A Two-Stage Least-Squares Finite Element Method for the Stress-Pressure-Displacement Elasticity Equations*

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A new stress-pressure-displacement formulation for the planar elasticity equations is proposed by introducing the auxiliary variables, stresses, and pressure. The resulting first-order system involves a nonnegative parameter that measures the material compressibility for the elastic body. A two-stage least-squares finite element procedure is introduced for approximating the solution to this system with appropriate boundary conditions. It is shown that the two-stage least-squares scheme is stable and, with respect to the order of approximation for smooth exact solutions, the rates of convergence of the approximations for all the unknowns are optimal both in the H^1-norm and in the L^2-norm. Numerical experiments with various values of the parameter are examined, which demonstrate the theoretical estimates. Among other things, computational results indicate that the behavior of convergence is uniform in the nonnegative parameter.
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I. INTRODUCTION

In the last ten years, the least squares finite element techniques have been extensively applied in many different fields such as fluid dynamics [1–12], elasticity [13–17], electromagnetism [18–19], and semiconductor device physics [20] (see also [21–23] and many references therein). The

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least-squares finite element approach represents a fairly general methodology that can produce a variety of algorithms. For the elasticity problem, Franca et al. [15, 16] proposed some finite element methods that are constructed by adding various least-squares terms to the classical mixed formulation. These methods can be subdivided into two categories for attaining stability, depending on whether the Babuška–Brezzi condition is circumvented or satisfied. Results for the full least-squares finite element methods applied to a stress-pressure-displacement formulation with the displacement boundary conditions have been recently reported in [17]. The present article investigates a two-stage least-squares finite element procedure for treating the elasticity equations with more general boundary conditions.

Introducing the auxiliary variables (stresses and pressure), we can recast the original two-dimensional elasticity system of second-order equations as an equivalent parameter-dependent first-order system with eight equations and six unknowns, in which the nonnegative parameter measures the material compressibility for the elastic body. This new stress-pressure-displacement formulation is different from that in [15, 24, 25] but is similar to the third formulation introduced in [16]. It can be further decomposed into two dependent subsystems, the stress-pressure system and the displacement system recovered from the stresses and pressure. Moreover, we can prove that the stress-pressure system with appropriate boundary conditions is an elliptic system in the sense of Petrovski and satisfies the Lopatinski condition [26]. Taking advantage of these properties, we propose a two-stage least-squares finite element procedure for these two subsystems to obtain approximations of all the unknowns in an orderly way.

The two-stage least-squares finite element presented approach offers many advantages:

- Because the two-stage procedure leads to two minimization problems (rather than the saddle point problem resulting from the mixed finite element procedure), the approximation spaces need not satisfy the Babuška–Brezzi condition, and a single continuous piecewise polynomial space can be used for approximating all the unknowns in both stages.
- Its discretization results in two symmetric and positive definite linear algebraic systems both with condition number $O(h^{-2})$, where h is the mesh parameter. This allows the use of efficient solvers such as the conjugate gradient method to solve the corresponding large linear systems.
- Accurate approximations of the stresses, pressure, and the displacements can be obtained in an orderly way according to the two-stage procedure.
- Under suitable regularity assumptions, the least-squares approximations for all the unknowns have optimal order of approximation in the H^1-norm and in the L^2-norm.
- Numerical experiments with various values of the parameter are examined, which confirm the theoretical error estimates. Among other things, computational results indicate that the behavior of convergence is uniform in the nonnegative parameter.

In addition, compared with the least-squares finite element methods developed in [17], the most significant features of the present approach are the following:

- The methods used in [17] work well only for the displacement boundary conditions, but the present two-stage least-squares method can be applied to the general stress-pressure-displacement boundary conditions, which are more useful in practical applications.
- Just as for the two-stage methods [5, 27], the proposed method has a computational advantage over the methods in [17]. Indeed, a linear system of size $6N$ must be solved for the methods in [17], where N is the dimension of the common approximation space. However, the two-stage procedure requires only the solution of a system of size $4N$, followed by the solution of a system of size $2N$, each with smaller bandwidths.

The remainder of the article is organized as follows. In Section II, we propose a new stress-pressure-displacement formulation for the elasticity equations. This is then decomposed into two subsystems, the stress-pressure system and the displacement system, with respective appropriate boundary conditions. In Section III, a two-stage least-squares finite element procedure is given, as well as its fundamental properties. In Section IV, *a priori* estimates for the stress-pressure system are derived. In Section V, error analysis is presented. In Section VI, the condition numbers of the resulting linear systems are estimated. Finally, in Section VII, some numerical experiments are examined to demonstrate this approach.

II. PRELIMINARIES

We shall consider the numerical solution of the boundary value problem,

$$-2\mu\left\{\nabla\cdot\varepsilon(\mathbf{u})+\frac{\nu}{1-2\nu}\nabla(\nabla\cdot\mathbf{u})\right\}=\mathbf{f}\quad\text{in }\Omega, \tag{2.1}$$

$$\mathbf{u}=\mathbf{0}\quad\text{on }\Gamma_1, \tag{2.2}$$

$$2\mu\left\{\varepsilon(\mathbf{u})+\frac{\nu}{1-2\nu}(\nabla\cdot\mathbf{u})\mathbf{I}\right\}\cdot\mathbf{n}=\mathbf{g}\quad\text{on }\Gamma_2, \tag{2.3}$$

with the following notation:

- $\Omega\subset\mathbf{R}^2$ is a bounded domain representing the region occupied by an elastic body.
- $\Gamma:=\partial\Omega$ is the smooth boundary of Ω, which is partitioned into two disjoint open parts, Γ_1 and Γ_2, such that $\Gamma=\bar{\Gamma}_1\cup\bar{\Gamma}_2$ and measure $(\Gamma_1)>0$.
- μ is the shear modulus given by

  $$\mu=\frac{E}{2(1+\nu)}>0,$$

  where ν is the Poisson ratio, $0<\nu<0.5$, and $E>0$ is the Young modulus. The upper limit of the Poisson ratio, i.e., $\nu\to 0.5^-$, corresponds to an incompressible material.
- $\mathbf{u}=(u_1,u_2)^t$ is the displacement field.
- $\mathbf{f}=(f_1,f_2)^t$ is the density of a body force acting on the body.
- $\mathbf{g}=(g_1,g_2)^t$ is the density of a surface force acting on Γ_2.
- $\mathbf{n}=(n_1,n_2)^t$ is the outward unit normal vector to $\partial\Omega$.
- $\varepsilon(\mathbf{u})$ is the strain tensor given by

  $$\varepsilon(\mathbf{u})=(\varepsilon_{ij}(\mathbf{u}))_{2\times 2}=(\frac{1}{2}(\partial_j u_i+\partial_i u_j))_{2\times 2}.$$

- $\mathbf{I}$ is the 2×2 identity matrix.

