Optimal Control Policies for a Periodic Review Inventory System with Emergency Orders

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Received August 1996; revised September 1997; accepted 9 September 1997

Abstract: We describe a periodic review inventory system where emergency orders, which have a shorter supply lead time but are subject to higher ordering cost compared to regular orders, can be placed on a continuous basis. We consider the periodic review system in which the order cycles are relatively long so that they are possibly larger than the supply lead times. Study of such systems is important since they are often found in practice. We assume that the difference between the regular and emergency supply lead times is less than the order-cycle length. We develop a dynamic programming model and derive a stopping rule to end the computation and obtain optimal operation parameters. Computational results are included that support the contention that easily implemented policies can be computed with reasonable effort. © 1998 John Wiley & Sons, Inc. Naval Research Logistics **45:** 187–204, 1998

1. INTRODUCTION

Most inventory models assume the use of a single supply mode. However, many companies do use alternative resupply modes. For example, a retailer could choose to replenish the inventory of an item under review by a fast resupply mode (by air, for example) if the inventory position of the item is dangerously low. In this paper, we study a periodic review inventory system in which there are two resupply modes: namely, a regular mode and an emergency mode. Orders placed via the emergency mode, compared to orders placed via the regular mode, have a shorter lead time but are subject to higher ordering cost.

Several studies in the literature address this problem. These studies divide into two groups: policy-evaluation studies and policy-optimization studies. The former assume a particular policy form and devise methods for evaluating it, while the latter compute the true optimal policy and solve specific instances of the problem under consideration. While the optimization results are stronger, the policy-evaluation studies typically use simpler policies and broader assumptions. This paper contributes to the optimization literature, and the model here is more general than those of early studies.

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 CCC 0894-069X/98/020187-18

The earlier works in this area are policy-optimization studies. Barankin [2] develops an one-product single-period inventory model where two supply options are available with fixed time lag of one and zero periods, respectively. Daniel [7], Neuts [15], Bulinskaya [3], Fukuda [8], and Veinott [18] extend the single-period analysis to the n-period case and derive similar forms of the optimal policy. Wright [20] further extends the analysis to the inventory system with multiple products. However, they all restrict attention to the case where the expedited and regular supply lead times differ by one period only. Whittmore and Saunders [19] analyze the n-period inventory problem by allowing the expedited and regular lead times to be of arbitrary lengths. Unfortunately, the form of the optimal policy they derive is extremely complex, relying on the use and solution of a multidimensional dynamic program. They are able to obtain explicit results only for the case where the expedited and regular lead times differ by one period only. All of these optimization studies on periodic review systems have focused on the situation where supply lead times are a multiple of a review period. Such models could be regarded as an approximation of continuous review inventory models, as the review periods can be modeled as small as, say, 1 working day. Different from these works, the periodic model we consider here allows the review periods to be larger than the supply lead times. Moreover, our model places virtually no restrictions on the difference between the expedited and regular lead times while developing easy-to-compute optimal policies.

Later works in this area are policy-evaluation studies. Moinzadeh and Nahmias [13] propose a heuristic policy which is of the reorder-level–order-quantity type, i.e., the (s_1, Q_1) type, except that an expedited order Q_2 is placed if on-hand inventory reduces to an expediting level s_2. Related, but somewhat different models, have also been considered by Hadley and Whitin [10], Allen and D'Esopo [1], Gross and Soriano [9], Rosenshine and Obee [16], and Moinzadeh and Schmidt [14].

In this paper, we consider a periodic review system where the review periods are relatively long such that they are possibly larger than the supply lead times. To the best of our knowledge, the utilization of emergency shipments for this class of problems has not been studied previously. However, this class of problems is important since periodic systems with relatively long review periods are commonly used in practice (see, e.g., Hax and Candea [11]). For example, a retailer may place regular replenishment orders every 2 weeks while the supply lead time is of the order of 1 week. There are several possible reasons for long review periods. One is to avoid large costs of reviewing, i.e., costs of aggregating inventory transactions, counting inventory, and so on. Another is to achieve economies (i.e., save on ordering and transportation costs) in the coordination and consolidation of orders for different items, which is particularly true if several items are purchased from the same source. In this case, the buyer typically determines first the length of review periods based on the nature of various products, the number of items, and the economies' consideration, and then decides the order size of each item. For example, Hotai Motor Co. Ltd., the distributor of Toyota products in Taiwan (Toyota Corolla and Camry are currently the best selling import cars), establishes its weekly ordering of about 15,000 auto parts from Toyota Motor Company in Japan. There may also exist some other practical and organizational considerations. For example, the housewares department in a department store often places regular orders at fixed epochs when a housewares distributor visits the department weekly and counts all the items it supplies to the department [4]. Also, one reason why Hotai Motor Co. Ltd. places regular orders weekly is to comply with the supplier's Just-In-Time (JIT) system. In this paper, as in much of the periodic-review literature, we assume that such considerations are handled outside the model, i.e., we will not be concerned with

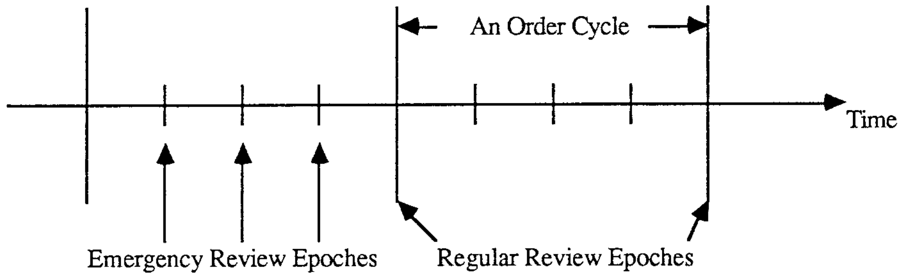

Figure 1. An order cycle with four periods.

the reviewing costs or how the coordination of orders for different items is accomplished. We simply assume that the length of review periods has already been determined by the organization and a regular order is placed after (and only after) every review. Such an operating policy, known as a replenishment cycle system, is in common use (see, e.g., Silver and Peterson [17]). As we consider review periods that are long, we will rename them *order cycles* thereafter.

