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The Shifting of Movable Eigenvalues in Uncontrollable Singular Systems

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ABSTRACT: *A feedback design method for an uncontrollable singular system is proposed by which a set of possible state feedback gains can be designed to shift certain eigenvalues to other real or complex scalars. The method can be applied to single- or multi-input systems, and if so the method obtains non-unique possible feedback gains. If the original system is regular, real and impulse free, the closed-loop system also preserves these properties.*

I. Introduction

Singular systems, also called descriptor systems, generalized systems, semi-state systems or implicit systems, can capture the dynamic behavior of many physical systems; therefore, they have attracted the attention of many researchers in recent decades (**1–13**). For a survey of singular systems, see Ref. (**1**) or (**2**).

The state feedback design for singular systems has been studied by many researchers (**3–10**). However, most previous studies have focused on controllable systems. It has been proven (**1–3**) that if a singular system is controllable, then a state feedback exists that can arbitrarily shift all (finite) eigenvalues to new locations and eliminate all impulsive modes. On the other hand, if a singular system is uncontrollable, some eigenvalues or impulsive modes cannot be shifted; however, it may still be possible to shift other eigenvalues.

Recently, Tornambe (**10**) proposed a method for dealing with the feedback design problem in uncontrollable systems, by which a state feedback can be designed to shift those eigenvalues that can be shifted to other places, and if the system is impulse free, to preserve this property in the closed-loop system. However, there are some restrictions on his approach. Firstly, real and complex eigenvalues can only be shifted to real scalars. Secondly, if some non-real eigenvalues must be shifted, the method can be

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applied only to single-input systems. Thirdly, only one solution can be found, no matter if it is a single- or multi-input system.

This paper proposes a more generalized method for feedback design for uncontrollable systems. If some eigenvalues can be shifted, a set of possible state feedback gains can be designed to shift them to other real or complex scalars. The method can be applied to multi-input systems, even when there are real or complex eigenvalues to be shifted. Also, the possible feedback gains obtained by this method are not unique for multi-input systems, providing a greater degree of freedom for control purposes. If the original system is regular, real and impulse free, these properties are preserved in the closed-loop system.

The organization of the paper is as follows. In Section II, some preliminary results are discussed. In Section III, the feedback gain design methodology for shifting eigenvalues is presented. An example showing design performance is illustrated in Section IV. Section V concludes the paper.

II. Some Preliminary Results

Consider the following linear time-invariant singular system:

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ and E may be singular. If a linear state feedback law defined by

$$u(t) = Kx(t) + r(t), \tag{2}$$

where $K \in R^{m\times n}$ and $r(t) \in R^n$, is applied to Eq. (1), the closed-loop system has the following form

$$E\dot{x}(t) = (A + BK)x(t) + Br(t). \tag{3}$$

Let

$$G_{E,A}(k) = \begin{bmatrix} E & & & \\ A & \ddots & & \\ & \ddots & E & \\ & & A & \end{bmatrix}, k = 1, 2, \cdots, \text{and } L_{E,A} = \begin{bmatrix} E & 0 \\ A & E \end{bmatrix}$$

We can then obtain the following lemma.

Lemma 1

(a) System Eq. (1) is regular if and only if rank $G_{E,A}(k) = nk$, $k = 1, 2, \cdots$.

(b) System Eq. (1) is impulse free if and only if rank $L_{E,A} = n + \text{rank } E$.

Proof: For a proof of point (a) see Ref. (**1**) or (**11**), and for a proof of point (b) see Ref. (**10**). ■

The following lemma concerns the properties of eigenvalues and eigenvectors in singular systems and the proof can be found in Ref. (**10**).

Lemma 2

Let h_i be a left eigenvector of an eigenvalue λ_i in system Eq. (1).

(a) If $h_i^{\mathrm{T}}B \neq 0$, then for all $\gamma_i \in C$, we can find a feedback (Eq. (2)) that shifts the eigenvalue λ_i to a new location γ_i and leaves the other eigenvalues in Eq. (1) unchanged.

(b) If $h_i^{\mathrm{T}}B = 0$, then for any feedback (Eq. (2)), λ_i is an eigenvalue of the closed-loop system.

In light of Lemma 2, the following definition can be given.

Definition 1

Let λ_i be an eigenvalue in Eq. (1) and h_i its left eigenvector. If $h_i^{\mathrm{T}}B \neq 0$, we call λ_i a *movable* eigenvalue; otherwise, we call λ_i an *immovable* eigenvalue. If all the immovable eigenvalues are stable (the real parts are negative), then the system is called *stabilizable*.

