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Application of center manifold reduction to nonlinear system stabilization

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Abstract

The Center Manifold Theorem is applied to the local feedback stabilization of nonlinear systems in critical cases. The paper addresses two particular critical cases of which the system linearization at the equilibrium point of interest is assumed to possess either a simple zero eigenvalue or a complex conjugate pair of simple and pure imaginary eigenvalues. In either case, the noncritical eigenvalues are taken to be stable. The results on stabilizability and stabilization are given explicitly in terms of the nonlinear model of interest in its original form, i.e., before reduction to the center manifold. Moreover, the formulation given in this paper uncovers connections between results obtained using the center manifold reduction and those of an alternative approach.

Keywords: Nonlinear systems; Stabilization; Center manifold reduction

1. Introduction

Recently, center manifold reduction (see, e.g. Refs. [2,4,11,12]) has been employed in nonlinear stabilization, resulting in stabilizing control law designs for various classes of nonlinear systems in the so-called "critical cases". Critical cases occur when the linearized system at an equilibrium point has at least one eigenvalue on the imaginary axis, with the remaining eigenvalues in the open left half of the complex plane. Aeyels [1], who initiated application of the center manifold reduction in nonlinear stabilization, investigated the existence of smooth stabilizing feedback control laws for a class of third-order nonlinear systems for which the linearized model possesses an uncontrollable pair of pure imaginary eigenvalues. Behtash and Sastry [10] used the same approach to study stabilization for nonlinear systems whose linearized model has two

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distinct pairs of complex conjugate pure imaginary eigenvalues, or a double pole at the origin, or a pole at the origin and a complex conjugate pair of pure imaginary eigenvalues. In Ref. [10], the design was undertaken for the reduced system on the center manifold using normal form calculations, and for simplicity, a scalar stable mode was assumed.

In the recent years, several practical examples (e.g., Refs. [13–17]) of which system may possess one simple zero and/or a pair of pure imaginary eigenvalues have demonstrated potential applications. Those applications also motivate the need of the design of stabilizing controller for the critical system with any finite number of stable modes as well as the expression of the control laws directly in terms of the original model rather than in terms of transformed versions. It is known that the stability of local simple bifurcations implies the stability of the system at criticality and vice versa. Local bifurcation control laws for simple stationary bifurcation and Hopf bifurcation are obtained, respectively, by using series expansion of the vector field of original system dynamics without normal form reduction [6,7]. The proposed control laws can hence be applied to the critical system. However, those designs are expressed in terms of the mathematical operation of both the left and right eigenvectors corresponding to the critical eigenvalue on the imaginary axis. No direct relationships between the stability coefficient and system states are explicitly given, which makes the application difficult and involving complicated calculation.

In this paper, we seek an alternative approach by employing center manifold reduction to design control laws explicitly in terms of the system states. A main goal of this paper is to derive stabilizing control algorithms for general nonlinear systems in critical cases. The development focuses on general nonlinear systems in two specific critical cases. In the first critical case of interest here, a simple zero eigenvalue occurs, while in the second case a pair of pure imaginary eigenvalues occurs. In either case, the critical eigenvalues of the linearized model need not be controllable. The feedback laws obtained include purely linear state feedbacks, purely nonlinear state feedbacks and feedback control laws containing both linear and nonlinear terms in the state. Results of this paper are also compared with those of Refs. [6,7].

2. Preliminaries

Consider a class of nonlinear autonomous systems given by

$$\dot{\eta} = A_{11}\eta + A_{12}\xi + F(\eta, \xi), \tag{1a}$$

$$\dot{\xi} = A_{21}\eta + A_{22}\xi + G(\eta, \xi), \tag{1b}$$

where $\eta \in \mathbb{R}^n, \xi \in \mathbb{R}^m$. In Eqs. (1a) and (1b), A_{ij} for $i, j = 1, 2$ are constant matrices, and the functions F,G are sufficiently smooth, with their values and first derivatives vanishing at the origin. If A_{12} and A_{21} vanish, the matrix A_{11} has all its eigenvalues on the imaginary axis, and A_{22} is Hurwitz, then the Center Manifold Theorem asserts the existence of a locally invariant manifold for Eqs. (1a) and (1b) near the origin. This manifold is given by the graph of a function $\xi = h(\eta)$.

In applying the Center Manifold Theorem to feedback stabilization problems, it is convenient to give a restatement of the theorem in a way that does not require vanishing of the "linear coupling" matrices A_{12} and A_{21}. This is especially true when the feedback is allowed to possess linear terms. For the purposes of this paper, a restatement allowing nonzero A_{21} but with $A_{12} = 0$ suffices. A linear transformation of variables is now employed to achieve this. Consider the equation

$$AM + MB = C, \tag{2}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $M, C \in \mathbb{C}^{m\times n}$. For $n = m$ and $B = A^{\mathrm{T}}$, Eq. (2) is a Liapunov matrix equation [5]. Let $\mathscr{F}$ denote the linear operator

$$\mathscr{F} : M \mapsto AM + MB \tag{3}$$

for $M \in \mathbb{C}^{m\times n}$.

The following result is a direct generalization of Ref. [5], Theorem F-1 and Corollary F-1a.