We assume that at a regular review epoch, an emergency order could be placed in addition to a regular order if justified by an urgent situation. In addition, since the order cycles are long, if an urgent situation should arise at any time between two regular review epochs, an emergency order can be placed immediately to have the inventory replenished and thus avoid any possible stockouts that might occur if replenishment is postponed until a regular review epoch. With the existing computer storage and updating capability, the number of applications in which the inventory position for each item is virtually known at any time increases significantly. To develop our model, we hypothesize that there are a number of *periods* (to be somewhat consistent with the literature), which are of length 1 working day, within an order cycle. Regular orders are placed at regular review epochs while emergency orders could be placed at both regular review epochs and period review epochs (called emergency review epochs, see Figure 1). This kind of inventory control system is common in the import auto industry in Taiwan. For example, Hotai Motor Co. Ltd. mentioned above replenishes the inventory of auto parts by ship as well as by air. In the former case, it weekly orders thousands of auto parts (there is an order-up-to level for each part) from Toyota Motor Company in Japan. In the latter case, if the inventory level of a part (usually an A item) falls below a warning point, it places an emergency order (whose lead time is about one week). Notice that, in Chiang and Gutierrez [6], a periodic system with long-order cycles is also considered. However, emergency orders as well as regular orders are placed only at regular review epochs due to the fact that inventory position of each item is not known at any time between two regular reviews. Also, Chiang and Gutierrez [6] considers only the case in which emergency orders have same variable costs (as the regular orders) but larger fixed order costs.

Although our model is closely related to the one in Moinzadeh and Nahmias [13], there are important differences between these two models. From an operational point of view, both models allow the placement of emergency orders at any time. However, while Moinzadeh and Nahmias' continuous review model places regular orders on continuous time, our model places regular orders only on periodic epochs (due to the fact of coordinated ordering or purchasing from the same source). No model subsumes the other. They simply describe different operating systems used in practice. From a theoretical point of view, while Moinzadeh and Nahmias evaluate an assumed policy, this research derives the form of the optimal policy and algorithms for computing it.

We analyze the problem within the framework of a stochastic dynamic program. We assume that the difference between the regular and emergency supply lead times is larger than one period (for the case in which the difference between the two lead times is exactly one period, see Chiang [5]), and that a larger unit item cost is associated with the use of the emergency supply channel. It is possible that emergency orders have both fixed order costs and larger variable costs. This paper, however, treats only a special case; i.e., we assume that emergency fixed costs are negligible. We will develop a recursive formulation for the finite horizon problem, and derive conditions to verify the convergence of the regular and emergency ordering policies as the planning horizon is extended. Hence, the ordering rules to which these policies respectively converge are optimal for the infinite horizon model.

The optimal control policy derived is a natural extension of the optimal order-up-to-R ordering policy for the ordinary one-supply-mode model. As we shall see, the optimal operation parameters are obtained once a stopping condition is satisfied. Thus, the optimal control policy is easy to compute.

2. A DYNAMIC PROGRAMMING MODEL

Assume that there are two supply modes available: namely, a regular mode and an emergency mode. The unit item costs for the regular and emergency supply modes are c_1 and c_0 respectively, where $c_1 < c_0$. Let m be the number of periods within an order cycle. We assume that the difference between the regular and emergency supply lead times is smaller than m periods but larger than one period. For notational simplicity, we assume that the emergency and regular supply lead times are one and τ periods respectively, where $3 \le \tau \le m$. Assume that all demand which is not filled immediately is backlogged. There are inventory holding and shortage costs which are charged based on the net inventory (i.e., inventory on hand minus backorder) at the end of each period according to a loss function $\ell(\cdot)$. Let $\Phi(t)$ be the probability distribution function for demand t during a period with mean μ. Demand is assumed to be nonnegative and independently distributed in disjoint time intervals. Let $a \vee b \equiv \max\{a, b\}$.

Suppose the net inventory at the beginning of a period is x; then the expected holding and shortage costs incurred in that period are given by

$$L(x) = \int_0^\infty \ell(x - t)\, d\Phi(t). \tag{1}$$

Other functional forms of $L(x)$ are allowed; however, for our analysis we need $L(x)$ to be a convex and differentiable function. Define $V_{i,j}(x, y)$ as the expected discounted cost with i order cycles and j periods remaining ($0 \le j \le m - 1$) until the end of the planning horizon, when the starting net inventory is x, inventory on order is y ($y \ge 0$), and an optimal ordering policy is used at every review epoch. For simplicity of formulation, only the costs affected by the decision made at the current epoch are included in $V_{i,j}(x, y)$. As inventory on order is zero when there are less than $m - \tau + 1$ periods remaining, $V_{i,j}(x, y)$ simplifies to $V_{i,j}(x, 0)$ for $j = 0, 1, \ldots, m - \tau$. Then, $V_{i,j}(x, y)$ satisfies the functional equation

$$V_{i,0}(x, 0) = \min_{x \le r, z \ge 0} \{c_0 r + c_1 z + \alpha E_t L(r - t) + \alpha E_t V_{i-1,m-1}(r - t, z)\} - c_0 x, \quad (2)$$

$$V_{i,j}(x, y) = \min_{x \le r} \{c_0 r + \alpha E_t L(r - t) + \alpha E_t V_{i,j-1}(r - t, y)\} - c_0 x,$$
$$j = 1, \ldots, m - 1 \quad \text{and} \quad j \ne m - \tau + 1, \quad (3)$$

$$V_{i,m-\tau+1}(x, y) = \min_{x \le r} \{c_0 r + \alpha E_t L(y + r - t) + \alpha E_t V_{i,m-\tau}(y + r - t, 0)\} - c_0 x, \quad (4)$$

where $V_{0,0}(x, 0) \equiv 0$, α $(0 < \alpha < 1)$ is the discount factor, r is the inventory position after a possible emergency order is placed at a review epoch, and z is the quantity ordered via the regular supply mode at a regular review epoch which becomes inventory on order thereafter. As we see, obtaining $V_{i,0}(x, 0)$ requires computations of the order of $O(imX^3)$, where X is the domain of the state x. Define

$$J_i(r) = \min_{z \ge 0} \{J_{i,0}(r, z)\}, \quad (5)$$

where

$$J_{i,0}(r, z) = c_1 z + \alpha E_t V_{i-1,m-1}(r - t, z). \quad (6)$$

Then (2) is expressed as

$$V_{i,0}(x, 0) = \min_{x \le r} \{c_0 r + \alpha E_t L(r - t) + \min\nolimits_{z \ge 0} \{J_{i,0}(r, z)\}\} - c_0 x$$
$$= \min_{x \le r} \{c_0 r + \alpha E_t L(r - t) + J_i(r)\} - c_0 x. \quad (7)$$

