9 ERSOY, O.K., and NOUIRA, A.: 'Image coding with the discrete cosine-III transform', *IEEE J. Sel. Areas Commun.*, 1992, **10**, (5), pp. 884–891

10 BI, GUOAN, and YU, L.W.: 'DCT algorithms for composite sequence lengths', to be published in *IEEE Trans. Signal Process.*

Multigrid computation for curve fitting in the wavelet domain

Wen-Jen Ho and Wen-Thong Chang

A multi-layer multigrid algorithm for curve fitting in the wavelet domain is presented. This algorithm is achieved by applying a wavelet transform to each grid of the conventional multigrid structure. Using a wavelet transform, the convergent rate in each grid is improved and the total system can converge more quickly.

Introduction: Curve fitting is widely used to derive a smooth curve from some known control points. The procedure of curve fitting commonly consists of three steps: the first step is to specify the known control points $[x_k, y_k]$; the second step is to calculate the smooth curve $[x(t), y(t)]$ to fit these control points; the third step is to sample the resultant curve for rendition. In this Letter, we emphasise the second step and discuss a fast algorithm. In curve fitting, the desired curve $[x(t), y(t)]$ is usually confined by an energy function E such as

$$E = \int [(x''(t))^2 + (y''(t))^2]\,dt + \beta \sum_{k\in C} [(x(t_k) - x_k)^2 + (y(t_k) - y_k)^2] \quad (1)$$

$$= \left[\int [x''(t)]^2 dt + \beta \sum_{k\in C} (x(t_k) - x_k)^2\right] + \left[\int [y''(t)]^2 dt + \beta \sum_{k\in C} (y(t_k) - y_k)^2\right] \quad (2)$$

$$= E_x + E_y \quad (3)$$

where $x''(t)$ denotes the second derivative of $x(t)$ with respect to t. C is the set of control points. The first term of eqn. 1 specifies how smooth the curve is; the second term specifies how well a curve fits the control points $[x_k, y_k]$. The parameter β determines the relative strength between these two factors. The function E considers both the x and y directions simultaneously, where E_x confines the curve along the x direction and E_y along the y direction. Since both of the terms are non-negative and independent, minimisation of E can be carried out by minimisation of E_x and E_y, separately. Since E_x and E_y, have the same form, only the minimisation of E_x is considered. The same method can be applied to the minimisation of E_y.

A parametric method has been proposed in [1] to discretise the form of eqn. 1. Suppose N bases are considered to expand the curve and that the cubic spline $\phi(t)$ is chosen as the basis function; that is, $x(t) = \Sigma_{i=0}^{N-1} v_i\phi(t - i)$. Substitution of this form into E_x will lead to the quadratic energy function:

$$\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v} - \mathbf{v}^T\mathbf{b} + c \quad (4)$$

where $\mathbf{v}$ is a column vector containing the variables v_i to be solved and $\mathbf{A}$ is a real symmetric matrix called the stiffness matrix, $\mathbf{b}$ and c are the associated column vector and constant. According to the Euler-Lagrange formula, optimisation of this quadratic function results in a linear equation system:

$$\mathbf{A}\mathbf{v} = \mathbf{b} \quad (5)$$

That is, the solution $x(t)$ of the curve fitting problem can be derived by a linear system. In this Letter, we present an efficient method to solve such a linear system, especially when $\mathbf{A}$ is sparse due to the finite support of the basis.