Theorem 1. *Let n, m be positive integers. If the sum of any eigenvalue of A and any eigenvalue of B is nonzero, then the linear matrix equation (2) has a unique solution for matrix M.*

We now apply the Center Manifold Theorem to the stability analysis of Eqs. (1a) and (1b) for the case of which $A_{12} = 0$, with A_{21} not necessarily zero. Let A_{22} be Hurwitz and A_{11} have all its eigenvalues on the imaginary axis. By Theorem 1, the equation

$$A_{22}E - EA_{11} + A_{21} = 0 \tag{4}$$

has a unique solution for the $m \times n$ matrix E. Letting $v := \xi - E\eta$, we can rewrite system (1) as

$$\dot{\eta} = A_{11}\eta + F(\eta, v + E\eta) \tag{5a}$$

$$\dot{v} = A_{22}v + G(\eta, v + E\eta) - E \cdot F(\eta, v + E\eta). \tag{5b}$$

The Center Manifold Theorem for Eqs. (1a) and (1b) can now be restated as follows:

Lemma 1. *Assume $A_{12} = 0$, A_{22} is Hurwitz, and all eigenvalues of A_{11} have zero real parts. Then the origin of Eqs. (1a) and (1b) is asymptotically stable (unstable) if the origin is asymptotically stable (unstable) for the reduced model*

$$\dot{\eta} = A_{11}\eta + F(\eta, h(\eta) + E\eta), \tag{6}$$

where h satisfies the partial differential equation

$$\mathrm{D}h(\eta)\{A_{11}\eta + F(\eta, h(\eta) + E\eta)\} = A_{22}h(\eta) + G(\eta, h(\eta) + E\eta) - E \cdot F(\eta, h(\eta) + E\eta) \tag{7}$$

with E the solution of Eq. (4) and boundary conditions: $h(0) = 0$ *and* $\mathrm{D}h(0) = 0$.

We employ Taylor series expansions in the development below, using multilinear function notation for the terms in these expansions. The definition of multilinear function is recalled as follows.

Definition 1 (e.g., Ref. [9]). Let $V_1, V_2, \ldots, V_k$ and W be vector spaces over the same field. A map $\psi: V_1 \times V_2 \times \cdots \times V_k \to W$ is multilinear (or k-linear) if it is linear in each of its arguments. That is, for any $v_i, \tilde{v}_i \in V_i$, $i = 1, \ldots, k$, and for any scalars $a, \tilde{a}$, we have

$$\psi(v_1, \ldots, av_i + \tilde{a}\tilde{v}_i, \ldots, v_k) = a\psi(v_1, \ldots, v_i, \ldots, v_k) + \tilde{a}\psi(v_i, \ldots, \tilde{v}_i, \ldots, v_k). \tag{8}$$

The integer k is the degree of the multilinear function ψ.

The next definition deals with the special case of which $V_1 = V_2 = \cdots = V_k = V$.

Definition 2 ([9]). A k-linear function $\psi: V \times V \times \cdots \times V \to W$ is symmetric if the vector $\psi(v_1, v_2, \ldots, v_k)$ is invariant under arbitrary permutations of the argument vectors v_i. A function $\phi: \mathbb{R}^n \to \mathbb{R}^m$ is homogeneous of degree k (k an integer), if for each scalar α, $\phi(\alpha\eta) = \alpha^k\phi(\eta)$ for all $\eta \in \mathbb{R}^n$.

Note that, in the sequel prime denotes the transpose of both vector and matrix and I denotes the identity matrix.

3. General framework

Consider a nonlinear control system

$$\dot{\eta} = A_{11}\eta + b_1 u + F(\eta, \xi), \tag{9a}$$

$$\dot{\xi} = A_{22}\xi + b_2 u + G(\eta, \xi), \tag{9b}$$

where η, ξ are real vectors, and a preliminary block diagonalization has been applied to remove any linear coupling term in the dynamics between η and ξ. For simplicity, u is supposed to be a scalar control. It is not difficult to extend the study to the case of which the input is a vector control. In the following, we apply the center manifold result in Lemma 1 to design stabilizing control laws for Eqs. (9a) and (9b) for which all eigenvalues of A_{11} lie on the imaginary axis.

Let us first consider the case of which b_1 is nonzero. For the simple critical cases of which A_{11} is the scalar 0 or is a 2×2 matrix with a pair of pure imaginary eigenvalues, linear theory will imply the existence of a linear stabilizing feedback control for Eqs. (9a) and (9b). In this paper, we consider next the existence of a purely nonlinear smooth feedback (i.e., one with vanishing linear part).

Since now we focus on purely nonlinear stabilizing controllers, system (9a) and (9b) retains the linear decoupling property upon control. Thus, if A_{22} is stable, then according to Center Manifold Theorem (e.g., Refs. [3,8]) there is a locally invariant manifold $\xi = h(\eta)$ for Eqs. (9a) and (9b). Furthermore, h satisfies

$$\begin{aligned} &\mathrm{D}h(\eta)\{A_{11}\eta + b_1 u(\eta, h(\eta)) + F(\eta, h(\eta))\} \\ &\quad = A_{22}h(\eta) + b_2 u(\eta, h(\eta)) + G(\eta, h(\eta)) \end{aligned} \tag{10}$$

with boundary conditions $h(0) = 0$ and $\mathrm{D}h(0) = 0$. Then, we seek a purely nonlinear stabilizing feedback control law by using stability conditions for the reduced model

$$\dot{\eta} = A_{11}\eta + b_1 u(\eta, h(\eta)) + F(\eta, h(\eta)). \tag{11}$$

Note that, for the case of which A_{22} is not stable, a linear state feedback $K_2\xi$ is needed to first stabilize $A_{22} + b_2K_2$.