Define

$$G_{i,0}(r) = c_0 r + \alpha E_t L(r - t) + J_i(r), \quad (8)$$

$$G_{i,j}(r, y) = c_0 r + \alpha E_t L(r - t) + \alpha E_t V_{i,j-1}(r - t, y),$$
$$j = 1, \ldots, m - 1 \quad \text{and} \quad j \ne m - \tau + 1, \quad (9)$$

$$G_{i,m-\tau+1}(r, y) = c_0 r + \alpha E_t L(y + r - t) + \alpha E_t V_{i,m-\tau}(y + r - t, 0). \quad (10)$$

[Note that as there is no inventory on order after the regular order arrives, $G_{i,j}(r, y)$ simplifies to $G_{i,j}(r, 0)$ for $j = 1, \ldots, m - \tau$.] Then (7), (3), and (4) can be rewritten as

$$V_{i,0}(x, 0) = \min_{x \le r} \{G_{i,0}(r)\} - c_0 x, \quad (11)$$

$$V_{i,j}(x, y) = \min_{x \le r} \{G_{i,j}(r, y)\} - c_0 x, \qquad j = 1, \ldots, m - 1. \quad (12)$$

We show in the following lemma that the cost function $V_{i,j}(x, y)$ is convex.

LEMMA 1: $V_{i,j}(x, y)$ for each (i, j) is a convex function.

PROOF: See the Appendix.

Let $r_{i,j}$ be the (smallest) value minimizing $G_{i,j}$ as given in expressions (8)–(10). For $j = m - \tau + 1, \ldots, m - 1$, $r_{i,j}$ [written as $r_{i,j}(y)$ if there is ambiguity] is a function of y, i.e., the optimal emergency order-up-to level is a function of the inventory on order. Also, let $z_i(r)$ be the (smallest) value of nonnegative z that minimizes $J_{i,0}(r, z)$ for a given r, i.e., the optimal regular order quantity at a regular review epoch is a function of the inventory position after a possible emergency order is placed at that epoch. Then the optimal policy at a regular review epoch is (i) order up to $r_{i,0}$ at unit cost c_0 if $x < r_{i,0}$ and (ii) order the amount $z_i(x \vee r_{i,0})$ at unit cost c_1; and the optimal policy at an emergency review epoch is (i) order up to $r_{i,j}$ at unit cost c_0 if $x < r_{i,j}$ and (ii) do not order if $x \geq r_{i,j}$.

3. PROPERTIES OF THE MODEL

In this section, we present important properties for the model developed in Section 2. We first introduce some observations that will be used extensively to establish the results of this section. Denote by Df the first derivative of the function f and by $D_i f$ the first derivative of the function f with respect to its ith variable. It follows from (11), (12), and the definition of $r_{i,j}$ that $D_1V_{i,j}(x, y) = -c_0$ for $x \leq r_{i,j}$, and $D_1V_{i,j}(x, y) \geq -c_0$ for all x. Also, as we see from (12), $V_{i,j}(x, y)$, for $j = m - \tau + 1, \ldots, m - 1$, can be written as

$$V_{i,j}(x, y) = G_{i,j}(x \vee r_{i,j}(y), y) - c_0 x, \qquad j = m - \tau + 1, \ldots, m - 1. \tag{13}$$

Consequently, we have that for $x \leq r_{i,j}(y)$, $D_2V_{i,j}(x, y) = D_2G_{i,j}(r_{i,j}(y), y) + D_1G_{i,j}(r_{i,j}(y), y) \cdot Dr_{i,j}(y) = D_2G_{i,j}(r_{i,j}(y), y)$ since $D_1G_{i,j}(r_{i,j}(y), y) = 0$ [$r_{i,j}(y)$ minimizes $G_{i,j}(r, y)$ by definition]. For $x \geq r_{i,j}(y)$, $D_2V_{i,j}(x, y) = D_2G_{i,j}(x, y)$.

We show in Theorem 1 that the emergency order-up-to levels within an order cycle, which can be separated into two groups (if $\tau < m$) according to whether or not there is inventory on order, are nondecreasing in each group.

THEOREM 1: $r_{i,m-\tau} \geq r_{i,m-\tau-1} \geq \cdots \geq r_{i,1}$ and $r_{i,m-1}(y) \geq r_{i,m-2}(y) \geq \cdots \geq r_{i,m-\tau+1}(y)$.

PROOF: See the Appendix.

We next show that the emergency order-up-to level $r_{i,m-\tau+1}$ is a linearly decreasing function of inventory on order y. Let Δ denote a positive real number.

THEOREM 2: $r_{i,m-\tau+1}(y)$ is decreasing in y. In addition, $r_{i,m-\tau+1}(y) = r_{i,m-\tau+1}(0) - y$.

PROOF: See the Appendix.

Theorem 2 confirms our intuition that we shall be less willing to use the emergency mode at the current review epoch if the inventory on order we will receive at the next epoch is larger. Next, we show that emergency order-up-to levels $r_{i,j}(y)$, $j = m - \tau + 2, m - \tau +$

3, . . . , $m - 1$, are nonincreasing in inventory on order, and that if they decrease as inventory on order increases, they decrease by an amount that is less than or equal to the amount by which inventory on order increases. We first state an important preliminary lemma.