The following lemma discusses the properties of movable eigenvalues.

Lemma 3

Let h_1 and $h_2 \in C^n$ be left eigenvectors of two distinct movable eigenvalues λ_1 and λ_2 in Eq. (1), and γ_1, $\gamma_2 \in C$ two other distinct complex scalars. Consider the following matrix

$$\begin{bmatrix} \dfrac{h_1^{\mathrm{T}}Bf_1}{(\lambda_1-\gamma_1)} & \dfrac{h_1^{\mathrm{T}}Bf_2}{(\lambda_1-\gamma_2)} \\ \dfrac{h_2^{\mathrm{T}}Bf_1}{(\lambda_2-\gamma_1)} & \dfrac{h_2^{\mathrm{T}}Bf_2}{(\lambda_2-\gamma_2)} \end{bmatrix}. \tag{4}$$

(a) We can find f_1 and $f_2 \in R^m$ such that the matrix in Eq. (4) is non-singular.

(b) We can find f_1 and $f_2 \in C^m$, where $f_2 = \bar{f}_1$, such that the matrix in Eq. (4) is non-singular.

Proof:

(a) If, for all f_1 and $f_2 \in R^m$, matrix Eq. (4) is singular, then, by evaluating its determinant, we can obtain

$$f_1^{\mathrm{T}}\left(\frac{(h_1^{\mathrm{T}}B)^{\mathrm{T}}(h_2^{\mathrm{T}}B)}{(\lambda_1-\gamma_1)(\lambda_2-\gamma_2)} - \frac{(h_2^{\mathrm{T}}B)^{\mathrm{T}}(h_1^{\mathrm{T}}B)}{(\lambda_2-\gamma_1)(\lambda_1-\gamma_2)}\right)f_2 = 0, \forall f_1, f_2 \in R^{m\times 1}. \tag{5}$$

Thus, the matrix between f_1^{T} and f_2 in Eq. (5) is a zero matrix, and so is its transpose. Adding the matrix and its transpose, we can obtain

$$\frac{((\lambda_2-\gamma_1)(\lambda_1-\gamma_2)-(\lambda_1-\gamma_1)(\lambda_2-\gamma_2))((h_1^{\mathrm{T}}B)^{\mathrm{T}}(h_2^{\mathrm{T}}B)+(h_2^{\mathrm{T}}B)^{\mathrm{T}}(h_1^{\mathrm{T}}B))}{(\lambda_1-\gamma_1)(\lambda_2-\gamma_2)(\lambda_2-\gamma_1)(\lambda_1-\gamma_2)} = 0.$$

Since, $(\lambda_2-\gamma_1)(\lambda_1-\gamma_2)-(\lambda_1-\gamma_1)(\lambda_2-\gamma_2) = (\lambda_1-\lambda_2)(\gamma_1-\gamma_2) \neq 0$, we obtain

$$(h_1^{\mathrm{T}}B)^{\mathrm{T}}(h_2^{\mathrm{T}}B)+(h_2^{\mathrm{T}}B)^{\mathrm{T}}(h_1^{\mathrm{T}}B) = 0.$$

If the relation is left-multiplied by $(h_1^{\mathrm{T}}B)$ and right-multiplied by $(h_2^{\mathrm{T}}B)^{\mathrm{T}}$, then

$$\|h_1^{\mathrm{T}}B\|_2^2\|h_2^{\mathrm{T}}B\|_2^2+((h_1^{\mathrm{T}}B)(h_2^{\mathrm{T}}B))^2 = 0,$$

where $\| \|_2$ represents the 2-norm. Therefore, $h_1^T B = 0$ or $h_2^T B = 0$. λ_1 or λ_2 is not movable, a contradiction.

(b) This can be proven using methods similar to those used for the proof of point (a). ■

III. Shifting Movable Eigenvalues

We now state the main problem dealt with in this paper.

Consider a real, regular, impulse free and stabilizable uncontrollable singular system Eq. (1). Let $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ be a set of distinct complex scalars chosen from the movable eigenvalues in Eq. (1), and let $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_p\}$ be another set of distinct complex scalars that satisfy $\Lambda \cap \Gamma = \emptyset$. We want to find possible feedback gains $K \in R^{m \times n}$ that shift the movable eigenvalues in Λ to elements in Γ while leaving the other eigenvalues in Eq. (1) unchanged. Furthermore, the resulting closed-loop system Eq. (3) must also be real, regular and impulse free.