Next, consider the case of $b_1 = 0$ and assume the feedback control to be of the form

$$u(\eta, \xi) = K_1\eta + K_2\xi + U(\eta, \xi), \tag{12}$$

where $U(\cdot, \cdot)$ is a smooth, purely nonlinear function whose first derivatives vanish at the origin. Rewrite the system dynamics (9a) and (9b) as

$$\dot{\eta} = A_{11}\eta + F(\eta, \xi), \tag{13}$$

$$\dot{\xi} = b_2K_1\eta + (A_{22} + b_2K_2)\xi + b_2U(\eta, \xi) + G(\eta, \xi). \tag{14}$$

From Eq. (14), the feedback has given rise to a linear coupling term between η and ξ in the dynamics. As discussed in Section 2, there is a constant matrix E such that, with $v := \xi - E\eta$, the transformed version of the control system (13) and (14) is in block diagonal form. Here, E is the (unique) solution of the Liapunov-like equation

$$b_2K_1 + (A_{22} + b_2K_2)E - EA_{11} = 0. \tag{15}$$

We assume that $A_{22} + b_2K_2$ is stable. Moreover, since all the eigenvalues of A_{11} lie on the imaginary axis, then Theorem 1 guarantees existence of a solution E to Eq. (15). The transformed dynamics in the states η and ξ is then

$$\dot{\eta} = A_{11}\eta + F(\eta, v + E\eta), \tag{16a}$$

$$\dot{v} = (A_{22} + b_2K_2)v + b_2U(\eta, v + E\eta) + G(\eta, v + E\eta) - E \cdot F(\eta, v + E\eta). \tag{16b}$$

Eqs. (16a) and (16b) have a center manifold given by the graph of the function $v = h(\eta)$, where h satisfies

$$\mathrm{D}h(\eta)\{A_{11}\eta + F(\eta, h(\eta) + E\eta)\} = (A_{22} + b_2K_2)h(\eta) + b_2U(\eta, h(\eta) + E\eta) + G(\eta, h(\eta) + E\eta) - E \cdot F(\eta, v + E\eta) \tag{17}$$

with boundary conditions $h(0) = 0$ and $\mathrm{D}h(0) = 0$.

Lemma 1 implies asymptotic stability of the origin for Eqs. (16a) and (16b) if the control gains K_1, K_2 and the nonlinear function U are chosen such that (i) $A_{22} + b_2K_2$ is Hurwitz, and (ii) the origin of reduced model (16a) with $v = h(\eta)$ is asymptotically stable.

We now proceed to consider two special cases in which the system has only simple critical modes (i.e., one zero eigenvalue or a pair of pure imaginary eigenvalues) and the rest of the eigenvalues are stabilizable.

4. One zero eigenvalue

In this section, we first consider stability conditions for scalar systems with a zero eigenvalue. These conditions are then employed in the design of stabilizing control laws for higher order systems with a simple zero eigenvalue.

Consider a scalar real nonlinear system as given by

$$\dot{x} = dx^2 + ex^3 + \cdots \tag{18}$$

Stability conditions for system (18) are given next.

Lemma 2. *The origin of system (18) is asymptotically stable if $d = 0$ and $e < 0$. The origin is unstable for Eq. (18) if $d \neq 0$.*

Now consider Eqs. (9a) and (9b), with the scalar x replacing the critical state η, and with

$$f(x, \xi) := F(x, \xi) = f_{xx}x^2 + xf_{x\xi}\xi + \xi' f_{\xi\xi}\xi + f_{xxx}x^3 + x^2 f_{xx\xi}\xi + x\xi' f_{x\xi\xi}\xi + f_{\xi\xi\xi}(\xi, \xi, \xi) + \mathrm{O}(\|(x, \xi\|^4), \tag{19}$$

$$G(x, \xi) = x^2 G_{xx} + xG_{x\xi}\xi + G_{\xi\xi}(\xi, \xi) + x^3 G_{xxx} + x^2 G_{xx\xi}\xi + xG_{x\xi\xi}(\xi, \xi) + G_{\xi\xi\xi}(\xi, \xi, \xi) + \mathrm{O}(\|(x, \xi)\|^4). \tag{20}$$

The coefficients in the Taylor series expansions (19) and (20) are either constants or symmetric multilinear functions of their arguments. For instance,

$f_{\xi\xi\xi}(\xi,\xi,\xi)$ and $G_{\xi\xi}(\xi,\xi)$ denote a symmetric trilinear scalar function and a bilinear vector function of ξ, respectively.

In the remainder of this section, stabilizing control laws will be obtained for system (9a) and (9b) under either of the following two hypotheses.

Hypothesis 1A. The matrix $A_{11} = 0$ is a scalar and $b_1 \neq 0$.

Hypothesis 1B. The matrix $A_{11} = 0$ is a scalar and $b_1 = 0$.

4.1. *The case of $b_1 \neq 0$*

In this section, we consider the case where Hypothesis 1A holds. The control law is taken to be purely nonlinear. Existence of linear stabilizing feedback for this case is evident. Nonlinear feedback controllers are none the less desirable in certain applications. Suppose A_{22} is stable and the scalar control input is of the form

$$u(x,\xi) = U(x,\xi) := u_{xx}x^2 + xu_{x\xi}\xi + \xi' u_{\xi\xi}\xi + u_{xxx}x^3 + x^2 u_{xx\xi}\xi + x\xi' u_{x\xi\xi}\xi + u_{\xi\xi\xi}(\xi,\xi,\xi). \quad (21)$$

According to Center Manifold Theorem, the stability of the origin for Eqs. (9a) and (9b) coincides with the stability of the origin for the reduced model

$$\dot{x} = b_1 u(x, h(x)) + f(x, h(x)). \quad (22)$$

Here, h solves Eq. (10) with η replaced by x and with boundary conditions $h(0) = 0$ and $\mathrm{D}h(0) = 0$. Indeed, solving Eq. (10) we have

$$h(x) = x^2 h_{xx} + \mathrm{O}(|x|^3), \quad (23)$$

where

$$h_{xx} = -A_{22}^{-1}(b_2 u_{xx} + G_{xx}). \quad (24)$$

From Lemma 2, we now have the following lemma.