Introducing the auxiliary variables, $\varphi_1,\varphi_2,\varphi_3$, and p, such that

$$\varphi_1=\frac{\partial u_1}{\partial x}, \tag{2.4}$$

$$\varphi_2=\frac{\partial u_1}{\partial y}, \tag{2.5}$$

$$\varphi_3 = \frac{\partial u_2}{\partial x}, \tag{2.6}$$

$$-\varphi_1 - \frac{1-2\nu}{\nu} p = \frac{\partial u_2}{\partial y} \tag{2.7}$$

on $\bar{\Omega}$ and letting

$$\epsilon = \frac{1-2\nu}{\nu} > 0, \quad 0 < \nu < \frac{1}{2},$$

we can rewrite (2.1) as

$$2\mu \left\{ -\frac{\partial \varphi_1}{\partial x} - \frac{1}{2}\frac{\partial \varphi_2}{\partial y} - \frac{1}{2}\frac{\partial \varphi_3}{\partial y} + \frac{\partial p}{\partial x} \right\} = f_1 \quad \text{in } \Omega, \tag{2.8}$$

$$2\mu \left\{ \frac{\partial \varphi_1}{\partial y} - \frac{1}{2}\frac{\partial \varphi_2}{\partial x} - \frac{1}{2}\frac{\partial \varphi_3}{\partial x} + (1+\epsilon)\frac{\partial p}{\partial y} \right\} = f_2 \quad \text{in } \Omega. \tag{2.9}$$

We call φ_i the *stresses* and p the *artificial pressure*, and remark that the ‘‘pressure’’ p gives the hydrostatic pressure only in the incompressible limit (cf. Remark 2.1 below). Note that a combination of $\varphi_1, \varphi_2, \varphi_3$ and p can represent the actual stresses σ_{ij} (for $i, j = 1, 2$), which are given by

$$\sigma(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))_{2\times 2} = 2\mu \left\{ \varepsilon(\mathbf{u}) + \frac{\nu}{1-2\nu}(\nabla \cdot \mathbf{u})\mathbf{I} \right\}.$$

Also, by (2.4)–(2.7), we obtain the following two compatibility equations:

$$\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x} = 0 \quad \text{in } \Omega, \tag{2.10}$$

$$\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_3}{\partial y} + \epsilon \frac{\partial p}{\partial x} = 0 \quad \text{in } \Omega. \tag{2.11}$$

System of Eqs. (2.4)–(2.11) is the so-called *stress-pressure-displacement formulation* for the two-dimensional elasticity equations, which is different from those in [15, 24, 25] but is similar to the third formulation introduced in [16]. Moreover, we can show that a sufficiently smooth solution of (2.1) solves system (2.4)–(2.11), and vice versa. It is interesting to observe that the relations between the stresses φ_i, pressure p, and the displacements u_i are defined by Eqs. (2.4)–(2.7), and the stress-pressure system (2.8)–(2.11) is independent of the displacements u_i. Therefore, if one can solve (2.8)–(2.11) with appropriate boundary conditions, then the displacements can be recovered from the stresses and pressure by solving Eqs. (2.4)–(2.7) with the boundary requirement $\mathbf{u} = \mathbf{0}$ on Γ_1. Our two-stage procedure is thus motivated.

Remark 2.1. For the incompressible limit, $\epsilon = 0$, the first-order system (2.4)–(2.11) is the system of *stress-pressure-velocity Stokes equations*, which have been studied in [5, 8]. In the context, $\mathbf{u}$ represents the velocity field for the Stokes flow, p expresses the pressure with appropriate scaling, and μ denotes the inverse of the Reynolds number. We also remark that all the results developed below still hold for the case of $\epsilon = 0$. ■

To deal with the boundary conditions, we note that (2.2) implies that the tangential derivatives of u_i vanish, $\nabla u_i \cdot (n_2, -n_1)^t = 0, i = 1, 2$, that is,

$$n_2\varphi_1 - n_1\varphi_2 = 0 \quad \text{on } \Gamma_1, \tag{2.12}$$

$$n_1\varphi_1 + n_2\varphi_3 + \epsilon n_1 p = 0 \quad \text{on } \Gamma_1. \tag{2.13}$$

Combining (2.12)–(2.13) with

$$n_1 u_1 + n_2 u_2 = 0 \quad \text{on } \Gamma_1,$$

we can verify (2.2) as well. Also, boundary conditions (2.3) can be written as

$$2\mu n_1\varphi_1 + \mu n_2\varphi_2 + \mu n_2\varphi_3 - 2\mu n_1 p = g_1 \quad \text{on } \Gamma_2. \tag{2.14}$$

$$-2\mu n_2\varphi_1 + \mu n_1\varphi_2 + \mu n_1\varphi_3 - 2\mu(1+\epsilon) n_2 p = g_2 \quad \text{on } \Gamma_2. \tag{2.15}$$

It is now clear that our strategy is to solve the stress-pressure system (2.8)–(2.11) with boundary conditions (2.12)–(2.15) at the first stage, i.e., to solve

$$\mathcal{L}_{sp}\Phi := A_{sp}\Phi_x + B_{sp}\Phi_y = F \quad \text{in } \Omega, \tag{2.16}$$

$$\mathcal{R}_{sp}\Phi := C_{sp}\Phi = G \quad \text{on } \Gamma, \tag{2.17}$$

where $\Phi = (\varphi_1, \varphi_2, \varphi_3, p)^t$, $F = (f_1, f_2, 0, 0)^t$,

$$A_{sp} = \begin{pmatrix} -2\mu & 0 & 0 & 2\mu \\ 0 & -\mu & -\mu & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & \epsilon \end{pmatrix},$$

$$B_{sp} = \begin{pmatrix} 0 & -\mu & -\mu & 0 \\ 2\mu & 0 & 0 & 2\mu(1+\epsilon) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$C_{sp} = \begin{pmatrix} n_2 & -n_1 & 0 & 0 \\ n_1 & 0 & n_2 & \epsilon n_1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \Gamma_1,$$

$$C_{sp} = \begin{pmatrix} 2\mu n_1 & \mu n_2 & \mu n_2 & -2\mu n_1 \\ -2\mu n_2 & \mu n_1 & \mu n_1 & -2\mu(1+\epsilon) n_2 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad \text{on } \Gamma_2.$$

The second stage is to solve the displacement system (2.4)–(2.7) with boundary conditions (2.2), i.e., to solve

$$\mathcal{L}_d\mathbf{u} := A_d\mathbf{u}_x + B_d\mathbf{u}_y = \Phi^\epsilon \quad \text{in } \Omega, \tag{2.18}$$

$$\mathcal{R}_d\mathbf{u} := C_d\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{2.19}$$

where $\Phi^\epsilon = (\varphi_1, \varphi_2, \varphi_3, -\varphi_1 - \epsilon p)^t$, the vector $\Phi = (\varphi_1, \varphi_2, \varphi_3, p)^t$ solves the stress-pressure problem (2.16)–(2.17), and

$$A_d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For constructing numerical solutions to these first-order problems, we will apply the least-squares principles in connection with finite element techniques to both stages.

It is interesting to point out that although the displacement system involves four first-order equations (2.4)–(2.7) with two unknown functions u_1 and u_2, those first-order equations are pairwise dependent, according to (2.10) and (2.11).

We shall require some function spaces defined on Ω throughout this article [28, 29]. We let $H^s(\Omega), s \geq 0$ integer, denote the Sobolev space of functions that have square-integrable derivatives of order up to s on Ω; as usual, $L^2(\Omega) := H^0(\Omega)$. The associated inner product and norm are given by

$$(u, v)_s = \sum_{|\alpha| \leq s} \int_\Omega \partial^\alpha u \cdot \partial^\alpha v,$$

$$\|u\|_s = \sqrt{(u, u)_s},$$

respectively. For the product space $[H^s(\Omega)]^m$, the corresponding usual inner product and norm are also denoted by $(\cdot, \cdot)_s$ and $\|\cdot\|_s$, respectively, when there is no chance for confusion. Let $H_0^s(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$, where $\mathcal{D}(\Omega)$ denotes the linear space of infinitely differentiable functions on Ω with compact support. We denote by $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$ normed by

$$\|u\|_{-s} = \sup_{0 \neq v \in H_0^s(\Omega)} \frac{\langle u, v \rangle}{\|v\|_s},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

The existence, uniqueness, and smoothness of the solution of original second-order problem (2.1)–(2.3) with smooth data are well-known (see, e.g., [30, 31]). Thus, it is reasonable to assume that the stress-pressure problem (2.16)–(2.17) and the displacement problem (2.18)–(2.19) have unique (strong) solutions $\Phi \in [H^1(\Omega)]^4$ and $\mathbf{u} \in [H^1(\Omega)]^2$, respectively, for given functions $F \in [L^2(\Omega)]^4$ and $G \in [L^2(\partial\Omega)]^2$. It is also understood that, when $\epsilon = 0$, we further require $\int_\Omega p = 0$ here, as well as in the approximations. Actually, the unique solvability of problem (2.18)–(2.19) can be ensured by virtue of Eqs. (2.4)–(2.7) and the boundary requirement $\mathbf{u} = \mathbf{0}$ on Γ_1, provided measure $(\Gamma_1) > 0$.