If an eigenvalue is movable, it can be shifted to any other complex scalar. However, if the closed-loop system must be real, the new eigenvalues chosen must be real or form self-conjugate pairs. Therefore, eigenvalues cannot in fact be shifted arbitrarily. In most practical applications, it is important to require that the system be real. When that is so, shifting a non-real movable eigenvalue to a real scalar also entails shifting its conjugate to a real scalar, and if a real movable eigenvalue is shifted to a non-real scalar, we must also shift another real movable eigenvalue to the conjugate of the non-real scalar. If there is no other real movable eigenvalue, the movement cannot be performed. This consideration motivated us to define the following four types of possible shift pairs.

1. An ordered pair $(\{\lambda\}, \{\gamma\})$ in which $\lambda, \gamma \in R$ is called a possible shift pair of the first type; we denote the set of all such ordered pairs as S_1.
2. An ordered pair $(\{\lambda, \bar{\lambda}\}, \{\gamma_1, \gamma_2\})$ in which $\gamma_1, \gamma_2 \in R$, $\lambda \in C$ and $\lambda \neq \bar{\lambda}$, is called a possible shift pair of the second type; we denote the set of all such ordered pairs as S_2.
3. An ordered pair $(\{\lambda_1, \lambda_2\}, \{\gamma, \bar{\gamma}\})$ in which $\lambda_1, \lambda_2 \in R$, $\gamma \in C$, $\gamma \neq \bar{\gamma}$, is called a possible shift pair of the third type; we denote the set of all such ordered pairs as S_3.
4. An ordered pair $(\{\lambda, \bar{\lambda}\}, \{\gamma, \bar{\gamma}\})$ in which $\lambda, \gamma \in C$, $\lambda \neq \bar{\lambda}$ and $\gamma \neq \bar{\gamma}$, is called a possible shift pair of the fourth type; we denote the set of all such ordered pairs as S_4.

Definition 2

For the two sets Λ and Γ defined above, if we can find disjoint decompositions $\Lambda = \bigcup_{i=1}^{q} \Lambda_i$ and $\Gamma = \bigcup_{i=1}^{q} \Gamma_i$ for which $\Lambda_i \cap \Lambda_j = \emptyset$, $\Gamma_i \cap \Gamma_j = \emptyset$, if $i \neq j$, and the ordered pair (Λ_i, Γ_i), $i \in \{1, \cdots, q\}$, is included in S_j, $j = 1,2,3,4$, then we call the order pair (Λ, Γ) a possible shift pair.

This yields the following lemma.

Lemma 4

Let system Eq. (1) be real. If we can find a feedback gain K that moves the set of movable eigenvalues in Λ to elements in Γ and makes the closed-loop system real, then (Λ, Γ) must be a possible shift pair.

In what follows, we first discuss the design of the feedback gains for the four special cases in which (Λ,Γ) is included in S_j, $j = 1,2,3,4$, then discuss the design of feedback gains for the general case in which (Λ,Γ) is a possible shift pair.

3.1. *Shifting one real movable eigenvalue*

Consider the case in which $(\Lambda,\Gamma)\in S_1$. $\Lambda = \{\lambda\}$ and $\Gamma = \{\gamma\}$, where $\lambda,\gamma\in R$. Denote the left eigenvector of λ as h. Since λ is movable, we can find an $f\in R^m$ such that $h^{\mathrm{T}}Bf\neq 0$. Let

$$t = \frac{(\gamma-\lambda)}{h^{\mathrm{T}}Bf}. \tag{6}$$

The feedback gain can then be obtained from

$$K = tfh^{\mathrm{T}}E. \tag{7}$$

Theorem I

If all possible feedback gains satisfying Eqs (6) and (7) are applied to Eq. (1), then the closed-loop system Eq. (3) will have the following properties.

(a) If Eq. (1) is real, then Eq. (3) will also be real.
(b) If Eq. (1) is regular and impulse free, then Eq. (3) will also be regular and impulse free.
(c) The feedback gains will shift the eigenvalue λ to γ, and leave the other eigenvalues of Eq. (1) unchanged.