Multi-resolution wavelet transform: The basic idea of our method is to apply the multi-resolution wavelet transform [2] to the conventional multigrid algorithm [3] when solving the linear system. The purpose of such a transform is to increase the connectivity among the elements in $\mathbf{v}$. Since the signal in one-layer of the multigrid is a lowpass version of its upper layer signal, the application of the transform to each layer can be implemented with a tree-structured filter bank. With a tree-structured transform, the vector $\mathbf{v}$ is transformed into its wavelet components $\{\mathbf{v}_J, \mathbf{w}_{j,(j=1\sim J)}\}$, where the signal $\mathbf{v}_J$ is the low frequency component and the details $\mathbf{w}_{j,(j=1\sim J)}$ are the high frequency components. We denote the discrete wavelet transform as $\mathbf{v} = \mathbf{R}\tilde{\mathbf{v}}$, with $\tilde{\mathbf{v}}$ containing the wavelet components $\{\mathbf{v}_J, \mathbf{w}_{j,(j=1\sim J)}\}$. The matrix $\mathbf{R}$ is the QMF matrix describing the synthesis filtering in the QMF structure. With this transform, the quadratic energy function in eqn. 4 can be rewritten as $\frac{1}{2}(\mathbf{R}\tilde{\mathbf{v}})^T \mathbf{A}(\mathbf{R}\tilde{\mathbf{v}}) - (\mathbf{R}\tilde{\mathbf{v}})^T \mathbf{b} + c$ and the subsequent linear system will be

$$\tilde{\mathbf{A}}\tilde{\mathbf{v}} = \tilde{\mathbf{b}} \quad (6)$$

where $\tilde{\mathbf{A}} \triangleq \mathbf{R}^T\mathbf{A}\mathbf{R}$ and $\tilde{\mathbf{b}} \triangleq \mathbf{R}^T\mathbf{b}$. The vector $\mathbf{b}$ is decomposed into $\{\mathbf{b}_J, \mathbf{z}_{j,(j=1\sim J)}\}$.

Multigrid computation: Consider the case of a two-grid algorithm with $J = 1$. Eqn. 6 can be described as the following block matrix form:

$$\begin{bmatrix} \tilde{\mathbf{A}}_{V_1V_1} & \tilde{\mathbf{A}}_{V_1W_1} \\ \tilde{\mathbf{A}}_{W_1V_1} & \tilde{\mathbf{A}}_{W_1W_1} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{w}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{z}_1 \end{bmatrix} \quad (7)$$

If the synthesis filters of the QMF filter banks are chosen as the decimation and interpolation filters in the conventional multigrid, then the submatrix equation:

$$\tilde{\mathbf{A}}_{V_1V_1}\mathbf{v}_1 = \mathbf{b}_1 \quad (8)$$

directly implements the desired coarse grid equation in the conventional two-grid algorithm. It can be seen that in the wavelet domain, the conventional coarser grid structure is already inherent in the matrix equation eqn. 7. From this fact, we know that eqn. 7 possesses a hierarchical structure suitable for two-grid implementation. Both eqns. 7 and 8 implement the two-layer multigrid in the wavelet domain. The desired solution can be obtained by transforming $\tilde{\mathbf{v}}$ back to $\mathbf{v}$ by $\mathbf{v} = \mathbf{R}\tilde{\mathbf{v}}$. The extension of the two-grid structure to a multigrid structure can be carried out by further splitting the submatrix $\tilde{\mathbf{A}}_{v_1v_1}$ using a tree-structured QMF with more than one stage. The advantage of such an approach is that the interpolation and decimation operation used in the multigrid transform can be done together with the wavelet transform. Also, with the use of a tree-structured QMF, the wavelet transform for each grid can be achieved with only one operation of the wavelet transform in the original finest grid. Thus, this structure successfully combines the advantages of the multi-resolution transform and the multigrid for best computational gain.

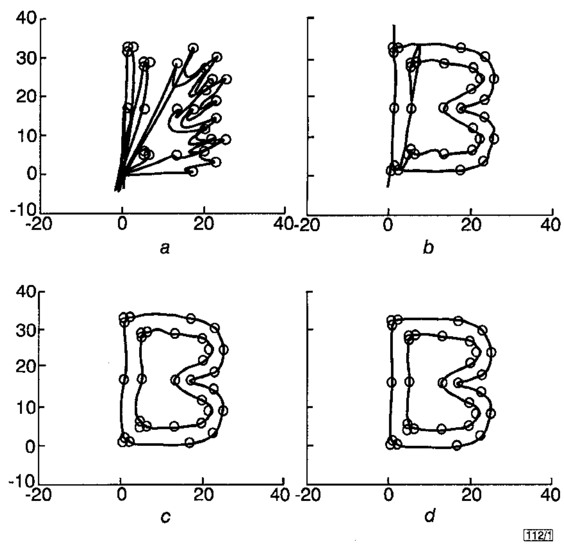

Fig. 1 *Curves generated with three different methods after 21 WUs*

a Gauss-Seidel method
b Multigrid method
c Our proposed method
d Final convergent curve

