamurthy, A group theoretical model for symmetric interconnection networks, in: Proc. Internat. Conf. on Parallel Processing, 1986, pp. 216–233.

[3] C. Berge, Principles of Combinatorics, Academic Press, New York, 1971, Chapter 4.

[4] K. Day, A. Tripathi, Characterization of node-disjoint paths in arrangement graphs, Tech. Rept. TR 91-43, Dept. of Computer Science, University of Minnesota, 1991.

[5] K. Day, A. Tripathi, Arrangement graphs: A class of generalized star graphs, Inform. Process. Lett. 42 (1992) 235–241.

[6] K. Day, A. Tripathi, Embedding of cycles in arrangement graphs, IEEE Trans. Comput. 42 (1993) 1002–1006.

[7] K. Day, A. Tripathi, Embedding grids, hypercubes, and trees in arrangement graphs, in: Proc. Internat. Conf. on Parallel Processing, 1993, pp. 65–72.

[8] J. Jwo, S. Lakshmivarahan, S.K. Dhall, A new class of interconnection networks based on the alternating group, Networks 23 (1993) 315–326.

[9] D.E. Knuth, The Art of Computer Programming, Vol. 1, Addison-Wesley, Reading, MA, 1973.