LEMMA 2: The mixed second derivative of $V_{i,m-\tau+1}(x, y)$ is nonnegative, i.e., $D_1V_{i,m-\tau+1}(x, y) \le D_1V_{i,m-\tau+1}(x, y + \Delta)$ [or $D_2V_{i,m-\tau+1}(x - \Delta, y) \le D_2V_{i,m-\tau+1}(x, y)$]. In addition, $D_1V_{i,m-\tau+1}(x - \Delta, y + \Delta) = D_1V_{i,m-\tau+1}(x, y) = D_2V_{i,m-\tau+1}(x, y) = D_2V_{i,m-\tau+1}(x - \Delta, y + \Delta)$.

PROOF: See the Appendix.

THEOREM 3: $r_{i,m-\tau+2}(y)$ is nonincreasing in y, i.e., $r_{i,m-\tau+2}(y) \ge r_{i,m-\tau+2}(y + \Delta)$. Moreover, $r_{i,m-\tau+2}(y) - r_{i,m-\tau+2}(y + \Delta) \le \Delta$.

PROOF: See the Appendix.

LEMMA 3: The mixed second derivative of $V_{i,m-\tau+2}(x, y)$ is nonnegative, i.e., $D_1V_{i,m-\tau+2}(x, y) \le D_1V_{i,m-\tau+2}(x, y + \Delta)$ [or $D_2V_{i,m-\tau+2}(x - \Delta, y) \le D_2V_{i,m-\tau+2}(x, y)$]. In addition, $D_1V_{i,m-\tau+2}(x - \Delta, y + \Delta) \le D_1V_{i,m-\tau+2}(x, y)$ and $D_2V_{i,m-\tau+2}(x, y) \le D_2V_{i,m-\tau+2}(x - \Delta, y + \Delta)$.

PROOF: See the Appendix.

COROLLARY: For $j = m - \tau + 3, \ldots, m - 1$, $r_{i,j}(y) \ge r_{i,j}(y + \Delta) \ge r_{i,j}(y) - \Delta$. Moreover, $D_1V_{i,j}(x - \Delta, y + \Delta) \le D_1V_{i,j}(x, y) \le D_1V_{i,j}(x, y + \Delta)$ and $D_2V_{i,j}(x - \Delta, y) \le D_2V_{i,j}(x, y) \le D_2V_{i,j}(x - \Delta, y + \Delta)$.

Next, we show that the quantity we order via the regular supply mode at a regular review epoch is nonincreasing in the starting net inventory (or the inventory position after a possible emergency order is placed at that epoch). This conforms the intuition that we shall order less if we start with more inventory. Moreover, if the quantity ordered via the regular mode decreases as the starting net inventory (or the inventory position after a possible emergency order) increases, it decreases by an amount that is less than or equal to the amount by which the starting net inventory increases.

THEOREM 4: $z_i(x \vee r_{i,0})$ is nonincreasing in x, i.e., $z_i(r) \le z_i(r - \Delta)$. In addition, $z_i(r - \Delta) - \Delta \le z_i(r)$.

PROOF: See the Appendix.

To illustrate, consider the example: $m = 10$ periods, $\tau = 6$ periods, $c_1 = \$10$, $c_0 = \$15$, $\mu = 2$ (with Poisson demand), $\ell(x) = .01x$ for $x > 0$ and $\ell(x) = -20x$ for $x < 0$ (this choice of the loss function implies that the holding and shortage costs are \$.01 and \$20 per unit respectively), and $\alpha = 0.999$. The resulting optimal policy $z_i(r)$ and $r_{i,j}(y)$ is

Table 1. Regular order quantity $z_i(r)$ as a function of r (i.e., inventory position after a possible emergency order is placed at a regular review epoch)[a].

r	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
11	8	25	29	30	30	30	30
12	8	25	29	29	29	29	29
13	7	24	29	29	29	29	29
14	7	23	29	29	29	29	29
15	6	23	28	28	28	28	28
16	5	22	27	28	28	28	28
17	5	21	27	27	27	27	27
18	4	20	26	26	26	26	26
19	3	19	25	25	25	25	25
20	2	18	24	25	25	25	25
21	1	17	23	24	24	24	24
22	0	16	22	23	23	23	23
23	0	15	21	22	22	22	22
24	0	14	20	21	21	21	21
25	0	13	19	20	20	20	20
26	0	12	18	19	19	19	19
27	0	11	17	18	18	18	18
28	0	10	16	17	17	17	17
29	0	9	15	16	16	16	16
30	0	8	14	15	15	15	15
31	0	7	13	14	14	14	14
32	0	6	12	13	13	13	13
33	0	5	11	12	12	12	12
34	0	4	10	11	11	11	11
35	0	3	9	10	10	10	10
36	0	2	8	9	9	9	9
37	0	1	7	8	8	8	8
38	0	0	6	7	7	7	7
39	0	0	5	6	6	6	6
40	0	0	4	5	5	5	5
41	0	0	3	4	4	4	4
42	0	0	2	3	3	3	3
43	0	0	1	2	2	2	2
44	0	0	0	1	1	1	1
45	0	0	0	0	0	0	0

[a] Data: $m = 10$, $c_1 = \$10$, $c_0 = \$15$, $\ell(x) = .01x$ for $x > 0$ and $\ell(x) = -20x$ for $x < 0$, $\mu = 2$ (Poisson demand), $\alpha = 0.999$, and $\tau = 6$.

reported in Tables 1 and 2. As we can observe from these tables, all policy parameters seem to converge after a few order cycles. For example, from Table 1 we see that $z_i(20) = 2, 18, 24, 25, 25, 25, 25$ for $i = 1, \ldots, 7$. From Table 2, we also see that $r_{i,6}(15) = 3, 7, 8, 8, 8, 8, 8, 8$ for $i = 0, 1, \ldots, 7$. Similar convergence properties can be observed for all other parameter values in Tables 1 and 2. A natural question at this point is whether or not this observed convergence property holds in general. We show in the final part of this section that this is indeed true if certain conditions are satisfied.