For simplicity, we shall also assume that the boundary data G in (2.17) is identical to $\mathbf{0}$, i.e., $\mathbf{g} = (g_1, g_2)^t = \mathbf{0}$ on Γ_2. This can be achieved under some suitable assumptions. For example, assume there exist $\psi_1, \psi_2 \in H^1(\Omega)$ such that the traces of ψ_1 and ψ_2 on Γ are given, respectively, by

$$\psi_1 = \begin{cases} 0 & \text{on } \Gamma_1; \\ g_1 n_1 - g_2 n_2 & \text{on } \Gamma_2, \end{cases}$$

and

$$\psi_2 = \begin{cases} 0 & \text{on } \Gamma_1; \\ g_2 n_1 + g_1 n_2 & \text{on } \Gamma_2. \end{cases}$$

Define

$$\varphi_1^* = \frac{\psi_1}{2\mu}, \quad \varphi_2^* = \frac{\psi_2}{\mu}.$$

By the change of variables $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{p})^t$, where

$$\tilde{\varphi}_1 = \varphi_1 - \varphi_1^*, \quad \tilde{\varphi}_2 = \varphi_2 - \varphi_2^*, \quad \tilde{\varphi}_3 = \varphi_3, \quad \tilde{p} = p,$$

the stress-pressure problem (2.16)–(2.17) can be transformed into the following form:

$$\mathcal{L}_{sp}\tilde{\Phi} := A_{sp}\tilde{\Phi}_x + B_{sp}\tilde{\Phi}_y = \tilde{F} \quad \text{in } \Omega,$$

$$\mathcal{R}_{sp}\tilde{\Phi} := C_{sp}\tilde{\Phi} = 0 \quad \text{on } \Gamma,$$

which is the desired result.

III. TWO-STAGE LEAST-SQUARES PROCEDURE

We first define two function spaces for our problems,

$$\mathcal{S} = \{\Psi \in [H^1(\Omega)]^4, \mathcal{R}_{sp}\Psi = \mathbf{0} \text{ on } \Gamma\}, \tag{3.1}$$

$$\mathcal{V} = \{\mathbf{v} \in [H^1(\Omega)]^2; \mathcal{R}_d\mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}, \tag{3.2}$$

and then define the least-squares quadratic functional $\mathcal{J}_{sp} : \mathcal{S} \to \mathbf{R}$ by

$$\mathcal{J}_{sp}(\Psi) = \|\mathcal{L}_{sp}\Psi - F\|_0^2 = \|(A_{sp}\Psi_x + B_{sp}\Psi_y) - F\|_0^2 \quad \forall\Psi \in \mathcal{S}. \tag{3.3}$$

It is evident that the exact solution $\Phi \in \mathcal{S}$ of the stress-pressure problem (2.16)–(2.17) minimizes (3.3), since $\mathcal{J}_{sp}(\Phi) = 0$, and a zero minimizer of the functional $\mathcal{J}_{sp}$ on $\mathcal{S}$ solves problem (2.16)–(2.17). Thus, the least-squares method for (2.16)–(2.17) is defined to be the following minimization problem:

$$\text{Seek } \Phi \in \mathcal{S} \text{ such that } \mathcal{J}_{sp}(\Phi) = \min_{\Psi\in\mathcal{S}} \mathcal{J}_{sp}(\Psi). \tag{3.4}$$

Taking the first variation, we can find that problem (3.4) is equivalent to

$$\text{Seek } \Phi \in \mathcal{S} \text{ such that } \mathcal{B}_{sp}(\Phi, \Psi) = \mathcal{F}_{sp}(\Psi) \quad \forall\Psi \in \mathcal{S}, \tag{3.5}$$

where

$$\mathcal{B}_{sp}(\Phi, \Psi) = \int_\Omega (A_{sp}\Phi_x + B_{sp}\Phi_y) \cdot (A_{sp}\Psi_x + B_{sp}\Psi_y) \quad \forall\Phi, \Psi \in \mathcal{S}, \tag{3.6}$$

$$\mathcal{F}_{sp}(\Psi) = \int_\Omega F \cdot (A_{sp}\Psi_x + B_{sp}\Psi_y) \quad \forall\Psi \in \mathcal{S}. \tag{3.7}$$

Therefore, a least-squares finite element approximation to the solution of problem (2.16)–(2.17) is defined by

$$\text{Seek } \Phi_h \in \mathcal{S}_h^r \text{ such that } \mathcal{B}_{sp}(\Phi_h, \Psi_h) = \mathcal{F}_{sp}(\Psi_h) \quad \forall\Psi_h \in \mathcal{S}_h^r, \tag{3.8}$$

where the finite-dimensional subspace $\mathcal{S}_h^r \subset \mathcal{S}$ (with $r \geq 0$) is assumed to possess the following approximation property: for every $\Psi \in \mathcal{S} \cap [H^{r+1}(\Omega)]^4$, there exists $\Psi_h \in \mathcal{S}_h^r$ such that

$$\|\Psi - \Psi_h\|_0 + h\|\Psi - \Psi_h\|_1 \leq Ch^{r+1}\|\Psi\|_{r+1}, \tag{3.9}$$

where C is a positive constant independent of Ψ and h. In what follows, C will denote a positive constant always independent of h, not necessarily the same in different occurrences.

After the stress problem (2.16)–(2.17) is solved by using the least-squares finite element scheme (3.8), our second stage is to solve the displacement problem (2.18)–(2.19) approximately. Define the following least-squares functional

$$\mathcal{J}_d(\mathbf{v}) = \|\mathcal{L}_d\mathbf{v} - \Phi^\epsilon\|_0^2 = \|(A_d\mathbf{v}_x + B_d\mathbf{v}_y) - \Phi^\epsilon\|_0^2 \quad \forall \mathbf{v} \in \mathcal{V}, \tag{3.10}$$

where $\Phi^\epsilon = (\varphi_1, \varphi_2, \varphi_3, -\varphi_1 - \epsilon p)^t$, with $\Phi = (\varphi_1, \varphi_2, \varphi_3, p)^t$ being the solution of problem (2.16)–(2.17). Similar to the least-squares method for the stress-pressure problem, we define the following minimization problem:

$$\text{Seek } \mathbf{u} \in \mathcal{V} \text{ such that } \mathcal{J}_d(\mathbf{u}) = \min_{\mathbf{v}\in\mathcal{V}} \mathcal{J}_d(\mathbf{v}), \tag{3.11}$$

or, equivalently,

$$\text{Seek } \mathbf{u} \in \mathcal{V} \text{ such that } \mathcal{B}_d(\mathbf{u}, \mathbf{v}) = \mathcal{F}_d(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{V}, \tag{3.12}$$

where

$$\mathcal{B}_d(\mathbf{u}, \mathbf{v}) = \int_\Omega (A_d\mathbf{u}_x + B_d\mathbf{u}_y) \cdot (A_d\mathbf{v}_x + B_d\mathbf{v}_y) \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \tag{3.13}$$

$$\mathcal{F}_d(\mathbf{v}) = \int_\Omega \Phi^\epsilon \cdot (A_d\mathbf{v}_x + B_d\mathbf{v}_y) \quad \forall \mathbf{v} \in \mathcal{V}. \tag{3.14}$$