Proof: Here we prove only the preservation of regularity. Other proofs can be obtained using methods similar to those used in Ref. (**10**). Let $Q(k) = -Btf\lambda^{k-1}h^{\mathrm{T}}\in R^{n\times n}$, $k = 1,2,\cdots$, and

$$R(k) = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ -Q(1) & I & \cdots & 0 & 0 \\ Q(2) & -Q(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^k Q(k) & (-1)^{k-1}Q(k-1) & \cdots & -Q(1) & I \end{bmatrix} \in R^{(k+1)n\times(k+1)n}, k = 1,2,\cdots. \tag{8}$$

It can be seen that $G_{E,A}+BK(k) = R(k)G_{E,K}(k)$. Thus, rank $G_{E,A+BK}(k) =$ rank $G_{E,A}(k)$ and, according to Lemma 1, if Eq. (1) is regular, then Eq. (3) must also be regular. ■

3.2. *Shifting two movable eigenvalues*

Consider the cases in which $(\Lambda,\Gamma)\in S_2,S_3$ or S_4. There are two distinct movable eigenvalues, λ_1 and λ_2, which will be moved to two distinct new ones, γ_1 and γ_2. Denote the left eigenvectors of λ_1 and λ_2 as h_1 and h_2, respectively. Then, according to Lemma 3, we can find (a) f_1 and $f_2\in R^{m\times 1}$ or (b) f_1 and $f_2\in C^{m\times 1}$, where $f_2 = \bar{f}_1$, such that the matrix

$$\begin{bmatrix} \dfrac{h_1^T B f_1}{(\lambda_1-\gamma_1)} & \dfrac{h_1^T B f_2}{(\lambda_1-\gamma_2)} \\ \dfrac{h_2^T B f_1}{(\lambda_2-\gamma_1)} & \dfrac{h_2^T B f_2}{(\lambda_2-\gamma_2)} \end{bmatrix} \tag{9}$$

is non-singular. If the feedback gain is chosen as follows:

$$K = -\begin{bmatrix} f_1 \; f_2 \end{bmatrix} T^{-1} \begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix} E \tag{10}$$

then we obtain following theorem.

Theorem II

If all possible feedback gains satisfying Eqs (9) and (10) are applied to Eq. (1), then the resulting closed-loop system Eq. (3) will have the following properties.

(a) If Eq. (1) is regular and impulse free, then Eq. (3) will also be regular and impulse free.

(b) The feedback gains will shift the eigenvalues $\{\lambda_1,\lambda_2\}$ to $\{\gamma_1,\gamma_2\}$, and leave the other eigenvalues in Eq. (1) unchanged.

(c) (1) In the case in which $(\{\lambda_1,\lambda_2\},\{\gamma_1,\gamma_2\}) \in S_2$, if $f_1, f_2 \in R^m$ then K will be real.
(2) In the case in which $(\{\lambda_1,\lambda_2\},\{\gamma_1,\gamma_2\}) \in S_3$, if $f_2 = \bar{f}_1$, then K will be real.
(3) In the case in which $(\{\lambda_1,\lambda_2\},\{\gamma_1,\gamma_2\}) \in S_4$, if $f_2 = \bar{f}_1$, then K will be real.

Proof:

(a) Let

$$Q(k) = B\begin{bmatrix} f_1 \; f_2 \end{bmatrix} T^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{k-1} \begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix} \in R^{n\times n}, k = 1,2, \cdots,$$

and $R(k) \in R^{(k+1)n\times(k+1)n}$, $k = 1,2, \cdots$ be as defined in Eq. (8). It can be shown that $G_{E,A+BK}(k) = R(k)G_{E,A}(k)$. Thus, rank $G_{E,A+BK}(k)$ = rank $G_{E,A}(k)$ and, according to Lemma 1, if Eq. (1) is regular, then Eq. (3) must also be regular. It can be seen that $L_{E,A+BK} = R(1)L_{E,A}$. Thus, rank $L_{E,A+BK}$ = rank $L_{E,A}$ and, according to Lemma 1, if Eq. (1) is impulse free, then Eq. (3) must also be impulse free.

(b) It can be shown that

$$T^{-1}\begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix}(A+BK) = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T^{-1} \begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix} E. \tag{11}$$

Thus, γ_1 and γ_2 are eigenvalues of Eq. (3). Assume that v_j is a right eigenvector of λ_j, where $\lambda_j \neq \lambda_1,\lambda_2$; this means $(A-\lambda_j I)v_j = 0$. It can be seen that $Kv_j = 0$. Thus, $(A+BK-\lambda_j I)v_j = 0$ and λ_j is still an eigenvalue of Eq. (3).