Let R_i be the minimum value of r for which $z_i(r) = 0$. We assume that $R_i > r_{i,0}$, i.e., $z_i(r_{i,0}) > 0$; otherwise, the regular supply mode is never used. It follows by Theorem 4 that, for all $r \le R_i$, $r + z_i(r) \le R_i + z_i(R_i) = R_i$. This implies that R_i is the maximum

Table 2. Emergency order-up-to levels $r_{i,j}$, $j = 6, \ldots, 9$, as a function of inventory on order y^{a}.

y	j	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
0	6	10	11	11	11	11	11	11	11
	7	10	11	11	11	11	11	11	11
	8	11	11	11	11	11	11	11	11
	9	11	11	11	11	11	11	11	11
1	6	9	11	11	11	11	11	11	11
	7	10	11	11	11	11	11	11	11
	8	11	11	11	11	11	11	11	11
	9	11	11	11	11	11	11	11	11
2	6	9	11	11	11	11	11	11	11
	7	10	11	11	11	11	11	11	11
	8	10	11	11	11	11	11	11	11
	9	11	11	11	11	11	11	11	11
3	6	8	11	11	11	11	11	11	11
	7	9	11	11	11	11	11	11	11
	8	10	11	11	11	11	11	11	11
	9	11	11	11	11	11	11	11	11
4	6	7	11	11	11	11	11	11	11
	7	9	11	11	11	11	11	11	11
	8	10	11	11	11	11	11	11	11
	9	10	11	11	11	11	11	11	11
5, 6	6	7	10	10	10	10	10	10	10
	7	8	11	11	11	11	11	11	11
	8	9	11	11	11	11	11	11	11
	9	10	11	11	11	11	11	11	11
7	6	6	10	10	10	10	10	10	10
	7	7	11	11	11	11	11	11	11
	8	9	11	11	11	11	11	11	11
	9	9	11	11	11	11	11	11	11
8	6	6	10	10	10	10	10	10	10
	7	7	11	11	11	11	11	11	11
	8	8	11	11	11	11	11	11	11
	9	9	11	11	11	11	11	11	11
9, 10	6	5	9	9	9	9	9	9	9
	7	7	10	10	10	10	10	10	10
	8	8	11	11	11	11	11	11	11
	9	9	11	11	11	11	11	11	11
11	6	4	9	9	9	9	9	9	9
	7	6	10	10	10	10	10	10	10
	8	8	11	11	11	11	11	11	11
	9	9	11	11	11	11	11	11	11
12	6	4	8	8	9	9	9	9	9
	7	6	10	10	10	10	10	10	10
	8	7	10	10	11	11	11	11	11
	9	8	11	11	11	11	11	11	11
13	6	4	8	8	8	8	8	8	8
	7	6	9	9	10	10	10	10	10
	8	7	10	10	10	10	10	10	10
	9	8	11	11	11	11	11	11	11
14	6	4	8	8	8	8	8	8	8
	7	6	9	9	9	9	9	9	9
	8	7	10	10	10	10	10	10	10
	9	8	11	11	11	11	11	11	11
15	6	3	7	8	8	8	8	8	8
	7	6	9	9	9	9	9	9	9
	8	7	10	10	10	10	10	10	10

Table 2. Continued

y	j	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
	9	8	10	11	11	11	11	11	11
16	6	3	7	7	7	7	7	7	7
	7	6	9	9	9	9	9	9	9
	8	7	10	10	10	10	10	10	10
	9	8	10	10	10	10	10	10	10
17	6	3	7	7	7	7	7	7	7
	7	6	8	8	9	9	9	9	9
	8	7	9	10	10	10	10	10	10
	9	8	10	10	10	10	10	10	10
18	6	3	7	7	7	7	7	7	7
	7	6	8	8	8	8	8	8	8
	8	7	9	9	9	9	9	9	9
	9	8	10	10	10	10	10	10	10
19	6	3	6	7	7	7	7	7	7
	7	5	8	8	8	8	8	8	8
	8	7	9	9	9	9	9	9	9
	9	8	10	10	10	10	10	10	10
20, 21	6	3	6	6	6	6	6	6	6
	7	5	8	8	8	8	8	8	8
	8	7	9	9	9	9	9	9	9
	9	8	10	10	10	10	10	10	10
22	6	3	6	6	6	6	6	6	6
	7	5	8	8	8	8	8	8	8
	8	7	9	9	9	9	9	9	9
	9	8	10	10	10	10	10	10	10
23	6	3	6	6	6	6	6	6	6
	7	5	7	8	8	8	8	8	8
	8	7	9	9	9	9	9	9	9
	9	8	10	10	10	10	10	10	10
24	6	3	5	6	6	6	6	6	6
	7	5	7	8	8	8	8	8	8
	8	7	9	9	9	9	9	9	9
	9	8	9	10	10	10	10	10	10
25	6	3	5	6	6	6	6	6	6
	7	5	7	7	7	7	7	7	7
	8	7	8	9	9	9	9	9	9
	9	8	9	10	10	10	10	10	10
26, 27	6	3	5	5	5	5	5	5	5
	7	5	7	7	7	7	7	7	7
	8	7	8	9	9	9	9	9	9
	9	8	9	10	10	10	10	10	10
28	6	3	5	5	5	5	5	5	5
	7	5	7	7	7	7	7	7	7
	8	7	8	8	9	9	9	9	9
	9	8	9	9	10	10	10	10	10
29	6	3	5	5	5	5	5	5	5
	7	5	7	7	7	7	7	7	7
	8	7	8	8	8	8	8	8	8
	9	8	9	9	9	9	9	9	9
30	6	3	4	5	5	5	5	5	5
	7	5	7	7	7	7	7	7	7
	8	7	8	8	8	8	8	8	8
	9	8	9	9	9	9	9	9	9

[a] Data: $m = 10$, $c_1 = \$10$, $c_0 = \$15$, $\ell(x) = .01x$ for $x > 0$ and $\ell(x) = -20x$ for $x < 0$, $\mu = 2$ (Poisson demand), $\alpha = 0.999$, and $\tau = 6$.