Since the data function Φ^ϵ can be obtained only through the numerical scheme (3.8), the associated least-squares approximate scheme for (2.18)–(2.19) is defined by

$$\text{Seek } \mathbf{u}_h \in \mathcal{V}_h^p \text{ such that } \mathcal{B}_d(\mathbf{u}_h, \mathbf{v}_h) = \tilde{\mathcal{F}}_d(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{V}_h^p, \tag{3.15}$$

where

$$\tilde{\mathcal{F}}_d(\mathbf{v}_h) = \int_\Omega \Phi_h^\epsilon \cdot (A_d\mathbf{v}_{hx} + B_d\mathbf{v}_{hy}) \quad \forall \mathbf{v}_h \in \mathcal{V}_h^p, \tag{3.16}$$

$\Phi_h^\epsilon = (\varphi_{1h}, \varphi_{2h}, \varphi_{3h}, -\varphi_{1h} - \epsilon p_h)^t$, the vector $\Phi_h = (\varphi_{1h}, \varphi_{2h}, \varphi_{3h}, p_h)^t$ is the solution of problem (3.8), and the finite-dimensional subspace $\mathcal{V}_h^p \subset \mathcal{V}$ (for $p \geq 0$) is also assumed to be equipped with the following approximation property: for every $\mathbf{v} \in \mathcal{V} \cap [H^{p+1}(\Omega)]^2$, there exists $\mathbf{v}_h \in \mathcal{V}_h^p$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 \leq Ch^{p+1}\|\mathbf{v}\|_{p+1}, \tag{3.17}$$

where C is a positive constant independent of $\mathbf{v}$ and h.

It is easily seen that $\mathcal{B}_{sp}(\cdot,\cdot)$ and $\mathcal{B}_d(\cdot,\cdot)$ define two inner products on $\mathcal{S} \times \mathcal{S}$ and $\mathcal{V} \times \mathcal{V}$, respectively. The positive-definiteness is ensured by the fact that the stress-pressure problem (2.16)–(2.17) and the displacement problem (2.18)–(2.19) possess a unique solution for each given smooth function $F \in [L^2(\Omega)]^4$. Denote the reduced norms, respectively, by

$$\|\Psi\|_{sp} = \sqrt{\mathcal{B}_{sp}(\Psi, \Psi)} \quad \forall \Psi \in \mathcal{S}, \tag{3.18}$$

$$\|\mathbf{v}\|_d = \sqrt{\mathcal{B}_d(\mathbf{v}, \mathbf{v})} \quad \forall \mathbf{v} \in \mathcal{V}. \tag{3.19}$$

Then, evidently, there exists a positive constant C such that

$$\|\Psi\|_{sp} \leq C\|\Psi\|_1 \quad \forall \Psi \in \mathcal{S}, \tag{3.20}$$

$$\|\mathbf{v}\|_d \le C\|\mathbf{v}\|_1 \quad \forall \mathbf{v} \in \mathcal{V}, \tag{3.21}$$

since both $\mathcal{L}_{sp}$ and $\mathcal{L}_d$ are first-order differential operators with constant coefficients.

We have the following fundamental properties of the first stage (3.8).

Theorem 3.1. *Let $\Phi \in \mathcal{S}$ be the solution of the stress-pressure problem* (2.16)–(2.17) *with the given functions $F \in [L^2(\Omega)]^4$ and $G = \mathbf{0}$.*

(i) *Problem* (3.8) *has a unique solution $\Phi_h \in \mathcal{S}_h^r$ satisfying the following stability estimate*:

$$\|\Phi_h\|_{sp} \le \|F\|_0. \tag{3.22}$$

(ii) *The matrix of the linear system associated with problem* (3.8) *is symmetric and positive definite*.

(iii) *The following orthogonality relation holds*:

$$\mathcal{B}_{sp}(\Phi - \Phi_h, \Psi_h) = 0 \quad \forall \Psi_h \in \mathcal{S}_h^r. \tag{3.23}$$

(iv) *The approximate solution Φ_h is a best approximation of Φ in the $\|\cdot\|_{sp}$-norm, that is*,

$$\|\Phi - \Phi_h\|_{sp} = \inf_{\Psi_h \in \mathcal{S}_h^r} \|\Phi - \Psi_h\|_{sp}. \tag{3.24}$$

(v) *If $\Phi \in \mathcal{S} \cap [H^{r+1}(\Omega)]^4$, then*

$$\|\mathcal{L}_{sp}\Phi_h - F\|_0 = \|\Phi - \Phi_h\|_{sp} \le Ch^r \|\Phi\|_{r+1}. \tag{3.25}$$

Proof. To prove the unique solvability, it suffices to prove the uniqueness of the solution, since $\mathcal{S}_h^r$ is a finite-dimensional space. Let Φ_h be a solution of (3.8). Then by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|\Phi_h\|_{sp}^2 &= \mathcal{B}_{sp}(\Phi_h, \Phi_h) = (F, \mathcal{L}_{sp}\Phi_h)_0 \\ &\le \|F\|_0 \|\mathcal{L}_{sp}\Phi_h\|_0 \\ &= \|F\|_0 \|\Phi_h\|_{sp},\end{aligned}$$

which implies (3.22). Consequently, the solution Φ_h of problem (3.8) is unique.

Assertion (ii) follows from the fact that the inner product $\mathcal{B}_{sp}(\cdot,\cdot)$ is symmetric and positive definite. Subtracting the equation in (3.8) from the equation in (3.5), we get (3.23). To prove (iv), by (3.23) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|\Phi - \Phi_h\|_{sp}^2 &= \mathcal{B}_{sp}(\Phi - \Phi_h, \Phi - \Phi_h) \\ &= \mathcal{B}_{sp}(\Phi - \Phi_h, \Phi - \Psi_h) \\ &\le \|\Phi - \Phi_h\|_{sp} \|\Phi - \Psi_h\|_{sp} \quad \forall \Psi_h \in \mathcal{S}_h^r,\end{aligned}$$

which implies (3.24).

Finally, let $\Psi_h \in \mathcal{S}_h^r$ such that (3.9) holds with Ψ replaced by Φ. Together with (3.24) and (3.20), we can obtain (3.25). ∎

Similar to Theorem 3.1 with minor modifications, we have the following results for second stage (3.15).

Theorem 3.2. *Let $\mathbf{u} \in \mathcal{V}$ be the solution of the displacement problem* (2.18)–(2.19).

(i) *Problem* (3.15) *has a unique solution* $\mathbf{u}_h \in \mathcal{V}_h^p$ *satisfying the following stability estimate*:

$$\|\mathbf{u}_h\|_d \le \|\Phi_h^\epsilon\|_0, \tag{3.26}$$

where $\Phi_h^\epsilon = (\varphi_{1h}, \varphi_{2h}, \varphi_{3h}, -\varphi_{1h} - \epsilon p_h)^t$, *and* $\Phi_h = (\varphi_{1h}, \varphi_{2h}, \varphi_{3h}, p_h)^t$ *is the solution of problem* (3.8).

(ii) *The matrix of the linear system associated with problem* (3.15) *is symmetric and positive definite*.

(iii) *The following relation holds*:

$$\mathcal{B}_d(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = (\Phi^\epsilon - \Phi_h^\epsilon, \mathcal{L}_d \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in \mathcal{V}_h^p, \tag{3.27}$$

where $\Phi^\epsilon = (\varphi_1, \varphi_2, \varphi_3, -\varphi_1 - \epsilon p)^t$, *and* $\Phi = (\varphi_1, \varphi_2, \varphi_3, p)^t$ *is the solution of problem* (2.16)–(2.17). ∎

In the following two sections, we will prove that the $\|\cdot\|_{sp}$-norm and the $\|\cdot\|_d$-norm are equivalent to the $\|\cdot\|_1$-norm in the respective spaces $\mathcal{S}$ and $\mathcal{V}$. By (3.22) and (3.26), we will have the following corollary.