Experiment: One numerical experiment is shown to demonstrate the performance of the proposed algorithm with a three-stage tree-structured ($J = 3$) biorthogonal filter bank. The impulse responses of the filter bank are $h_0(z) = 1$, $h_1(z) = \frac{1}{4}(-1 + 2z - z^2)$, $g_0(z) = \frac{1}{4}(z^{-1} + 2 + z)$ and $g_1(z) = z^{-1}$, respectively. Filters h_0 and h_1 represent the analysis filters; filters g_0 and g_1 represent the synthesis filters. The value of β is set at 10 and the Gauss-Seidel is used as the iterative algorithm [4]. For the multigrid computation, the V-cycle computation strategy is used [3]. The computation complexity is normalised with respect to the number of non-zero elements in the system matrix. To consider the computation complexity, one *work unit* (WU) is defined as the cost of performing one iteration of eqn. 5 on the finest grid. For example, the number of non-zero elements of $\tilde{\mathbf{A}}$ with $J = 3$ is ~ 2.16 times that for $\mathbf{A}$ with $J = 0$. So, the cost of performing one iteration of $\tilde{\mathbf{A}}$ on the finest grid is 2.16 WUs. The curves generated with the proposed method after 21 WUs are shown in Fig. 1*c*. For comparison, the curves generated with eqn. 5 (single-grid) and with the conventional multigrid are also shown in Fig. 1*a* and *b*, respectively. Fig. 1*d* shows the final convergent curve (after 200 WUs with the proposed method). The shape of the curve depends on the placement of the control points

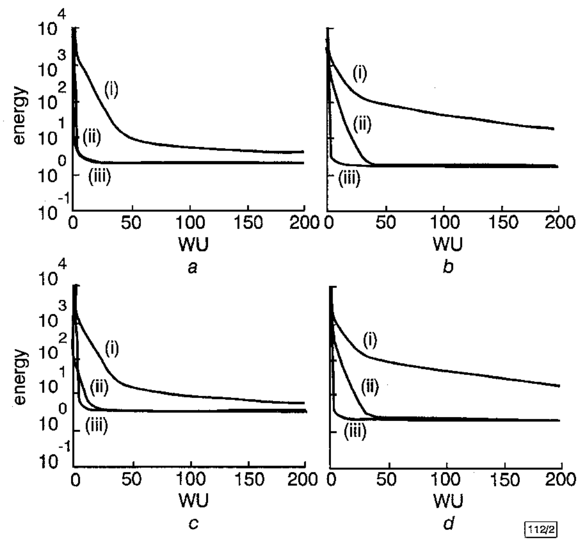

Fig. 2 *Convergence status corresponding to Fig. 1*

a Outer curve of character B in *x*-direction
b Outer curve of character B in *y*-direction
c Inside curve of character B in *x*-direction
d Inside curve of character B in *y*-direction
(i) Gauss-Seidel
(ii) multigrid
(iii) proposed method

and is beyond the scope of this Letter. This example clearly indicates that the convergence rate in solving the linear system can be improved by our proposed computation structure. The convergence status of the energy for these different methods are plotted in Fig. 2. In this Figure, the logarithm of the energy is used to denote the convergence status. As shown in Fig. 2, our proposed method performs better than the multigrid method, which in turn performs better than the single grid method.

