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Theory and Methodology

An approximately global optimization method for assortment problems

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Abstract

Assortment problems occur when we want to cut a number of small rectangular pieces from a large rectangle to get the minimum area within the rectangle. Recently, Chen et al. proposed a useful model for assortment problems. Although Chen et al.'s model is quite promising to find solutions, there are two inadequacies in their model: firstly, the objective function in their model is a polynomial term, which may not lead to a globally optimal solution; secondly, too many 0-1 variables are used to formulate the non-overlapping constraints. We propose a new method to reformulate an assortment model. Our model is not only able to find the approximately global optimal solution, but involves less 0-1 variables for formulating non-overlapping constraints. © 1998 Elsvier Science B.V.

Keywords: Assortment; Optimization

1. Introduction

An assortment optimization problem is the problem of placing a given set of rectangles within a rectangle which has minimum area. Assortment problems occur when a number of small rectangular pieces need to be cut from a large rectangle to get minimum area. Methods for assortment problems can be classified as the exact approach and the heuristic approach. The exact approach guarantees to find the optimal solution, while the heuristic approach can only find solutions which are 'good enough'. This paper emphasizes on discussing the exact methods of assortment problems.

Page [5] proposed a dynamic programming for solving cutting problems of rectangle steel bars. Beaslay [2] proposed an integer model to solve a guillotine cutting problem. Recently, Chen et al. [1] presented a mixed integer programming for assortment problems. Compared to previous models, Chen et al.'s model is more promising in solving practical problems. However, Chen at al.'s model may only finds locally optimal solutions. In addition, Chen et al. use many 0-1 variables to formulate non-overlapping constraints in their models, which causes an extra computational burden.

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This paper proposes a new method for solving assortment optimization problems. Two advantages of this method are listed below:

1. it can find the solution which can be as close as possible to the globally optimum, instead of obtaining a local optimum as found by Chen et al.'s model;
2. it adopts less 0-1 variables to reformulate the non-overlapping constraints than used in Chen et al.'s model.

2. Problem formulation

Given n rectangles with fixed lengths and widths. An assortment optimization problem is to allocate all of these rectangles within an enveloping rectangle which has minimum area. Denote x and y as the width and the length of the enveloping rectangle, the assortment optimization problem is stated briefly as follows:

Minimize xy

Subject to

1. all of n rectangles are non-overlapping;
2. all of n rectangles are within the range of x and y.

The related terminologies used in assortment models, by referring to Chen et al. [1] are described below:

(p_i, q_i): Dimension of rectangle i, p_i is the long side and q_i is the short side. p_i and q_i are constants, $i \in J$, J is the set of given rectangles.
(x_i, y_i): The top right corner coordinates of rectangle i, $i \in J$, x_i and y_i are variables.
(x, y): The top right corner coordinates of the enveloping rectangle, x and y are variables.
s_i: An orientation indicator for rectangle i, $i \in J$. $s_i = 1$ if p_i (the longer dimension of rectangle i) is parallel to the x-axis; $s_i = 0$ if p_i is parallel to y-axis. Take Fig. 1 for example, $s_1 = 0$, $s_2 = 1$, and $s_3 = 1$.

For any pair of rectangles i and k, the relative position indicators are denoted as follows:

a_{ik}: The 'left' position indicator for rectangles i and k; a_{ik} equals 1 if rectangle i is to the left of rectangle k, and a_{ik} equals 0 otherwise.
b_{ik}: The 'right' position indicator for rectangles i and k; b_{ik} equals 1 if rectangle i is to the right of rectangle k, and b_{ik} equals 0 otherwise.

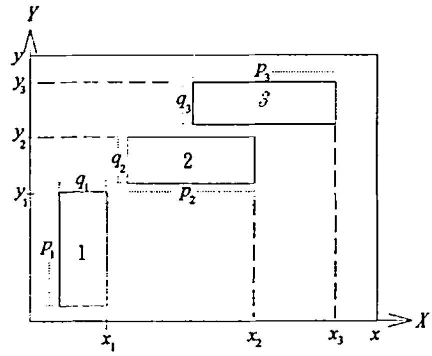


Fig. 1. Graphical illustration of the Chen et al. model.

c_{ik}: The 'below' position indicator for rectangles i and k; c_{ik} equals 1 if rectangle i is to the below of rectangle k, and c_{ik} equals 0 otherwise.
d_{ik}: The 'above' position indicator for rectangles i and k; d_{ik} equals 1 if rectangle i is to the above of rectangle k, and d_{ik} equals 0 otherwise.
J: Set of all rectangles within the enveloping rectangle xy.