Corollary 3.1. *The two-stage least-squares finite element scheme* (3.8) *and* (3.15) *is stable with respect to the* $\|\cdot\|_1$*-norm, i.e.*,

$$\|\Phi_h\|_1 \le C\|F\|_0, \tag{3.28}$$

$$\|\mathbf{u}_h\|_1 \le C\|F\|_0, \tag{3.29}$$

where C *is a positive constant independent of* h. ∎

IV. *A PRIORI* ESTIMATES

In this section we shall apply the theory given by Wendland [26] to derive coercive type *a priori* estimates for the solution Φ to the stress-pressure problem (2.16)–(2.17). Following these estimates, the error estimates for the least-squares finite element approximation (3.8) can be obtained.

We shall show that $\mathcal{L}_{sp}$ is an elliptic operator in the sense of Petrovski, and that the boundary operator $\mathcal{R}_{sp}$ in (2.17) satisfies the Lopatinski condition. Then $(\mathcal{L}_{sp}, \mathcal{R}_{sp})$ is a regular elliptic system and so, by [26], it is a Fredholm operator with zero nullity, which enables us to get the *a priori* estimates (cf. Theorem 4.1 below).

For all $(\xi, \eta) \in \mathbf{R}^2$ and $(\xi, \eta) \neq (0, 0)$, we have

$$\det(\xi A_{sp} + \eta B_{sp}) = 2\mu^2(1+\epsilon)(\xi^2 + \eta^2)^2 \neq 0.$$

Thus, (2.16) is an elliptic system in the sense of Petrovski. Taking $(\xi, \eta) = (1, 0)$, we find that A_{sp} is nonsingular and

$$A_{sp}^{-1} = \begin{pmatrix} -\frac{\epsilon}{2\mu(1+\epsilon)} & 0 & 0 & \frac{1}{1+\epsilon} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{\mu} & 1 & 0 \\ \frac{1}{2\mu(1+\epsilon)} & 0 & 0 & \frac{1}{1+\epsilon} \end{pmatrix}.$$

Then the original elliptic system (2.16) can be transformed into the following form:

$$\Phi_x + \tilde{B}_{sp}\Phi_y = \tilde{F} \quad \text{in } \Omega,$$

where

$$\tilde{B}_{sp} = A_{sp}^{-1}B_{sp} = \begin{pmatrix} 0 & \frac{\epsilon}{2(1+\epsilon)} & \frac{2+\epsilon}{2(1+\epsilon)} & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2(1+\epsilon) \\ 0 & -\frac{1}{2(1+\epsilon)} & \frac{1}{2(1+\epsilon)} & 0 \end{pmatrix},$$

$$\tilde{F} = A_{sp}^{-1}F = \begin{pmatrix} -\frac{\epsilon}{2\mu(1+\epsilon)}f_1 \\ 0 \\ -\frac{1}{\mu}f_2 \\ \frac{1}{2\mu(1+\epsilon)}f_1 \end{pmatrix}.$$

We now check the Lopatinski condition as follows: after elementary operations, we can find that the eigenvalues of matrix $\tilde{B}_{sp}^t$ are i and $-i$, each with multiplicity two. Consider the eigenvalue $\tau_+ = i$ in the upper half-plane, to which there is a pair of linearly independent generalized eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of $\tilde{B}_{sp}^t$ obtained from

$$\tilde{B}_{sp}^t\mathbf{p}_1 - \tau_+\mathbf{p}_1 = \mathbf{0},$$

$$\tilde{B}_{sp}^t\mathbf{p}_2 - \tau_+\mathbf{p}_2 = \mathbf{p}_1,$$

where

$$\mathbf{p}_1 = (0, 1, -1, -2(1+\epsilon)i)^t,$$

$$\mathbf{p}_2 = \left(-\frac{4(1+\epsilon)}{2+\epsilon}, 0, \frac{4(1+\epsilon)}{2+\epsilon}i, -\frac{2(1+\epsilon)(2+3\epsilon)}{2+\epsilon}\right)^t.$$

Notice that

$$\mathcal{P} = (\mathbf{p}_1, \bar{\mathbf{p}}_1, \mathbf{p}_2, \bar{\mathbf{p}}_2)^t$$

is nonsingular. Let

$$\mathcal{Q} = (\mathbf{q}_1, \bar{\mathbf{q}}_1, \mathbf{q}_2, \bar{\mathbf{q}}_2)$$

be the inverse matrix of $\mathcal{P}$. Then

$$\mathcal{Q} = \begin{pmatrix} -\frac{2+3\epsilon}{8(1+\epsilon)}i & \frac{2+3\epsilon}{8(1+\epsilon)}i & -\frac{2+\epsilon}{8(1+\epsilon)} & -\frac{2+\epsilon}{8(1+\epsilon)} \\ \frac{1}{2} & \frac{1}{2} & -\frac{2+\epsilon}{8(1+\epsilon)}i & \frac{2+\epsilon}{8(1+\epsilon)}i \\ 0 & 0 & -\frac{2+\epsilon}{8(1+\epsilon)}i & \frac{2+\epsilon}{8(1+\epsilon)}i \\ \frac{1}{4(1+\epsilon)}i & -\frac{1}{4(1+\epsilon)}i & 0 & 0 \end{pmatrix}.$$

Now we have

$$\det\{2C_{sp}(\mathbf{q}_1,\mathbf{q}_2)\} = \begin{cases} \frac{(2+\epsilon)(2+3\epsilon)}{16(1+\epsilon)^2}(n_1+n_2 i)^2 \neq 0 & \text{on } \Gamma_1; \\ -\frac{(2+\epsilon)^2}{16(1+\epsilon)^2}(n_1+n_2 i)^2 \neq 0 & \text{on } \Gamma_2, \end{cases}$$

for all $\epsilon \geq 0$ and $(n_1, n_2) \neq (0, 0)$. That is, the Lopatinski condition is fulfilled for our boundary conditions (2.17). Thus, we have the following theorem.

Theorem 4.1. *For the boundary value problem* (2.16)–(2.17), *we have the following a priori estimates*: *for each* $l \geq 0$ *there is a constant* $C > 0$ *such that if* $\Psi \in [H^{l+1}(\Omega)]^4$, *then*

$$\|\Psi\|_{l+1} \leq C\{\|\mathcal{L}_{sp}\Psi\|_l + \|\mathcal{R}_{sp}\Psi\|_{l+\frac{1}{2}}\}. \tag{4.1}$$

■

By an interpolation argument in [35] (see also [26, Lemma 8.2.1]), the inequalities (4.1) can be extended to the case $l \geq -1$. Taking $l = 1, l = 0$, and $l = -1$ in (4.1), we have, respectively,

$$\|\Psi\|_2 \leq C\|\mathcal{L}_{sp}\Psi\|_1 \quad \forall \Psi \in \mathcal{S} \cap [H^2(\Omega)]^4, \tag{4.2}$$

$$\|\Psi\|_1 \leq C\|\mathcal{L}_{sp}\Psi\|_0 \quad \forall \Psi \in \mathcal{S}, \tag{4.3}$$

$$\|\Psi\|_0 \leq C\|\mathcal{L}_{sp}\Psi\|_{-1} \quad \forall \Psi \in \mathcal{S}. \tag{4.4}$$

The *a priori* estimates (4.2), (4.3), and (4.4) play crucial roles for the least-squares error estimates of the stresses and pressure in the next section.

Remark 4.1. It is unclear whether the constant C in (4.2)–(4.4) is independent of the non-negative parameter ϵ, since the constant in (4.1) is not explicitly known (cf. [32], page 74, Remark 2). ■

V. ERROR ESTIMATES

It is easily seen that the bilinear form $\mathcal{B}_{sp}$ is continuous on $\mathcal{S} \times \mathcal{S}$; the coercivity of $\mathcal{B}_{sp}$ follows from (4.3). Thus, we first obtain the following result.