(c) In all three case, it can be shown that $K = \bar{K}$, and thus, K is also real. ■

3.3. *General case*

If we want to shift the movable eigenvalues in Λ to elements in Γ and the closed-loop system is real, then, according to Lemma 4, (Λ,Γ) must be a possible shift pair.

From this fact and the preceding discussion, we may conclude that the feedback K can be obtained using the following algorithm.

Algorithm for shifting movable eigenvalues

Step 1: Find disjoint decompositions $\Lambda = \bigcup_{i=1}^{q} \Lambda_i$ and $\Gamma = \bigcup_{i=1}^{q} \Gamma_i$, for which $\Lambda_i \cap \Lambda_j = \varnothing$, $\Gamma_i \cap \Gamma_j = \varnothing$, $i \neq j$, and the ordered pair (Λ_i, Γ_i), $i \in \{1, \cdots, q\}$ is included in S_j, $j = 1,2,3,4$.

Step 2: Let $i = 1$ and $A_0 = A$.

Step 3: Design the feedback gain K_i for the system with system matrices E, A_{i-1}, B in order to shift the movable eigenvalues in Λ_i to elements in Γ_i. If $(\Lambda_i, \Gamma_i) \in S_1$, the feedback gain K_i can be designed using Eqs (6) and (7). If $(\Lambda_i, \Gamma_i) \in S_2, S_3$, or S_4, the feedback gain K_i can be designed using Eqs (9) and (10).

Step 4: Let $A_i = A_{i-1} + BK_i$. If $i = q$, go to Step 5; otherwise, let $i = i + 1$ and go to Step 3.

Step 5: The feedback gain K for shifting the movable eigenvalues in Λ to elements in Γ can be obtained from

$$K = \sum_{i=1}^{q} K_i.$$

The algorithm presented above and the feedback design methods for the four special cases lead to the main theorem of this paper.

Theorem III

If all possible feedback gains obtained using the algorithm presented above are applied to Eq. (1), then the resulting closed-loop system Eq. (3) will have the following properties.

(a) If Eq. (1) is real, then Eq. (3) will also be real.

(b) If Eq. (1) is regular and impulse free, then Eq. (3) will also be regular and impulse free.

(c) The feedback gains will shift the movable eigenvalues in Λ to elements in Γ, and leave the other eigenvalues in Eq. (1) unchanged. ■

IV. Example

Consider the following real system.

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to find

$$\det(A-\lambda E) = -(\lambda+1)(\lambda^2+1) \text{ and rank } L_{E,A} = 4+\text{rank } E.$$

Hence, the system is regular and impulse free.

The system has three eigenvalues $-1, j, -j$. The left eigenvector of -1 is $h_1 = [0\ 0\ 0\ 1]^T$. Since $h_1^T B = 0$, -1 is an immovable eigenvalue and the system is an uncontrollable system. The left eigenvector of j is $h = [-j\ 1\ \ -1+j\ 1-j]^T$. Since $h^T B = [-1\ j] \neq 0$, j is a movable eigenvalue. This means $-j$ is also a movable eigenvalue and its left eigenvector is $\bar{h}$. We want to design a feedback gain K to move the eigenvalues $\{j,-j\}$ to $\{-1+j,-1-j\}$. It can be seen that $(\{j,-j\},\{-1+j,-1-j\}) \in S_4$. Let $f = [1\ j]^T$; then, from Eqs (9) and (10), we obtain

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The feedback gain is real. Furthermore, we have

$$\det(A+BK-\lambda E) = -(\lambda+1)(\lambda^2+2\lambda+1) \text{ and rank } L_{E,A+BK} = 4+\text{rank } E.$$

Hence, the closed-loop system is also regular and impulse free.

V. Conclusion

In this paper, we propose a feedback design method for uncontrollable singular systems. Four special types of shift pairs are defined according to the properties of the eigenvalues as candidates for shifting, and possible feedback gains are given for each type. If the resulting closed-loop system must be real, the shifting pair can be decomposed as combinations of the four special types of possible shift pairs. Therefore, the overall feedback gain can be obtained by iterating the design of each special shift pair. Tornambe's approach (**10**) proposes only feedback gains for shifting one eigenvalue. Thus, eigenvalues can only be shifted to real scalars, and if some eigenvalues to be shifted are non-real, the approach can only be applied to single-input systems. Our approach considers feedback gains for shifting two eigenvalues. Therefore, the eigenvalues can be shifted to other real or complex scalars and the approach can be applied to multi-input systems. Also, if the system has multiple inputs, the solutions our approach yields are not unique. This provides a greater degree of control freedom.

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