The conditions for rectangles i and k to be non-overlapping are stated below

(1) $a_{ik}+b_{ik}+c_{ik}+d_{ik} \geq 1$,

(2) $a_{ik}+b_{ik}<2$ and $c_{ik}+d_{ik}<2$.

Take Fig. 1 for example, rectangles 1, 2 and 3 are non-overlapping, where

$$a_{12}+b_{12}+c_{12}+d_{12}=1+0+0+1=2,$$

and

$$a_{23}+b_{23}+c_{23}+d_{23}=0+0+0+1=1.$$

Chen et al. [1] formulated an assortment model as follows (Fig. 1):

Chen et al.'s Model:

Minimize xy

subject to, for all $i,\ k \in J,\ i<k$:

$$x_k - p_k s_k - q_k(1-s_k) + (1-a_{ik})M \geq x_i, \tag{1}$$

$$x_i - p_i s_i - q_i(1-s_i) + (1-b_{ik})M \geq x_k, \tag{2}$$

$$y_k - q_k s_k - p_k(1-s_k) + (1-c_{ik})M \geq y_i, \tag{3}$$

$$y_i - q_i s_i - p_i(1-s_i) + (1-d_{ik})M \geq y_k, \tag{4}$$

$$a_{ik}+b_{ik}+c_{ik}+d_{ik} \geq 1, \tag{5}$$

where M is a large positive numbers: for all $i \in J$:

$$x \geq x_i, \tag{6}$$

$$y \geq y_i, \tag{7}$$

$$x_i - p_i s_i - q_i(1-s_i) \geq 0, \tag{8}$$

$$y_i - q_i s_i - p_i(1-s_i) \geq 0, \tag{9}$$

$$a_{ik}, b_{ik}, c_{ik}, d_{ik} = 0 \text{ or } 1$$

for all $i,\ k \in J,\ i<k$,

$$s_i = 0 \text{ or } 1 \text{ for all } i \in = J.$$

The objective function is to minimize the enveloping area. Constraints (1)–(5) ensure that the rectangles will not overlap. Constraints (6)–(9) ensure that all rectangles are within the enveloping rectangle.

Chen et al.'s model is a program with a nonlinear objective function, which is difficult to solve. By fixing the value of y in the objective function, Chen et al. solve their model to obtain the solution. Clearly, there are two defects within Chen et al.'s model:

1. Firstly, y is assigned as a fixed value in solution process to reduce the computational complexity of the problem. Chen et al.'s model therefore may only find a locally optimal solution.

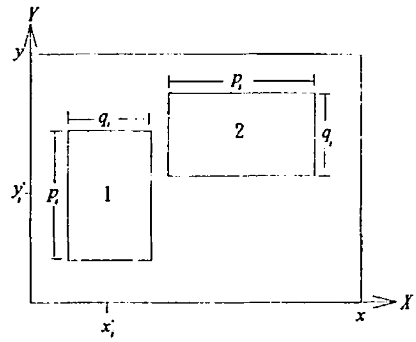


Fig. 2. Graphical illustration of proposed model.

2. Secondly, extravagant 0-1 variables are used to ensure the non-overlapping between rectangles, this will increase the computational burden in the solution process

In order to overcome above defects, we propose a new model for assortment optimization problem. First, denote x_i' and y_i' below:

x_i': Distance between center of rectangle i and original point along the x-axis, $i \in J$ (Fig. 2);
y_i': Distance between center of rectangle i and original point along the y-axis, $i \in J$ (Fig. 2).

The conditions of non-overlapping between rectangles i and k can be reformulated by introducing two binary variables u_{ik} and v_{ik} as follows (Fig. 3):

Condition 1. $u_{ik} = 0$ and $v_{ik} = 0$ if and only if rectangle i is at the right of rectangle k.

Condition 2. $u_{ik} = 1$ and $v_{ik} = 0$ if and only if rectangle i is at the left of rectangle k.

Condition 3. $u_{ik} = 0$ and $v_{ik} = 1$ if and only if rectangle i is at the above of rectangle k.

Condition 4. $u_{ik} = 1$ and $v_{ik} = 1$ if and only if rectangle i is at the below of rectangle k.