Conclusion: A multi-resolution multi-layer algorithm is proposed to solve the linear equation for the curve fitting problem. The convergent rate improvement in generating the parametric curve has been shown. The algorithm is very general and can be applied to problems with a similar linear property.

22 December 1997
Electronics Letters Online No: 19980294

Wen-Jen Ho and Wen-Thong Chang (*Department of Communication Engineering, National Chiao Tung University, Hsinchu, Taiwan, Republic of China*)

E-mail: wtchang@cc.nctu.edu.tw

Corresponding author: Wen-Thong Chang

References

1 YAOU, M.H., and CHANG, W.T.: 'Fast surface interpolation using multiresolution wavelet transform', *IEEE Trans. Pattern Anal. Mach. Intell.*, 1994, **PAMI-16**, (7), pp. 673–688
2 COHEN, A., DAUBECHIES, I., and FEAUVEAU, J.-C.: 'Biorthogonal bases of compactly supported wavelets', *Commun. Pure Appl. Math.*, 1992, **45**, pp. 485–560
3 HACKBUSCH, W., and TROTTENBERG, U. (Eds.): 'Multigrid methods' (Springer-Verlag, New York, 1982)
4 HAGEMAN, L.A., and YOUNG, D.M.: 'Applied iterative methods' (Academic Press, New York, 1981)

Signal-adapted wavelet filter bank design

Susu Yao

A method for designing wavelet filter banks that are adapted to the given signal is proposed. The method is based on optimising a certain cost function with constraint conditions. Gradient-descent optimisation techniques are not adequate for the minimisation of such a cost function. Evolutionary programming is used to resolve this difficult optimisation problem. Simulation results are given.

Introduction: Wavelet-type multiresolution transforms have been introduced recently in digital speech and image coding [1]. The wavelet transform of a given signal may be interpreted as the decomposition of a signal into a set of frequency channels which have equal bandwidth on a logarithmic scale. The wavelet decomposition can be realised using orthonormal multirate filter banks [2, 3]. This Letter is mainly concerned with the problem of designing optimal wavelet filter banks that are adapted to the given signal, in the sense that they maximise the energy of the projection of the signal on the low frequency band. In other words, this problem is based on optimising a certain cost function with the constraint conditions. Because the cost function is not strictly convex or concave, generally having a number of local minima, it is difficult to use gradient-descent optimisation techniques to minimise the cost function. To resolve this problem, a kind of guided random technique called evolutionary programming is used in our approach. The design method of orthogonal wavelet filter banks and results are presented in this Letter.

Discrete orthogonal wavelet filter banks: Discrete orthogonal compactly supported wavelets of support size equal to or less than N, where N is an integer, are completely characterised by a lowpass filter H with impulse response $\{h_k\}$ $(-(N/2 \le k \le N/2))$ and highpass filter G with impulse response $\{g_k\}$ $(-(N/2 + 1 \le k \le N/2))$. The impulse response of filter G is related to the impulse response of filter H by $g_k = (-1)^{1-k} h_{1-k}$

Let $H(\omega)$ and $G(\omega)$ be the Fourier transform of $\{h_k\}$ and $\{g_k\}$, respectively. $H(\omega)$ is defined by

$$H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-j\omega k} \tag{1}$$

and G is the quadrature mirror filter of H, $G(\omega) = e^{-j\omega}H(\omega+\pi)$. They satisfy the following two orthogonality conditions:

$$|H(0)| = 1 \qquad |H(\pi)| = 0 \tag{2}$$

and

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1 \tag{3}$$

or equally

$$\sum_k h_{2k} = \sum_k h_{2K+1} = \frac{\sqrt{2}}{2} \tag{4}$$

$$\sum_k h_{k-2n} h_{k-2m} = \delta_{n,m} \qquad \sum_k h_{k-2n} g_{k-2m} = 0 \tag{5}$$

Any square integrable signal f can be decomposed into approximation and detail signals in multiresolution analysis space, as introduced by Mallat [2]. Assuming that $A_{j,k}f$ represents the approximation signal of the original signal at resolution j, the