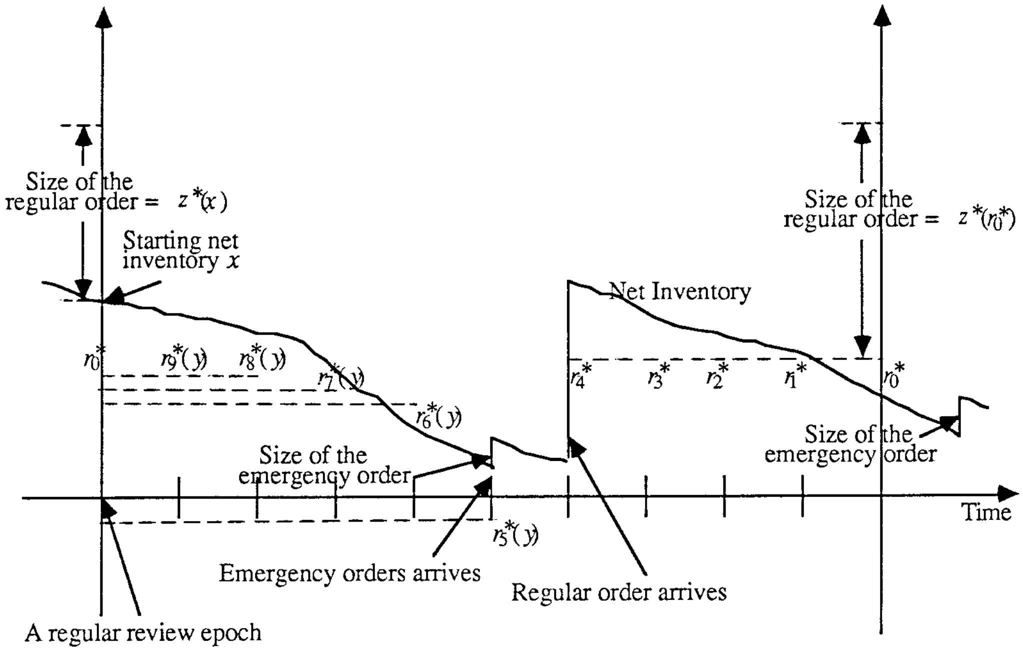

Figure 2. A realization of the inventory process for the periodic system in which the emergency and regular supply lead times are one and six periods, respectively.

possible order-up-to level (or inventory position) with i order cycles remaining if a regular order is placed at that epoch. Let Z_i be the maximum possible regular order quantity with i cycles remaining. Then, $Z_i = z_i(r_{i,0})$ by Theorem 4 and for all $y \le Z_i$, $y + r_{i-1,m-1}(y) \le Z_i + r_{i-1,m-1}(Z_i)$ by the corollary. We show in Theorem 5 that if, for a regular review epoch, the two consecutive maximum order-up-to levels are equal to each other, i.e., $R_i = R_{i-1}$, and greater than every intermediate emergency order-up-to level (plus, if any, a regular order quantity), i.e., $R_i \ge r_{i-1,m-\tau}$ and $R_i \ge r_{i-1,m-1}(Z_i) + Z_i$, and the first derivatives for these two consecutive cost functions are equal, i.e., $D_1V_{i,0}(x, 0) = D_1V_{i-1,0}(x, 0)$ for $x \le R_i$, then the sequences $\{r_{n,0}\}$, $\{r_{n,j}\}$ for $j = 1, \ldots, m - \tau$, $\{r_{n,j}(y)\}$ for $j = m - \tau + 1, \ldots, m - 1$, and $\{z_n(r), r \ge r_{n,0}\}$ converge respectively to $r_0^* = r_{i,0}$, $r_j^* = r_{i-1,j}$ for $j = 1, \ldots, m - \tau$, $r_j^*(y) = r_{i-1,j}(y)$ for $j = m - \tau + 1, \ldots, m - 1$, and $z^*(r) = z_i(r)$ for $r \ge r_0^*$. As a result, r_j^* for $j = 0, 1, \ldots, m - \tau$, $r_j^*(y)$ for $j = m - \tau + 1, \ldots, m - 1$, and $z^*(r)$ for $r \ge r_0^*$, are, respectively, the optimal emergency order-up-to levels and regular order quantities for the infinite horizon model (see Figure 2 for an example of a realization of the inventory process over an infinite horizon).

THEOREM 5: If there exists some i such that

(a) $R_i = R_{i-1}$ $(R_i > r_{i,0})$,
(b) $R_i \ge r_{i-1,m-\tau}$, $R_i \ge r_{i-1,m-1}(Z_i) + Z_i$,
(c) $D_1V_{i,0}(x, 0) = D_1V_{i-1,0}(x, 0)$ for $x \le R_i$,

then the following convergence properties hold

(i)

$$r_{n,j} = r_{i-1,j}, \qquad j = 1, \ldots, m - \tau \text{ (if } \tau < m), \qquad n \geq i,$$
$$r_{n,j}(y) = r_{i-1,j}(y) \qquad \text{for } y \leq R_i - r_{i-1,j}(y), \qquad j = m - \tau + 1, \ldots, m - 1, \qquad n \geq i; \tag{14}$$

(ii)

$$z_{n+1}(r) = z_i(r) \qquad \text{for } r \leq R_i, \qquad n \geq i,$$
$$r_{n+1,0} = r_{i,0} \text{ and } R_{n+1} = R_i \text{ (thus } Z_{n+1} = Z_i), \qquad n \geq i. \tag{15}$$

PROOF: See the Appendix.

Theorem 5 states that if conditions (a), (b), and (c) are satisfied, convergence of all operation parameters has been obtained and thus computation can be stopped. Referring to the example discussed above, as we see from Tables 1 and 2, convergence is obtained after just five order cycles (as a note, $R_5 = 45$, $r_0^* = r_{5,0} = 11$, and $Z_5 = 30$). We note that condition (a) can be easily satisfied, and condition (b) is expected to hold. The latter is because R_i (as explained before Theorem 5) should be large enough to cover demand over an order cycle plus the regular supply lead time as it will take $m + \tau$ periods for the next regular order to arrive, and similarly, $r_{i-1,m-1}(Z_i) + Z_i$, for example, should meet demand over a time interval of $m + \tau - 1$ periods (which is shorter by one period).