Lemma 3. *Let A_{22} be stable. Then under Hypothesis 1A, the origin is asymptotically stable for Eqs. (9a) and (9b) if $f_{xx} + b_1 u_{xx} = 0$ and $f_{xxx} + b_1 u_{xxx} - (f_{x\xi} + b_1 u_{x\xi})A_{22}^{-1}(G_{xx} + b_2 u_{xx}) < 0$.*

It is obvious from Lemma 3 that a purely quadratic stabilizing control law exists.

Corollary 1. *Assume that A_{22} is stable. Under Hypothesis 1A, the origin of Eqs. (9a) and (9b) is asymptotically stabilizable by a* purely quadratic *feedback of the form $u = u_{xx}x^2 + xu_{x\xi}\xi$ if the vector $A_{22}^{-1}G_{xx} \neq 0$.*

Furthermore, below we have a purely cubic stabilizing controller for system (9a) and (9b) while $f_{xx} = 0$.

Corollary 2. *Assume that A_{22} is stable and $f_{xx} = 0$. Under Hypothesis 1A, the origin of Eqs. (9a) and (9b) is asymptotically stabilizable by a purely cubic feedback of the form $u = u_{xxx}x^3$.*

For the case of which A_{22} is not stable, a linear feedback $K_2\xi$ is first needed to guarantee the existence of a locally invariant manifold. Then the design of stabilizing control laws proposed in Lemma 3 and Corollaries 1 and 2 can be applied directly.

4.2. The case of $b_l = 0$

Next, we consider the case of which Hypothesis 1B holds and feedback control has the form of

$$u(x, \xi) = k_1 x + K_2\xi + U(x, \xi), \tag{25}$$

where k_1 denotes a scalar control gain and the nonlinear control function U is defined as in Eq. (21).

Suppose $A_{22} + b_2K_2$ is stable. As discussed in Section 3, the stability of control system (9a) and (9b) in this critical case coincides with the stability of the reduced model

$$\dot{x} = f(x, h(x) + Ex), \tag{26}$$

where E and $h(\,)$ solve Eqs. (15) and (17), respectively.

Solving Eqs. (15) and (17), we have

$$E = -(A_{22} + b_2K_2)^{-1}b_2k_1, \tag{27}$$

$$h_{xx} = -(A_{22} + b_2K_2)^{-1}\{-[f_{xx} + f_{x\xi}E + E'f_{\xi\xi}E]E \\ + [b_2u_{xx} + G_{xx} + (b_2u_{x\xi} + G_{x\xi})E + b_2E'u_{\xi\xi}E + G_{\xi\xi}(E, E)\}. \tag{28}$$

The reduced model (26) is then given by

$$\dot{x} = \{f_{xx} + f_{x\xi}E + E'f_{\xi\xi}E\}x^2 + \{f_{x\xi}h_{xx} + 2E'f_{\xi\xi}h_{xx} + f_{xxx} + f_{xx\xi}E \\ + E'f_{x\xi\xi}E + f_{\xi\xi\xi}(E, E, E)\}x^3 + \mathrm{O}(|x|^4). \tag{29}$$

Employing Lemma 2, we then have the next result.

Lemma 4. *Let the control input u be of the form as in Eq. (25). Then under Hypothesis 1B, the origin of the closed-loop system (9a) and (9b) is asymptotically stable if $A_{22} + b_2K_2$ is stable and the following two conditions hold:*

$$f_{xx} + f_{x\xi}E + E'f_{\xi\xi}E = 0, \tag{30}$$

$$f_{x\xi}h_{xx} + 2E'f_{\xi\xi}h_{xx} + f_{xxx} + f_{xx\xi}E + E'f_{x\xi\xi}E + f_{\xi\xi\xi}(E,E,E) < 0, \tag{31}$$

where E and h_{xx} are given in Eqs. (27) and (28).

The stability criterion for the uncontrolled version of system (9a) and (9b) follows readily from Lemma 4.

Corollary 3. *Suppose Hypothesis 1B holds. Then the origin is asymptotically stable for Eqs. (9a) and (9b) (with $u = 0$) if A_{22} is stable, $f_{xx} = 0$ and $f_{xxx} - f_{x\xi}A_{22}^{-1}G_{xx} < 0$.*

In the rest of this section, we assume that the stability conditions given in Corollary 3 do not hold, and seek stabilizing control laws for system (9a) and (9b).

Linear stabilizing control laws follow readily from Lemma 4, and are as given next.