These four conditions can be represented as a proposition below:

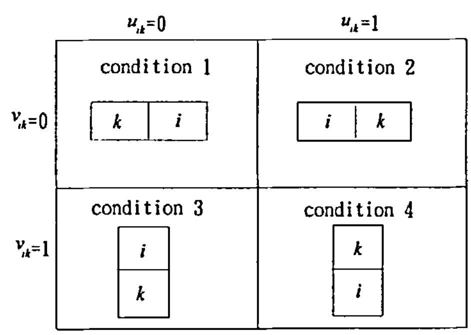


Fig. 3. Graphical illustration of non-overlapping conditions.

Table 1
Non-overlapping and position indication

	u_{ik}	v_{ik}	s_i	s_k	Corresponding conditions in Fig. 3	Coordinates paralleled to longer dimension	
						i	k
1	0	0	0	0	1	y-axis	y-axis
2	0	0	0	1	1	y-axis	x-axis
3	0	0	1	0	1	x-axis	y-axis
4	0	0	1	1	i	x-axis	x-axis
5	0	1	0	0	2	y-axis	y-axis
6	0	1	0	1	2	y-axis	x-axis
7	0	1	1	0	2	x-axis	y-axis
8	0	1	1	1	2	x-axis	x-axis
9	1	0	0	0	3	y-axis	y-axis
10	1	0	0	1	3	y-axis	x-axis
11	1	0	1	0	3	x-axis	y-axis
12	1	0	1	1	3	x-axis	x-axis
13	1	1	0	0	4	y-axis	y-axis
14	1	1	0	1	4	y-axis	x-axis
15	1	1	1	0	4	x-axis	y-axis
16	1	1	1	1	4	x-axis	x-axis

Proposition 1. *Rectangles i and k are non-overlapping if the following conditions are satisfied:*

$$(x'_i - x'_k) + u_{ik}M + v_{ik}M \geq \frac{1}{2}\left[p_i s_i + q_i(1-s_i) + p_k s_k + q_k(1-s_k)\right], \tag{10}$$

$$(x'_k - x'_i) + (1-u_{ik})M + v_{ik}M \geq \frac{1}{2}\left[p_i s_i + q_i(1-s_i) + p_k s_k + q_k(1-s_k)\right], \tag{11}$$

$$(y'_i - y'_k) + u_{ik}M + (1-v_{ik})M \geq \frac{1}{2}\left[p_i(1-s_i) + q_i s_i + p_k(1-s_k) + q_k s_k\right], \tag{12}$$

$$(y'_k - y'_i) + (1-u_{ik})M + (1-v_{ik})M \geq \frac{1}{2}\left[p_i(1-s_i) + q_i s_i + p_k(1-s_k) + q_k s_k\right], \tag{13}$$

where all of variables are the same as defined before.

The corresponding binary variables u_{ik}, v_{ik}, s_i and s_k have 16 combinations shown in Table 1. From the basis of Proposition 1 the proposed model can be formulated as follows:

Proposed model:

Minimize xy (14)

subject to

constraints (10)–(13), (15)

$$y \geq y'_i + \frac{1}{2}\left[p_i(1-s_i) + q_i s_i\right], \quad i = 1,2,\ldots,N, \tag{16}$$

$$x \geq x'_i + \frac{1}{2}\left[p_i s_i + q_i(1-s_i)\right], \quad i = 1,2,\ldots,N, \tag{17}$$

Table 2
Size of non-overlapping constraints and variables

	Chen et al.'s model	Proposed model
No. of 0-1 variables	$2n^2 - n$	n^2
No. of constraints	$\frac{5}{2}n(n-1)+4n$	$2n(n-1)+4n$

$$x_i' - \frac{1}{2}[p_i s_i + q_i(1-s_i)] \geq 0, \quad i = 1,2,\ldots,N, \tag{18}$$

$$y_i' - \frac{1}{2}[p_i(1-s_i) + q_i s_i] \geq 0, \quad i = 1,2,\ldots,N. \tag{19}$$

The number of variables and constraints used in Chen et al.'s model ((1)–(9)) and the proposed model ((15)–(19)) is listed in Table 2. Table 2 shows that the proposed model uses less 0-1 variables to reformulate the non-overlapping constraints.

3. Linear strategies

This section aims at linearizing the polynomial term xy appear in the objective function of (14). Consider two bounded variables x, y where $0 \leq x \leq \bar{x}$ and $0 \leq y \leq \bar{y}$, $\bar{x}$ and $\bar{y}$ are constants. x and y can be represented as follows:

$$x = \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1}\theta_g + \varepsilon_x, \tag{20}$$

$$y = \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1}\delta_h + \varepsilon_y, \tag{21}$$

where ε_x and ε_y are small positive variables. $\bar{\varepsilon}_x$ and $\bar{\varepsilon}_y$ are the pre-specified constants which are the upper bounds of ε_x and ε_y respectively. θ and δ are 0-1 variables, and G, H are integers which denote the number of required 0-1 variables for representing x or y.