Theorem 5.1. *Let* $\Phi \in \mathcal{S}$ *and* $\Phi_h \in \mathcal{S}_h^r$ *be the solutions of problems* (2.16)–(2.17) *and* (3.8), *respectively. Assume that* $\Phi \in [H^{r+1}(\Omega)]^4$. *Then there exists a positive constant* C *independent of* h *and* Φ *such that*

$$\|\Phi - \Phi_h\|_1 \leq Ch^r\|\Phi\|_{r+1}. \tag{5.1}$$

Proof. Since the $\|\cdot\|_{sp}$-norm is equivalent to the $\|\cdot\|_1$-norm on the space $\mathcal{S}$, the assertion follows from (3.25) immediately. ■

For deriving the optimal L^2-estimates, we need the following regularity assumption that we shall use in the subsequent result.

Assumption (A1). *For any* $\Psi \in [H_0^1(\Omega)]^4$, *the unique solution* $\tilde{\Phi}$ *of the problem*

$$\mathcal{L}_{sp}\tilde{\Phi} = \Psi \quad \text{in } \Omega,$$

$$\mathcal{R}_{sp}\tilde{\Phi} = \mathbf{0} \quad \text{on } \Gamma \tag{5.2}$$

belongs to $\mathcal{S} \cap [H^2(\Omega)]^4$.

Evidently, this is a reasonable assumption because the differential operator $\mathcal{L}_{sp}$ is of first order and the data function Ψ is in $[H_0^1(\Omega)]^4$.

Theorem 5.2. *Let* $\Phi \in \mathcal{S}$ *and* $\Phi_h \in \mathcal{S}_h^r$ *be the solutions of problems* (2.16)–(2.17) *and* (3.8), *respectively. Assume that* $\Phi \in [H^{r+1}(\Omega)]^4$ *and that assumption* (A1) *holds. Then there exists a positive constant* C *independent of* h *and* Φ *such that*

$$\|\Phi - \Phi_h\|_0 \leq Ch^{r+1}\|\Phi\|_{r+1}. \tag{5.3}$$

Proof. Let $\Psi \in [H_0^1(\Omega)]^4$ and let $\tilde{\Phi} \in \mathcal{S} \cap [H^2(\Omega)]^4$ be the corresponding solution to problem (5.2). Then

$$\begin{aligned}
|(\mathcal{L}_{sp}(\Phi - \Phi_h), \Psi)_0| &= |(\mathcal{L}_{sp}(\Phi - \Phi_h), \mathcal{L}_{sp}\tilde{\Phi})_0| \\
&= |(\mathcal{L}_{sp}(\Phi - \Phi_h), \mathcal{L}_{sp}(\tilde{\Phi} - \Psi_h))_0| \quad \forall \Psi_h \in \mathcal{S}_h^r \quad \text{(by (3.23))} \\
&\leq \|\mathcal{L}_{sp}(\Phi - \Phi_h)\|_0 \|\mathcal{L}_{sp}(\tilde{\Phi} - \Psi_h)\|_0 \quad \forall \Psi_h \in \mathcal{S}_h^r \\
&\leq C\|\Phi - \Phi_h\|_1 \|\tilde{\Phi} - \Psi_h\|_1 \quad \forall \Psi_h \in \mathcal{S}_h^r \\
&\leq Ch\|\Phi - \Phi_h\|_1 \|\tilde{\Phi}\|_2 \quad \text{(by (3.9))} \\
&\leq Ch\|\Phi - \Phi_h\|_1 \|\mathcal{L}_{sp}\tilde{\Phi}\|_1 \quad \text{(by (4.2))} \\
&= Ch\|\Phi - \Phi_h\|_1 \|\Psi\|_1.
\end{aligned}$$

In addition, the L^2 inner product $(\mathcal{L}_{sp}(\Phi - \Phi_h), \Psi)_0$ defines a bounded linear functional on $[H_0^1(\Omega)]^4$, since

$$|(\mathcal{L}_{sp}(\Phi - \Phi_h), \Psi)_0| \leq \|\mathcal{L}_{sp}(\Phi - \Phi_h)\|_0 \|\Psi\|_1 \quad \forall \Psi \in [H_0^1(\Omega)]^4.$$

Therefore,

$$\|\mathcal{L}_{sp}(\Phi - \Phi_h)\|_{-1} \leq Ch\|\Phi - \Phi_h\|_1. \tag{5.4}$$

Combining (5.4) with (4.4) and (5.1), we can readily conclude estimate (5.3). ■

The results of Theorem 5.1 and Theorem 5.2 indicate that the rates of convergence for the stresses and pressure are optimal, both in the H^1-norm and in the L^2-norm. We now estimate the rates of convergence of the approximations for the displacements. The continuity of the bilinear form $\mathcal{B}_d$ can be obtained easily. For any $\mathbf{v} = (v_1, v_2)^t \in \mathcal{V} = \{\mathbf{v} \in [H^1(\Omega)]^2; \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}$, we have

$$\begin{aligned}
\mathcal{B}_d(\mathbf{v}, \mathbf{v}) &= \int_\Omega \left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_1}{\partial y}\right)^2 + \left(\frac{\partial v_2}{\partial x}\right)^2 + \left(\frac{\partial v_2}{\partial y}\right)^2 \\
&= \|\nabla \mathbf{v}\|_0^2.
\end{aligned}$$

It follows from the Poincaré–Friedrichs inequality that

$$\mathcal{B}_d(\mathbf{v}, \mathbf{v}) \geq C\|\mathbf{v}\|_1^2, \tag{5.5}$$

i.e., $\mathcal{B}_d$ is coercive on $\mathcal{V} \times \mathcal{V}$. Similar to Theorem 5.1, we have the optimal order of convergence in the H^1-norm for the displacements.

Theorem 5.3. *Let* $\Phi \in \mathcal{S}, \mathbf{u} \in \mathcal{V}$, *and* $\mathbf{u}_h \in \mathcal{V}_h^p$ *be the solutions of problems* (2.16)–(2.17), (2.18)–(2.19), *and* (3.15), *respectively. Assume* $\Phi \in [H^{r+1}(\Omega)]^4$ *and* $\mathbf{u} \in [H^{p+1}(\Omega)]^2$. *Then there exists a positive constant* C *independent of* $h, \mathbf{u}$, *and* Φ *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leq C(h^p\|\mathbf{u}\|_{p+1} + h^r\|\Phi\|_{r+1}). \tag{5.6}$$

Proof. For any $\mathbf{v}_h \in \mathcal{V}_h^p$, by (5.5) and (3.27) we have

$$\begin{aligned} C\|\mathbf{u}_h - \mathbf{v}_h\|_1^2 &\leq \mathcal{B}_d(\mathbf{u}_h - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) \\ &= \mathcal{B}_d(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) - \mathcal{B}_d(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) \\ &= \mathcal{B}_d(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) - (\Phi^\epsilon - \Phi_h^\epsilon, \mathcal{L}_d(\mathbf{u}_h - \mathbf{v}_h))_0 \\ &\leq C\|\mathbf{u} - \mathbf{v}_h\|_1\|\mathbf{u}_h - \mathbf{v}_h\|_1 + C\|\Phi^\epsilon - \Phi_h^\epsilon\|_0\|\mathbf{u}_h - \mathbf{v}_h\|_1 \\ &\leq C(\|\mathbf{u} - \mathbf{v}_h\|_1 + \|\Phi - \Phi_h\|_0)\|\mathbf{u}_h - \mathbf{v}_h\|_1, \end{aligned}$$

which implies

$$\|\mathbf{u}_h - \mathbf{v}_h\|_1 \leq C(\|\mathbf{u} - \mathbf{v}_h\|_1 + \|\Phi - \Phi_h\|_0) \quad \forall \mathbf{v}_h \in \mathcal{V}_h^p.$$

Thus,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_1 &\leq \|\mathbf{u} - \mathbf{v}_h\|_1 + \|\mathbf{u}_h - \mathbf{v}_h\|_1 \\ &\leq C(\|\mathbf{u} - \mathbf{v}_h\|_1 + \|\Phi - \Phi_h\|_0) \quad \forall \mathbf{v}_h \in \mathcal{V}_h^p. \end{aligned}$$