Proposition 1. *Suppose hypothesis 1B holds and let $M := (A_{22} + b_2K_2)^{-1}$. Then there is a purely linear feedback which asymptotically stabilizes the origin of* Eqs. (9a) and (9b) *if there exist feedback gains k_1 and K_2 for which $(A_{22} + b_2K_2)$ is stable,*

$$f_{xx} - k_1 f_{x\xi}Mb_2 + k_1^2 b_2'M'f_{\xi\xi}Mb_2 = 0, \tag{32}$$

$$\begin{aligned} & f_{xxx} - f_{x\xi}MG_{xx} + k_1\{f_{x\xi}MG_{x\xi} + 2G_{xx}'M'f_{\xi\xi}' - f_{xx\xi} - f_{xx}f_{x\xi}M\}Mb_2 \\ & + k_1^2\{b_2'M'f_{x\xi\xi}Mb_2 - f_{x\xi}MG_{\xi\xi}(Mb_2, Mb_2) - 2b_2'M'f_{\xi\xi}MG_{x\xi}Mb_2 \\ & + (f_{x\xi}Mb_2)^2 + 2f_{xx}(b_2'M')f_{\xi\xi}M^2b_2\} - k_1^3\{f_{\xi\xi\xi}(Mb_2, Mb_2, Mb_2) \\ & - 2b_2'M'f_{\xi\xi}MG_{\xi\xi}(Mb_2, Mb_2) + 3f_{x\xi}Mb_2(b_2'M'f_{\xi\xi}Mb_2)\} \\ & + 2k_1^4(b_2'Mf_{\xi\xi}Mb_2)^2 < 0. \end{aligned} \tag{33}$$

The linear stabilizing control rule proposed in Proposition 1 is a composite-type controller design. First, the feedback gain K_2 is chosen to stabilize state ξ. Then the remaining feedback gain k_1 is selected to satisfy the conditions (32) and (33) based on the chosen gain K_2.

According to the stability conditions given in Lemma 4, the cubic terms of both the function G and the control input u do not contribute to the stability criteria of system (9a) and (9b). A general linear-plus-quadratic feedback control law can then be abstracted as

$$u(x,\xi) = k_1x + K_2\xi + u_{xx}x^2 + xu_{x\xi}\xi + \xi'u_{\xi\xi}\xi, \tag{34}$$

while the control gains satisfying the conditions of Lemma 4.

From Lemmas 1 and 2 and the discussions above, we then have the next result.

Lemma 5. *Suppose A_{22} is stable and Hypothesis 1B holds. Then there exists no* purely quadratic *feedback stabilizer for the origin of system (9a) and (9b) if $f_{xx} \neq 0$. However, the origin of (9) is asymptotically stabilizable by a* purely *quadratic feedback of the form $u = u_{xx}x^2$ if $f_{xx} = 0$ and $f_{x\xi}A_{22}^{-1}b_2 \neq 0$.*

Note that the stabilization results given in Corollaries 1 and 2 and Lemma 5 agree with those obtained in Ref. [7].

5. Pair of pure imaginary eigenvalues

In this section, we consider system (9a) and (9b), specifically, of which A_{11} has a pair of pure imaginary eigenvalues and is in the form of Eq. (36) below.

First, however, consider the stability of a planar system

$$\dot{\eta} = A_{11}\eta + Q(\eta,\eta) + C(\eta,\eta,\eta) + \cdots, \tag{35}$$

where $\eta = (x,y)'$, and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 \\ -\Omega_2 & 0 \end{pmatrix} \tag{36}$$

with $\Omega_1\Omega_2 > 0$ and $Q(\eta,\eta)$ and $C(\eta,\eta,\eta)$ represent the quadratic and cubic terms, respectively. Without loss of generality, we may express $Q(\eta,\eta)$ and $C(\eta,\eta,\eta)$ in the form of

$$Q(\eta,\eta) = \begin{pmatrix} q_{11}x^2 + q_{12}xy + q_{13}y^2 \\ q_{21}x^2 + q_{22}xy + q_{23}y^2 \end{pmatrix}, \tag{37}$$

$$C(\eta,\eta,\eta) = \begin{pmatrix} c_{11}x^3 + c_{12}x^2y + c_{13}xy^2 + c_{14}y^3 \\ c_{21}x^3 + c_{22}x^2y + c_{23}xy^2 + c_{24}y^3 \end{pmatrix}, \tag{38}$$

respectively. Note the linearization of Eq. (35) at the origin has the pair of pure imaginary eigenvalues $\pm\mathrm{i}\sqrt{\Omega_1\Omega_2}$, where $\mathrm{i} = \sqrt{-1}$.

Applying a general stability criterion for planar systems undergoing Hopf bifurcation (see, e.g., Ref. [8]), we find that a sufficient condition for the stability of the origin for Eq. (35) is:

$$\frac{1}{8}\left\{ q_{22}\left(\frac{1}{\Omega_2}q_{21} + \frac{1}{\Omega_1}q_{23}\right) - q_{12}\left(\frac{1}{\Omega_1}q_{11} + \frac{\Omega_2}{\Omega_1^2}q_{13}\right) + \frac{2}{\Omega_2}q_{11}q_{21} - \frac{2\Omega_2}{\Omega_1^2}q_{13}q_{23} + 3\left(c_{11} + \frac{\Omega_2}{3\Omega_1}c_{13} + \frac{1}{3}c_{22} + \frac{\Omega_2}{\Omega_1}c_{24}\right)\right\} < 0. \tag{39}$$

In the following, we apply the stability criterion (39) to the design of stabilizing control laws for the more general (nonplanar) system (9a) and (9b) in which both $\eta = (x, y)'$ and $b_1 := (b_{11}, b_{12})'$ are two-dimensional vectors, and $F(\eta, \xi) = (f(x, y, \xi), g(x, y, \xi))'$.

Results obtained in this section will apply under either of the following two hypotheses.