Proposition 2. *Referring to* (20)–(21), *a polynomial term xy is represented as*

$$\begin{aligned} xy &= \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1}\theta_g y + \varepsilon_x y \\ &= \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1}\theta_g y + \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1}\delta_h \varepsilon_x + \varepsilon_x \varepsilon_y. \end{aligned} \tag{22}$$

Let e_{xy} as a linear approximation of $\varepsilon_x \varepsilon_y$, expressed as

$$e_{xy} = \frac{1}{2}\left(\bar{\varepsilon}_x \varepsilon_y + \bar{\varepsilon}_y \varepsilon_x\right). \tag{23}$$

The error of approximating $\varepsilon_x \varepsilon_y$ is then computed as

$$0 \leq e_{xy} - \varepsilon_x \varepsilon_y = \frac{1}{2}\left(\bar{\varepsilon}_x \varepsilon_y + \bar{\varepsilon}_y \varepsilon_x\right) - \varepsilon_x \varepsilon_y \leq \frac{\bar{\varepsilon}_x \bar{\varepsilon}_y}{4}.$$

The maximal difference between e_{xy} and $\varepsilon_x \varepsilon_y$ is $\bar{\varepsilon}_x \bar{\varepsilon}_y/4$, which occurs at

$$\varepsilon_x = \frac{\bar{\varepsilon}_x}{2} \text{ and } \varepsilon_y = \frac{\bar{\varepsilon}_y}{2}.$$

Substitute $\varepsilon_x \varepsilon_y$ in (22) by e_{xy} in (23), the polynomial term xy in (22) with $0 \le x \le \bar{x}$ and $0 \le y \le \bar{y}$ can be approximately linearized as

$$\widetilde{xy} = \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1} \theta_g y + \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1} \delta_h \varepsilon_x + \frac{1}{2}\left(\bar{\varepsilon}_x \varepsilon_y + \bar{\varepsilon}_y \varepsilon_x\right), \tag{24}$$

where $0 \le \widetilde{xy} - xy \le \bar{\varepsilon}_x \bar{\varepsilon}_y/4$.

The term $\theta_g y$ and $\delta_h \varepsilon_x$ in (24) can be fully linearized based on Proposition 3 below

Proposition 3. *An optimization problem of {Minimize θy, where $\theta \in (0,1)$, $0 \le y \le \bar{y}$, $y \in F$ (a feasible set)} can be linearized as*

Minimize z

subject to

$$y + \bar{y}(\theta - 1) \le z,$$
$$z \ge 0, \quad \theta \in (0,1),$$
$$y \in F.$$

From the basis of Proposition 1 to Proposition 3, the proposed model of (14)–(19) can be linearized as follows:

$$\text{Minimize } \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1} z_g + \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1} u_h$$

subject to

$$z_g \ge y + \bar{y}(\theta_g - 1), \quad g = 1,2,\ldots,G,$$
$$z_g \ge 0$$
$$u_h \ge \varepsilon_x + \bar{\varepsilon}_x(\delta_h - 1), \quad h = 1,2,\ldots,H,$$
$$u_h \ge 0,$$
$$(15)\text{–}(19),$$
$$\theta_g, \delta_h \in (0,1).$$