Choose $\mathbf{v}_h \in \mathcal{V}_h^p$ such that (3.17) holds with $\mathbf{v}$ replaced by $\mathbf{u}$. Then we obtain

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_1 &\leq C(\|\mathbf{u} - \mathbf{v}_h\|_1 + \|\Phi - \Phi_h\|_1) \\ &\leq C(h^p\|\mathbf{u}\|_{p+1} + h^r\|\Phi\|_{r+1}). \end{aligned}$$

This completes the proof. ∎

Similar to the derivation of Theorem 5.2, we shall use the Aubin–Nitsche trick [28, 33] to establish the optimal L^2-estimates for the displacements. For each h, consider the following adjoint problem:

$$\text{Find } \tilde{\mathbf{u}} \in \mathcal{V} \text{ such that } \mathcal{B}_d(\tilde{\mathbf{u}}, \mathbf{v}) = (\mathbf{u} - \mathbf{u}_h, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathcal{V}. \tag{5.7}$$

Note that the right-hand side of the equation in (5.7) defines a bounded linear functional on $\mathcal{V}$. Thus, the unique solvability of problem (5.7) is ensured by the Lax–Milgram lemma, since $\mathcal{B}_d$ is coercive on $\mathcal{V} \times \mathcal{V}$. We now assume the following regularity assumption (cf. [14]).

Assumption (A2). *Assume the unique solution $\tilde{\mathbf{u}}$ of problem* (5.7) *belongs to $[H^2(\Omega)]^2 \cap \mathcal{V}$ and there exists a positive constant C independent of $\tilde{\mathbf{u}}$ and $\mathbf{u} - \mathbf{u}_h$ such that*

$$\|\tilde{\mathbf{u}}\|_2 \leq C\|\mathbf{u} - \mathbf{u}_h\|_0. \tag{5.8}$$

Then we have the following optimal L^2-estimates for the displacements.

Theorem 5.4. *Let $\Phi \in \mathcal{S}, \mathbf{u} \in \mathcal{V}$, and $\mathbf{u}_h \in \mathcal{V}_h^p$ be the solutions of problems* (2.16)–(2.17), (2.18)–(2.19), *and* (3.15), *respectively. Assume that $\Phi \in [H^{r+1}(\Omega)]^4, \mathbf{u} \in [H^{p+1}(\Omega)]^2$, and regularity assumptions* (A1) *and* (A2) *hold. Then there exists a positive constant C independent of $h, \mathbf{u}$, and Φ such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C(h^{p+1}\|\mathbf{u}\|_{p+1} + h^{r+1}\|\Phi\|_{r+1}). \tag{5.9}$$

Proof. Choosing $\mathbf{v} = \mathbf{u} - \mathbf{u}_h \in \mathcal{V}$ in (5.7), we have

$$\mathcal{B}_d(\tilde{\mathbf{u}}, \mathbf{u} - \mathbf{u}_h) = \|\mathbf{u} - \mathbf{u}_h\|_0^2,$$

which together with (3.27) enables us to obtain

$$\mathcal{B}_d(\tilde{\mathbf{u}} - \mathbf{w}_h, \mathbf{u} - \mathbf{u}_h) = \|\mathbf{u} - \mathbf{u}_h\|_0^2 - (\Phi^\epsilon - \Phi_h^\epsilon, \mathcal{L}_d\mathbf{w}_h)_0 \quad \forall \mathbf{w}_h \in \mathcal{V}_h^p.$$

It follows that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_0^2 &= \mathcal{B}_d(\tilde{\mathbf{u}}-\mathbf{w}_h,\mathbf{u}-\mathbf{u}_h)+(\Phi^\epsilon-\Phi_h^\epsilon,\mathcal{L}_d\mathbf{w}_h)_0\\ &= \mathcal{B}_d(\tilde{\mathbf{u}}-\mathbf{w}_h,\mathbf{u}-\mathbf{u}_h)+(\Phi^\epsilon-\Phi_h^\epsilon,\mathcal{L}_d(\mathbf{w}_h-\tilde{\mathbf{u}}))_0+(\Phi^\epsilon-\Phi_h^\epsilon,\mathcal{L}_d\tilde{\mathbf{u}})_0\\ &\le \|\tilde{\mathbf{u}}-\mathbf{w}_h\|_1\|\mathbf{u}-\mathbf{u}_h\|_1+\|\Phi^\epsilon-\Phi_h^\epsilon\|_0\|\tilde{\mathbf{u}}-\mathbf{w}_h\|_1+\|\Phi^\epsilon-\Phi_h^\epsilon\|_0\|\tilde{\mathbf{u}}\|_2,\end{aligned}$$

for all $\mathbf{w}_h\in\mathcal{V}_h^p$. Choose $\mathbf{w}_h\in\mathcal{V}_h^p$ so that

$$\|\tilde{\mathbf{u}}-\mathbf{w}_h\|_1\le Ch\|\tilde{\mathbf{u}}\|_2.$$

Hence, together with (5.8) we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0^2\le Ch\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{u}-\mathbf{u}_h\|_1+Ch\|\Phi-\Phi_h\|_0\|\mathbf{u}-\mathbf{u}_h\|_0+C\|\Phi-\Phi_h\|_0\|\mathbf{u}-\mathbf{u}_h\|_0,$$

which implies

$$\|\mathbf{u}-\mathbf{u}_h\|_0\le C(h\|\mathbf{u}-\mathbf{u}_h\|_1+h\|\Phi-\Phi_h\|_0+\|\Phi-\Phi_h\|_0).$$

This completes the proof. ∎

VI. CONDITION NUMBERS

In this section, we shall give estimates for the condition numbers of the linear systems arising from problem (3.8) and problem (3.15). Recall that the condition number for a symmetric and positive definite $m\times m$ matrix $\mathcal{M}$ is defined by

$$\textit{condition number of } \mathcal{M}=\frac{\lambda_{\max}}{\lambda_{\min}}=\frac{\max\rho(\Xi)}{\min\rho\,(\Xi)},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathcal{M}$, and $\rho(\Xi)$ is the Rayleigh quotient,

$$\rho(\Xi):=\frac{\Xi^t\mathcal{M}\Xi}{\Xi^t\Xi}\quad\forall\Xi=(\xi_1,\dots,\xi_m)^t\in\mathbf{R}^m,\quad\Xi\ne\mathbf{0}.$$

We shall assume that the respective bases $\{\Phi_1,\dots,\Phi_K\}$ and $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ of the finite element spaces $\mathcal{S}_h^r$ and $\mathcal{V}_h^p$ are chosen so that the following conditions hold: there exist positive constants Λ_1 and Λ_2 such that for all $\xi_1,\dots,\xi_K,\eta_1,\dots,\eta_k\in\mathbf{R}$,

$$\Lambda_1h^2\sum_{i=1}^K\xi_i^2\le\left\|\sum_{i=1}^K\xi_i\Phi_i\right\|_0^2\le\Lambda_2h^2\sum_{i=1}^K\xi_i^2,\tag{6.1}$$

$$\Lambda_1h^2\sum_{i=1}^k\eta_i^2\le\left\|\sum_{i=1}^k\eta_i\mathbf{u}_i\right\|_0^2\le\Lambda_1h^2\sum_{i=1}^k\eta_i^2.\tag{6.2}$$

The above conditions hold for most finite element spaces $\mathcal{S}_h^r$ and $\mathcal{V}_h^p$. If, in addition, the corresponding regular family $\{\mathcal{T}_h\}$ of triangulations of $\bar{\Omega}$ is quasi-uniform [28, 34], i.e., there exists a positive constant C independent of h such that

$$h\le C\,\mathrm{diam}(\Omega_i^h)\quad\forall\Omega_i^h\in\mathcal{T}_h,\mathcal{T}_h\in\{\mathcal{T}_h\},$$

TABLE I. The approximations Φ_h with $E = 2.5$ and $\nu = 0.25$ ($\epsilon = 2.0$).