Hypothesis 2A. The matrix A_{11} (appearing in Eqs. (9a) and (9b)) is a 2×2 matrix of the form (36) above, and the vector $b_1 \neq 0$.

Hypothesis 2B. The matrix A_{11} (appearing in Eqs. (9a) and (9b)) is a 2×2 matrix of the form (36) above, and the vector $b_1 = 0$.

5.1. The case of $b_1 \neq 0$

First, we consider the case of which at least one of b_{11} and b_{12} is nonzero. Although this assumption guarantees the controllability of the subsystem (9a) and (9b), here we focus on the design of purely nonlinear control laws only. Suppose A_{22} is stable and the control input $u = U(x, y, \xi)$ is a smooth and purely nonlinear function. According to the discussions in Section 3, the stability of the origin of Eqs. (9a) and (9b) coincides with the stability of the origin of the reduced model:

$$\dot{x} = \Omega_1 y + b_{11} U(x, y, h(x, y)) + f(x, y, h(x, y)), \tag{40}$$

$$\dot{y} = -\Omega_2 x + b_{12} U(x, y, h(x, y)) + g(x, y, h(x, y)), \tag{41}$$

where h solves Eq. (10) with η replaced by $(x, y)'$ and with boundary conditions $h(0) = 0$ and $\mathrm{D}h(0) = 0$. Indeed, h takes the form

$$h(x, y) = x^2 h_{xx} + xy h_{xy} + y^2 h_{yy} + \mathrm{O}(\|(x, y)\|^3), \tag{42}$$

where h_{xx}, h_{xy}, h_{yy} are constant vectors.

In the following, we restrict the nonlinear control function U to be a function of x and y only, as follows:

$$U(x, y, \xi) = u_{xx} x^2 + u_{xy} xy + u_{yy} y^2 + u_{xxx} x^3 + u_{xxy} x^2 y + u_{xyy} xy^2 + u_{yyy} y^3. \tag{43}$$

A stability criterion for the control system (9a) and (9b) in this case is given next.

Lemma 6. *Suppose A_{22} is stable and Hypothesis 2A holds. Then the origin is asymptotically stable for Eqs. (9a) and (9b) if*

$$
\begin{aligned}
&(b_{12}u_{xy}+g_{xy})\left\{\frac{1}{\Omega_2}(b_{12}u_{xx}+g_{xx})+\frac{1}{\Omega_1}(b_{12}u_{yy}+g_{yy})\right\} \\
&-(b_{12}u_{xy}+f_{xy})\left\{\frac{1}{\Omega_1}(b_{11}u_{xx}+f_{xx})+\frac{\Omega_2}{\Omega_1^2}(b_{11}u_{yy}+f_{yy})\right\} \\
&+\frac{2}{\Omega_2}(b_{11}u_{xx}+f_{xx})(b_{12}u_{xx}+g_{xx})-\frac{2\Omega_2}{\Omega_1^2}(b_{11}u_{yy}+f_{yy})(b_{12}u_{yy}+g_{yy}) \\
&+3\left\{b_{11}u_{xxx}+f_{xxx}+f_{x\xi}h_{xx}+\frac{\Omega_2}{3\Omega_1}(b_{11}u_{xyy}+f_{xyy}+f_{x\xi}h_{yy}+f_{y\xi}h_{xy})\right. \\
&\left.+\frac{1}{3}(b_{12}u_{xxy}+g_{xxy}+g_{x\xi}h_{xy}+g_{y\xi}h_{xx})+\frac{\Omega_2}{\Omega_1}(b_{12}u_{yyy}+g_{yyy}+g_{y\xi}h_{yy})\right\}<0,
\end{aligned} \tag{44}
$$

where

$$
\begin{aligned}
h_{xy} &= \{A_{22}^2+4\Omega_1\Omega_2 I\}^{-1}\{2\Omega_2(u_{yy}b_2+G_{yy})-2\Omega_1(u_{xx}b_2+G_{xx}) \\
&\quad -A_{22}(u_{xy}b_2+G_{xy})\}, &(45)\\
h_{xx} &= -A_{22}^{-1}(u_{xx}b_2+G_{xx}+\Omega_2 h_{xy}), &(46)\\
h_{yy} &= -A_{22}^{-1}(u_{yy}b_2+G_{yy}-\Omega_1 h_{xy}). &(47)
\end{aligned}
$$

It is observed from Lemma 6, generically there exists a quadratic-plus- cubic feedback stabilizer for system (9a) and (9b). In addition, a purely quadratic state feedback stabilizing control law and a purely cubic state feedback stabilizing control law follow readily from Lemma 6 as given in the next two corollaries.

Corollary 4. *Let A_{22} be stable and Hypothesis 2A hold. Then the origin of system (9a) and (9b) is stabilizable by a* purely quadratic *state feedback of the form $u = u_{xy}xy$ if*

$$
\begin{aligned}
&b_{12}\left\{\frac{1}{\Omega_1}(g_{yy}-f_{xx})+\frac{1}{\Omega_2}g_{xx}-\frac{\Omega_2}{\Omega_1^2}f_{yy}\right\} \\
&-\frac{1}{3}\left\{\Omega_2(2g_{y\xi}-8f_{x\xi})A_{22}^{-1}+\frac{\Omega_2}{\Omega_1}f_{y\xi}+g_{x\xi}\right\}(A_{22}^2+4\Omega_1\Omega_2 I)^{-1}A_{22}b_2\neq 0.
\end{aligned} \tag{48}
$$

Corollary 5. *Let A_{22} be stable and Hypothesis 2A hold. Then the origin of system (9a) and (9b) is stabilizable by a* purely cubic *state feedback of the form $u = u_{xxx}x^3 + u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3$.*

The results of Corollary 5 agrees with the one in Theorem 1 of Ref. [5].