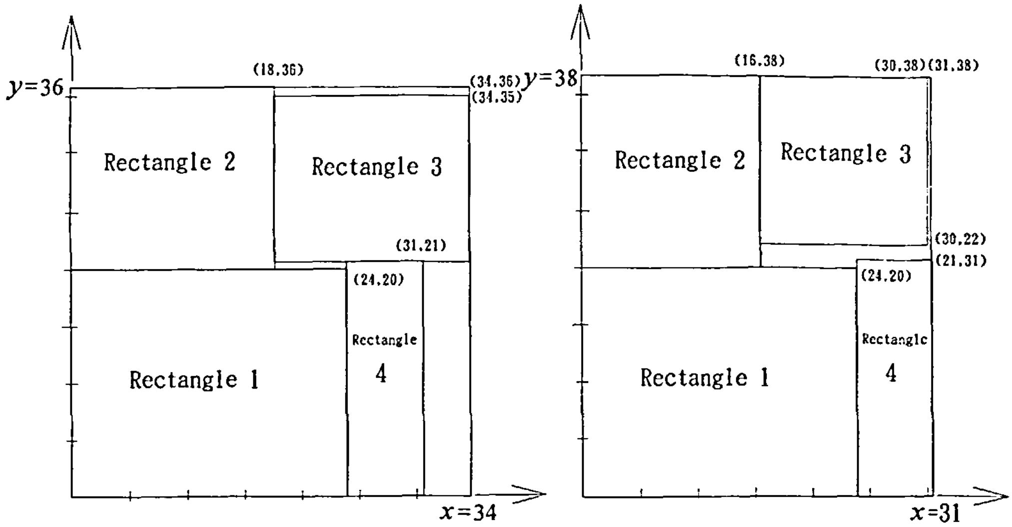
4. Examples

Consider the following assortment optimization problems adopted from Chen et al. [1]: The sizes of pieces of rectangles are given in Table 3.

Minimize xy

subject to

$$(1)\text{–}(9).$$

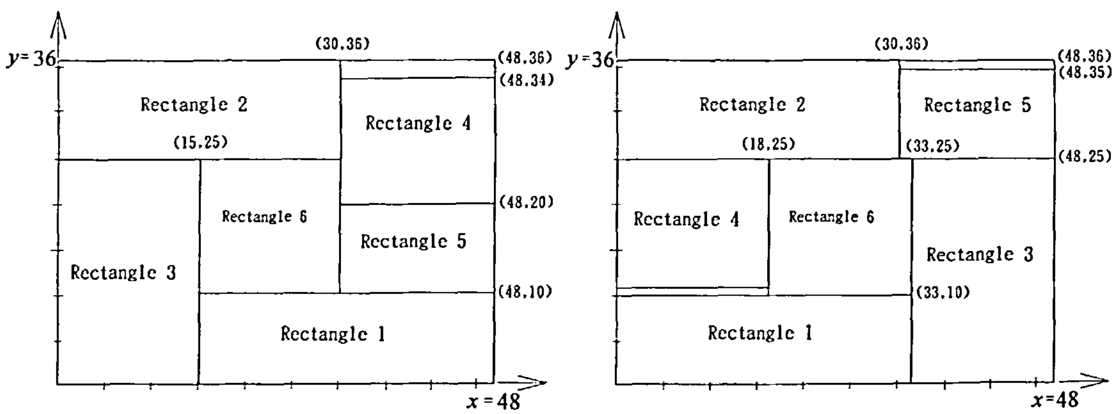


(a) Result of Chen et al. model (b) Result of proposed model

Fig. 4. Solutions of Problem 1.

Table 3
Sample data

Problem number	Number of rectangles	p_i	q_i
1	4	24	20
		18	16
		16	14
		21	7
2	6	33	10
		30	11
		25	15
		18	14
		18	10
		15	15



(a) Result of Chen et al. model (b) Result of proposed model

Fig. 5. Solutions of Problem 2.

Table 4
Computational comparison of two models

Problem number	Chen et al.'s model		Proposed model	
	No. of 0-1 variables in non-overlapping constraints	Objective value	No. of 0-1 variables in non-overlapping constraints	Objective value
1	28	1224	16	1178
2	58	1728	36	1728

Chen et al. treated Problem 1 by fixing the value of y as $y = 36$, then they solved a linear mixed 0-1 program to obtain the solution depicted in Fig. 4(a) with an objective value that equals 1224. Similarly, Chen et al. treated Problem 2 by assuming $y = 36$, then they solved the problem to obtain the results depicted in Fig. 5(a). By specifying $\bar{\varepsilon}_x = \bar{\varepsilon}_y = 0.1$, the proposed model solves these two problems by LINDO [4] on an IBM-PC/AT 586 to find the solution shown in Fig. 4(b) and Fig. 5(b) in which we obtained an approximately global optimal solution.

Table 4 is the computational comparison of two models, which demonstrates that

1. the proposed model is guaranteed to find a approximately global optimal solution;
2. the proposed model can solve an optimal program directly, while Chen et al.'s model needs to assign (or to guess) the value of y; and
3. the proposed model uses less number of 0-1 variables to formulate the non-overlapping constraints.

5. Conclusion

This paper proposes a new method to solve assortment optimization problems. By approximately linearizing the polynomial objective function, the proposed method with linear strategies can reach a solution which is close to the globally optimal solution. In addition, less 0-1 variables are used to reformulate the non-overlapping constraints of assortment optimization problems.

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