$1/h$	L^2-error	RelErr	L^2-rate	$\|\cdot\|_{sp}$-error	RelErr	$\|\cdot\|_{sp}$-rate
2	0.85600	2.91285e-1	—	7.15156	4.58279e-1	—
4	0.24756	8.42423e-2	1.79	3.52245	2.25722e-1	1.02
8	0.06657	2.26540e-2	1.89	1.76541	1.13130e-1	1.00
16	0.01708	5.81339e-3	1.96	0.88398	5.66466e-2	1.00
32	0.00431	1.46587e-3	1.99	0.44219	2.83361e-2	1.00

then we have the following inverse estimates:

$$\left\|\sum_{i=1}^{K}\xi_i\Phi_i\right\|_1^2 \le Ch^{-2}\left\|\sum_{i=1}^{K}\xi_i\Phi_i\right\|_0^2 \le C\Lambda_2\sum_{i=1}^{K}\xi_i^2, \tag{6.3}$$

$$\left\|\sum_{i=1}^{k}\eta_i\mathbf{u}_i\right\|_1^2 \le Ch^{-2}\left\|\sum_{i=1}^{k}\eta_i\mathbf{u}_i\right\|_0^2 \le C\Lambda_2\sum_{i=1}^{k}\eta_i^2, \tag{6.4}$$

where C is a positive constant independent of h.

Theorem 6.1. *Under conditions* (6.1) *and* (6.3) [*respectively*, (6.2) *and* (6.4)] *the condition number of the linear system arising from problem* (3.8) [*respectively*, *problem* (3.15)] *is* $O(h^{-2})$.

Proof. Let $\Phi_h := \sum_{i=1}^{K}\xi_i\Phi_i \in \mathcal{S}_h^r$. Since the bilinear form $\mathcal{B}_{sp}(\cdot,\cdot)$ is coercive on $\mathcal{S}\times\mathcal{S}$, by (6.1) we have

$$\mathcal{B}_{sp}(\Phi_h,\Phi_h) \ge C\|\Phi_h\|_1^2 \ge C\|\Phi_h\|_0^2 \ge C\Lambda_1 h^2\sum_{i=1}^{K}\xi_i^2.$$

On the other hand, by the continuity of $\mathcal{B}_{sp}(\cdot,\cdot)$ on $\mathcal{S}\times\mathcal{S}$, we get from (6.3) that

$$\mathcal{B}_{sp}(\Phi_h,\Phi_h) \le C\|\Phi_h\|_1^2 \le C\Lambda_2\sum_{i=1}^{K}\xi_i^2.$$

Thus, $\lambda_{\max} \le C\Lambda_2$ and $\lambda_{\min} \ge C\Lambda_1 h^2$, and so the condition number for problem (3.8) is $O(h^{-2})$. The estimates of the condition number for problem (3.15) can be achieved in a similar way. ∎

TABLE II. The approximations $\mathbf{u}_h$ with $E = 2.5$ and $\nu = 0.25$ ($\epsilon = 2.0$).

$1/h$	L^2-error	RelErr	L^2-rate	$\|\cdot\|_d$-error	RelErr	$\|\cdot\|_d$-rate
2	0.17245	2.43881e-1	—	1.40921	4.48564e-1	—
4	0.04299	6.08011e-2	2.00	0.70905	2.25697e-1	0.99
8	0.01075	1.52010e-2	2.00	0.35569	1.13221e-1	1.00
16	0.00269	3.79905e-3	2.00	0.17801	5.66632e-2	1.00
32	0.00067	9.48164e-4	2.00	0.08903	2.83383e-2	1.00

TABLE III. Rates of convergence in the $\|\cdot\|_{sp}$-norm with $E = 2.5$ and small ϵ.

$1/h$	$\nu = 0.49$ $\epsilon \simeq$ 4.1e-2	$\nu = 0.499$ $\epsilon \simeq$ 4.0e-3	$\nu = 0.4999$ $\epsilon \simeq$ 4.0e-4	$\nu = 0.49999$ $\epsilon \simeq$ 4.0e-5	$\nu = 0.499999$ $\epsilon \simeq$ 4.0e-6
2	—	—	—	—	—
4	0.97	0.96	0.96	0.96	0.96
8	0.98	0.98	0.97	0.97	0.97
16	0.99	0.99	0.99	0.99	0.99
32	1.00	1.00	1.00	1.00	1.00

VII. NUMERICAL EXPERIMENTS

We shall present a simple example solved by using our two-stage least-squares finite element scheme (3.8) and (3.15). To simplify the numerical implementation, we shall assume that $\Omega = (0,1) \times (0,1), \Gamma_1 = \partial\Omega$, and the square domain Ω is uniformly partitioned into a set of $1/h^2$ square subdomains Ω_i^h with side-length h. The problem we present has the smooth exact solution,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ p \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \pi\cos(\pi x)\sin(\pi y) \\ \pi\sin(\pi x)\cos(\pi y) \\ \pi\cos(\pi x)\sin(\pi y) \\ -\frac{\pi}{\epsilon}(\cos(\pi x)\sin(\pi y) + \sin(\pi x)\cos(\pi y)) \\ \sin(\pi x)\sin(\pi y) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}. \tag{7.1}$$

Substituting (7.1) into (2.16)–(2.17), we have $G = (0,0)^t, F = (f_1, f_2, 0, 0)^t$, and

$$f_1 = 2\mu\pi^2\left\{\left(\frac{3}{2} + \frac{1}{\epsilon}\right)\sin(\pi x)\sin(\pi y) - \left(\frac{1}{2} + \frac{1}{\epsilon}\right)\cos(\pi x)\cos(\pi y)\right\},$$

$$f_2 = 2\mu\pi^2\left\{\left(\frac{3}{2} + \frac{1}{\epsilon}\right)\sin(\pi x)\sin(\pi y) - \left(\frac{1}{2} + \frac{1}{\epsilon}\right)\cos(\pi x)\cos(\pi y)\right\}.$$

Piecewise bilinear finite elements are applied for all the unknowns. For the case of Poisson's ratio $\nu = 0.25$ and Young's modulus $E = 2.5$, the results are collected in Table I and Table II, where RelErr denotes the relative error and, for simplicity, the data function Φ_h^ϵ is replaced by the exact function Φ^ϵ in the second stage (3.15). Since the $\|\cdot\|_{sp}$-norm and the $\|\cdot\|_d$-norm are both equivalent to the H^1-norm on the spaces $\mathcal{S}$ and $\mathcal{V}$, respectively, the numerical results in Table I and Table II indicate that the two-stage least-squares procedure (3.8) (3.15) achieves optimal convergence both in the L^2-norm and in the H^1-norm for all the unknowns.

TABLE IV. Rates of convergence in the L^2-norm with $E = 2.5$ and small ϵ.

$1/h$	$\nu = 0.49$ $\epsilon \simeq$ 4.1e-2	$\nu = 0.499$ $\epsilon \simeq$ 4.0e-3	$\nu = 0.4999$ $\epsilon \simeq$ 4.0e-4	$\nu = 0.49999$ $\epsilon \simeq$ 4.0e-5	$\nu = 0.499999$ $\epsilon \simeq$ 4.0e-6
2	—	—	—	—	—
4	1.78	1.76	1.76	1.76	1.76
8	1.89	1.89	1.89	1.89	1.89
16	1.96	1.95	1.95	1.95	1.95
32	1.99	1.98	1.98	1.98	1.98

The behavior of convergence for the stresses and pressure influenced by the nonnegative parameter ϵ is particularly examined. Table III and Table IV exhibit that, except on very coarse meshes, the optimal convergence is still essentially ensured for various values of the parameter, even for nearly incompressible elasticity. That is, computational results in Table III and Table IV indicate that the behavior of convergence is uniform in the nonnegative parameter.

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