5.2. The case of $b_l = 0$

Next, we consider the case of which Hypothesis 2B holds, i.e., b_1 is a zero vector and A_{11} is as in Eq. (36). Let the control input be of the form

$$u = k_{11}x + k_{12}y + K_2\xi + U(x, y, \xi), \tag{49}$$

where U is defined in Eq. (43).

Suppose $A_{22} + b_2K_2$ is stable. Then from Section 3, the stability of the origin of Eqs. (9a) and (9b) agrees with the stability of the origin of the reduced model:

$$\dot{x} = \Omega_1 y + f(x, y, E_1x + E_2y + h(x, y)), \tag{50}$$

$$\dot{y} = -\Omega_2 x + g(x, y, E_1x + E_2y + h(x, y)). \tag{51}$$

Here, $E = (E_1, E_2)$ and $h(x, y)$ are the solutions of Eqs. (15) and (17), respectively, with $K_1 = (k_{11}, k_{12})$.

Let

$$\begin{aligned} H(x,y) := {} & b_2U(x,y,E_1x+E_2y) + G(x,y,E_1x+E_2y) \\ & - f(x,y,E_1x+E_2y)E_1 - g(x,y,E_1x+E_2y)E_2 \\ = {} & x^2H_{xx} + xyH_{xy} + y^2H_{yy} + \mathrm{O}(\|(x,y)\|^3). \end{aligned} \tag{52}$$

Similarly, we take h to be of the form (42). Solving Eqs. (15) and (17), we have

$$E_1 = -\{(A_{22}+b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{11}(A_{22}+b_2K_2) - \Omega_2 k_{12} I\}b_2 \tag{53}$$

$$E_2 = -\{(A_{22}+b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{12}(A_{22}+b_2K_2) + \Omega_1 k_{11} I\}b_2 \tag{54}$$

and

$$h_{xy} = \{(A_{22}+b_2K_2)^2 + 4\Omega_1\Omega_2 I\}^{-1}\{2\Omega_2(H_{yy} - 2\Omega_1 H_{xx} - (A_{22}+b_2K_2)H_{xy}\}, \tag{55}$$

$$h_{xx} = -(A_{22}+b_2K_2)^{-1}(H_{xx} + \Omega_2 h_{xy}), \tag{56}$$

$$h_{yy} = -(A_{22}+b_2K_2)^{-1}(H_{yy} - \Omega_1 h_{xy}). \tag{57}$$

Note that, matrices $(A_{22}+b_2K_2)^2 + \Omega_1\Omega_2 I$ and $(A_{22}+b_2K_2)^2 + 4\Omega_1\Omega_2 I$ are both invertible since the matrix $(A_{22}+b_2K_2)$ is stable.

The reduced model (50) and (51) is hence obtained as

$$\begin{aligned} \dot{x} = {} & \Omega_1 y + \hat{f}_{xx}x^2 + \hat{f}_{xy}xy + \hat{f}_{yy}y^2 + \hat{f}_{xxx}x^3 + \hat{f}_{xxy}x^2y + \hat{f}_{xyy}xy^2 \\ & + \hat{f}_{yyy}y^3 + \mathrm{O}(\|(x,y)\|^4) \end{aligned} \tag{58}$$

$$\begin{aligned} \dot{y} = {} & -\Omega_2 x + \hat{g}_{xx}x^2 + \hat{g}_{xy}xy + \hat{g}_{yy}y^2 + \hat{g}_{xxx}x^3 + \hat{g}_{xxy}x^2y + \hat{g}_{xyy}xy^2 \\ & + \hat{g}_{yyy}y^3 + \mathrm{O}(\|(x,y)\|^4). \end{aligned} \tag{59}$$

Here $\hat{f}_{ij}$, $\hat{g}_{ij}$, $\hat{f}_{ijk}$ and $\hat{g}_{ijk}$, $i,j,k \in \{x,y,z\}$, denote the controlled version of the quadratic terms and cubic terms, respectively. The values of which are given in Appendix A.

Referring to the stability criterion (39) and the preceding discussions, we summarize the stabilizability conditions for system (9a) and (9b) as below.

Lemma 7. *Suppose Hypothesis 2B holds and that the control input is in the form of Eq. (49). Then the origin of Eqs. (9a) and (9b) is asymptotically stable if* $A_{22} + b_2K_2$ *is stable and*

$$\hat{g}_{xy}\left(\frac{1}{\Omega_2}\hat{g}_{xx} + \frac{1}{\Omega_1}\hat{g}_{yy}\right) - \hat{f}_{xy}\left(\frac{1}{\Omega_1}\hat{f}_{xx} + \frac{\Omega_2}{\Omega_1^2}\hat{f}_{yy}\right) + \frac{2}{\Omega_2}\hat{f}_{xx}\hat{g}_{xx} - \frac{2\Omega_2}{\Omega_1^2}\hat{f}_{yy}\hat{g}_{yy} + 3\left(\hat{f}_{xxx} + \frac{\Omega_2}{3\Omega_1}\hat{f}_{xyy} + \frac{1}{3}\hat{g}_{xxy} + \frac{\Omega_2}{\Omega_1}\hat{g}_{yyy}\right) < 0. \tag{60}$$

Note that, it is observed from Eq. (60) and Appendix A that only quadratic terms of the function G, and the linear and quadratic terms of the control input u contribute to the stability conditions. A linear and/or quadratic feedback stabilizing control law readily follows from Lemma 7. Moreover, a stability criterion for the uncontrolled version of system (9a) and (9b) is also implied by Lemma 7 by letting $u = 0$.