4. COMPUTATIONAL RESULTS

In this section, we test a set of problems to investigate the speed of convergence of operation parameters $r_{i,j}$, $j = 0, 1, \ldots, m - \tau$, $r_{i,j}(y)$, $j = m - \tau + 1, \ldots, m - 1$, and $z_i(r)$. To this end, we define $\ell(x) = hx$ for $x > 0$, $\ell(0) = 0$, and $\ell(x) = -px$ for $x < 0$ (i.e., the holding and shortage costs are charged at the rates of h and p, respectively). Also, we use the following end-of-horizon condition (to speed the computation): $V_{0,0}(x, 0) \equiv -c_0 x$ for $x < 0$, $V_{0,0}(0, 0) \equiv 0$, and $V_{0,0}(x, 0) \equiv -c_1 x$ for $x > 0$.

For the purposes of this experiment, we say that the first derivatives of two consecutive cost functions (at regular review epoches) could be regarded as equal when

$$\operatorname*{Max}_{x \leq R_i} |D_1 V_{i,0}(x, 0) - D_1 V_{i-1,0}(x, 0)| \leq \varepsilon, \tag{16}$$

where ε is set to 0.02 for all problems we solve (using FORTRAN on an IBM-3081 computer with a VMS operating system).

Table 3. CPU time (s) and number of order cycles required for the convergence of operation parameters[a].

μ	τ	Average CPU time	Average number of order cycles required
1	4	0.140	3.0
	6	0.233	3.0
	8	0.319	3.0
2	4	0.481	3.4
	6	0.772	3.2
	8	1.012	3.0
5	4	2.939	3.6
	6	4.732	3.2
	8	6.302	3.0
10	4	11.169	3.2
	6	19.564	3.0
	8	30.516	3.2
20	4	43.297	3.0
	6	84.422	3.0
	8	127.743	3.0

[a] Common data: $m = 10$, $c_1 = \$10$, $h = \$0.01$, $\alpha = 0.999$. For each combination of μ and τ, five problems with different values of c_0 and p are solved, i.e., $(c_0, p) = (15, 20), (15, 40), (15, 60), (20, 40)$, or $(20, 60)$.

As shown in Table 3, 15 combinations of different values of μ and τ are formed, and, for each combination, there are five problems varying on values of c_0 and p. Thus, there are a total of 75 problems. For each problem, the CPU time (in seconds) and number of order cycles required (for the convergence to occur) are recorded. The average CPU time and the average number of order cycles required for each combination are reported in Table 3. As we see, the average number of order cycles required is between 3.0 and 3.4. In fact, the maximum and minimum number of order cycles required across all problems are 5 and 3, respectively. However, the average CPU time required increases (approximately) linearly in τ and (approximately) quadratically in μ. The latter implies that if mean demand (per period) is very large, we may need to rescale the data to accommodate the problem so that it can be solved within reasonable time.

5. CONCLUSION

In this paper, we study a periodic review inventory system in which regular orders are placed periodically while emergency orders can be placed continuously. Assuming that the difference between the regular and emergency channel lead times is greater than one period but less than the order-cycle length, we develop a dynamic programming model and derive a stopping rule to end the computation and obtain optimal operation parameters. Computational results are included that support the contention that easily implemented policies can be computed with reasonable effort.

Extension of the analysis to the case in which the two channel lead times differ by more than an order cycle appears to be difficult. The problem lies in the need of keeping track of every amount of inventory on order, which would increase the state space from two dimensions to multiple dimensions. This is left for future research.

APPENDIX

PROOF OF LEMMA 1: $V_{0,0}(x, 0)$ is convex. Assuming that $V_{0,j-1}(x, 0)$, $j \le m - \tau$, is convex, we show that $V_{0,j}(x, 0)$ is convex. It follows from (9) that $G_{0,j}(r, 0)$ is a convex function. Hence, $V_{0,j}(x, 0)$ is convex by Proposition B-4 of Heyman and Sobel [12]. Since $V_{0,m-\tau}(x, 0)$ is convex, $G_{0,m-\tau+1}(r, y)$ is jointly convex in r and y. Thus, $V_{0,m-\tau+1}(x, y)$ is jointly convex in x and y by Proposition B-4 of Heyman and Sobel [12], and so are $V_{0,j}(x, y)$, $j = m - \tau + 2, \ldots, m - 1$. Hence, $J_1(r)$ is convex in r and $V_{1,0}(x, 0)$ is convex in x by their Proposition B-4. Convexity is established by induction for $V_{i,j}(x, y)$, $i \ge 1$. □

PROOF OF THEOREM 1: We first show that $r_{i,m-\tau} \ge r_{i,m-\tau-1} \ge \cdots \ge r_{i,1}$. We show that $r_{i,2} \ge r_{i,1}$. The remaining inequalities can be established similarly. To show $r_{i,2} \ge r_{i,1}$, we show that $D_1G_{i,2}(r, 0) < 0$ for $r < r_{i,1}$. It follows from (9) and definition of $r_{i,j}$ that $D_1G_{i,1}(r, 0) = c_0 + \alpha E_tDL(r - t) + \alpha E_tD_1V_{i,0}(r - t, 0) < 0$ for $r < r_{i,1}$. Also, from the observations mentioned in the text, $D_1G_{i,2}(r, 0) = c_0 + \alpha E_tDL(r - t) + \alpha E_tD_1V_{i,1}(r - t, 0) = c_0 + \alpha E_tDL(r - t) - \alpha c_0$ for $r < r_{i,1}$. Hence, $D_1G_{i,2}(r, 0) < -\alpha[c_0 + E_tD_1V_{i,0}(r - t, 0)] \le 0$, for $r < r_{i,1}$.