Although Lemma 7 addresses the design of a linear feedback stabilizing control law, such a linear stabilizing control law may not exist. In the next result, we consider a special case of which the noncritical state ξ of system (9a) and (9b) is a scalar. Since ξ is a scalar, as observed from Eqs. (53) and (54), we always have solutions for the control gains k_{11} and k_{12} for arbitrary given values of E_1, E_2 and K_2. According to the formulations as in Appendix A, we can select $E_1 = 0$ and E_2 large enough ($E_2 = 0$ and E_1 large enough) such that the condition (60) in Lemma 7 holds while $g_{\xi\xi\xi} < 0$ ($f_{\xi\xi\xi} < 0$). We have the next result.

Corollary 6. *Suppose the noncritical state* ξ *is a scalar and Hypothesis 2B holds. Then there is a* purely linear *feedback which asymptotically stabilizes the origin of Eqs. (9a) and (9b) if either* $f_{\xi\xi\xi} < 0$ *or* $g_{\xi\xi\xi} < 0$.

Referring to Eqs. (52)–(54), for the general case of which the state ξ of system (9a) and (9b) may not be a scalar, we have $H(x,y) = b_2U(x,y,0) + G(x,y,0)$ while $k_{11} = k_{12} = 0$. A purely quadratic stabilizing control law is then obtained as follows.

Corollary 7. *Suppose* A_{22} *is stable and Hypothesis 2B holds. Then a* purely quadratic *stabilizing feedback in the form of* $u = u_{xx}x^2 + u_{xy}xy + u_{yy}y^2$ *exists* *for the origin of Eqs. (9a) and (9b) if one of the following three conditions holds:*

1. $M_0 A_{22} b_2 \neq 0$, *or*
2. $\{(3f_{x\xi} + \frac{1}{3}g_{y\xi})A_{22}^{-1} + 2\Omega_1 M_0\}b_2 \neq 0$, *or*
3. $\{2\Omega_2 M_0 - \frac{\Omega_2}{\Omega_1}(\frac{1}{3} + f_{x\xi} + g_{y\xi})A_{22}^{-1}\}b_2 \neq 0$, *where*

$$M_0 = \frac{1}{3}\left\{\Omega_2(2g_{y\xi} - 8f_{x\xi})A_{22}^{-1} + \frac{\Omega_2}{\Omega_1}f_{y\xi} + g_{x\xi}\right\}(A_{22}^2 + 4\Omega_1\Omega_2 I)^{-1}. \tag{61}$$

For the case of which A_{22} is not stable, an additional linear feedback $K_2\xi$ is needed to ensure the existence of a locally invariant manifold and the stability of the Jacobian matrix of (9b). Then Corollary 7 can be applied. We note that Aeyel's stabilization conditions for a third-order system [1] are special cases of those given in Corollary 7. Moreover, similar results for quadratic feedback stabilization of Eqs. (9a) and (9b) were obtained by Abed and Fu [6], where an asymptotic expansion method based on bifurcation analysis is used for controller design.

6. Conclusions

In this paper, the center manifold reduction technique has been proposed for the design of smooth feedback stabilization of nonlinear systems in critical cases. The stabilizing control laws involving a two step composite-type design were also obtained for two critical cases. Linear stability of the noncritical state ξ is first ensured, then the remaining control gains are chosen to stabilize the origin of the reduced model whose eigenvalues all lie on the imaginary axis. Stabilizing control laws have been designed in linear and/or nonlinear feedback forms. The stabilization conditions of the overall system are explicitly expressed in terms of system states, which will make the applications easy.

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Appendix A

The coefficients in the Taylor expansions of $\hat{f}$, $\hat{g}$ are given below in terms of those of f, g. Here, ρ denotes either f or g, and $i \neq j$ for $i, j \in \{x, y\}$ with $E_{[x]} = E_1$, and $E_{[y]} = E_2$.

$$\hat{\rho}_{ii} = \rho_{ii} + \rho_{i\xi}E_{[i]} + E'_{[i]}\rho_{\xi\xi}E_{[i]},$$

$$\hat{\rho}_{ij} = \rho_{ij} + \rho_{i\xi}E_{[j]} + \rho_{j\xi}E_{[i]} + 2E'_{[i]}\rho_{\xi\xi}E_{[j]},$$

$$\hat{\rho}_{iii} = \rho_{iii} + \rho_{ii\xi}E_{[i]} + E'_{[i]}\rho_{i\xi\xi}E_{[i]} + \rho_{\xi\xi\xi}(E_{[i]}, E_{[i]}E_{[i]}) + \rho_{i\xi}h_{ii} + 2E'_{[i]}\rho_{\xi\xi}h_{ii},$$

$$\hat{\rho}_{iij} = \rho_{j\xi}h_{ii} + \rho_{i\xi}h_{ij} + 2E'_{[j]}\rho_{\xi\xi}h_{ii} + 2E'_{[i]}\rho_{\xi\xi}h_{ij} + \rho_{iij} + \rho_{ij\xi}E_{[i]} + \rho_{ii\xi}E_{[j]} + E'_{[i]}\rho_{j\xi\xi}E_{[i]} + 2E'_{[i]}\rho_{i\xi\xi}E_{[j]} + 3\rho_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[j]}).$$

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