We next show that $r_{i,m-1}(y) \ge r_{i,m-2}(y) \ge \cdots \ge r_{i,m-\tau+1}(y)$. We show that $r_{i,m-\tau+2}(y) \ge r_{i,m-\tau+1}(y)$ is true. The remaining inequalities can be established similarly. To show $r_{i,m-\tau+2}(y) \ge r_{i,m-\tau+1}(y)$, we show that $D_1G_{i,m-\tau+2}(r, y) < 0$ for $r < r_{i,m-\tau+1}(y)$. It follows from (9) and (10) that

$$D_1G_{i,m-\tau+1}(r, y) = c_0 + \alpha E_tDL(y + r - t) + \alpha E_tD_1V_{i,m-\tau}(y + r - t, 0), \tag{17}$$

$$D_1G_{i,m-\tau+2}(r, y) = c_0 + \alpha E_tDL(r - t) + \alpha E_tD_1V_{i,m-\tau+1}(r - t, y). \tag{18}$$

As $r_{i,m-\tau+1}(y)$ is the smallest value minimizing $G_{i,m-\tau+1}(r, y)$, $D_1G_{i,m-\tau+1}(r, y) < 0$ for $r < r_{i,m-\tau+1}(y)$, i.e., $\alpha E_tDL(y + r - t) < -c_0 - \alpha E_tD_1V_{i,m-\tau}(y + r - t, 0)$ for $r < r_{i,m-\tau+1}(y)$. However, $D_1G_{i,m-\tau+2}(r, y) = c_0 + \alpha E_tDL(r - t) - \alpha c_0$ for $r < r_{i,m-\tau+1}(y)$ (by the observations in the text). It follows then from the convexity of $L(\cdot)$ and $y \ge 0$ that $D_1G_{i,m-\tau+2}(r, y) \le c_0 + \alpha E_tDL(y + r - t) - \alpha c_0 < -\alpha[c_0 + E_tD_1V_{i,m-\tau}(y + r - t, 0)] \le 0$ for $r < r_{i,m-\tau+1}(y)$. □

PROOF OF THEOREM 2: As we see from (17), $r_{i,m-\tau+1}(y)$ minimizing $G_{i,m-\tau+1}(r, y)$ will decrease as y increases. In fact, a change of variables in (17) (i.e., $y + r$ is denoted by a single variable) shows that $r_{i,m-\tau+1}(y) = r_{i,m-\tau+1}(0) - y$ and $r_{i,m-\tau+1}(y + \Delta) = r_{i,m-\tau+1}(y) - \Delta$. □

PROOF OF LEMMA 2: $r_{i,m-\tau+1}(y + \Delta) = r_{i,m-\tau+1}(y) - \Delta < r_{i,m-\tau+1}(y)$ by Theorem 2. Hence, (i) for $x \le r_{i,m-\tau+1}(y + \Delta)$, $D_1V_{i,m-\tau+1}(x, y) = D_1V_{i,m-\tau+1}(x, y + \Delta) = -c_0$; (ii) for $x \in (r_{i,m-\tau+1}(y + \Delta), r_{i,m-\tau+1}(y))$, $D_1V_{i,m-\tau+1}(x, y) = -c_0 \le D_1V_{i,m-\tau+1}(x, y + \Delta)$; and (iii) for $x \ge r_{i,m-\tau+1}(y)$,

$$\begin{aligned} D_1V_{i,m-\tau+1}(x, y) &= \alpha E_tDL(y + x - t) + \alpha E_tD_1V_{i,m-\tau}(y + x - t, 0) \\ &\le \alpha E_tDL(y + \Delta + x - t) + \alpha E_tD_1V_{i,m-\tau}(y + \Delta + x - t, 0) \\ &= D_1V_{i,m-\tau+1}(x, y + \Delta), \end{aligned}$$

due to (13), (17), and Lemma 1. We have shown that the mixed second derivative of $V_{i,m-\tau+1}(x, y)$ is nonnegative. In addition, it follows from (17) and the observations mentioned in the text that for $x \le r_{i,m-\tau+1}(y)$ or $x - \Delta \le r_{i,m-\tau+1}(y + \Delta)$,

$$\begin{aligned} D_1V_{i,m-\tau+1}(x, y) &= D_1V_{i,m-\tau+1}(x - \Delta, y + \Delta) = -c_0 \\ &= \alpha E_tDL(y + r_{i,m-\tau+1}(y) - t) + \alpha E_tD_1V_{i,m-\tau}(y + r_{i,m-\tau+1}(y) - t, 0), \end{aligned}$$

$$\begin{aligned} D_2V_{i,m-\tau+1}(x, y) &= D_2V_{i,m-\tau+1}(x - \Delta, y + \Delta) \\ &= \alpha E_tDL(y + \Delta + r_{i,m-\tau+1}(y + \Delta) - t) + \alpha E_tD_1V_{i,m-\tau}(y + \Delta + r_{i,m-\tau+1}(y + \Delta) - t, 0). \end{aligned}$$

As $r_{i,m-\tau+1}(y + \Delta) + \Delta = r_{i,m-\tau+1}(y)$, the above two expressions are equal. Also, for $x > r_{i,m-\tau+1}(y)$ or $x - \Delta > r_{i,m-\tau+1}(y + \Delta)$,

$$D_1V_{i,m-\tau+1}(x, y) = \alpha E_tDL(y + x - t) + \alpha E_tD_1V_{i,m-\tau}(y + x - t, 0)$$

$$
\begin{aligned}
&= D_2V_{i,m-\tau+1}(x, y),\\
D_1V_{i,m-\tau+1}(x-\Delta, y+\Delta) &= D_2V_{i,m-\tau+1}(x-\Delta, y+\Delta)\\
&= \alpha E_t DL(y+\Delta+x-\Delta-t) + \alpha E_t D_1V_{i,m-\tau}(y+\Delta+x-\Delta-t, 0)\\
&= \alpha E_t DL(y+x-t) + \alpha E_t D_1V_{i,m-\tau}(y+x-t, 0).
\end{aligned}
$$

As a result, $D_1V_{i,m-\tau+1}(x, y) = D_2V_{i,m-\tau+1}(x, y) = D_1V_{i,m-\tau+1}(x-\Delta, y+\Delta) = D_2V_{i,m-\tau+1}(x-\Delta, y+\Delta)$. □

PROOF OF THEOREM 3: It follows from Lemma 2 that

$$
\begin{aligned}
D_1G_{i,m-\tau+2}(r, y) &= c_0 + \alpha E_t DL(r-t) + \alpha E_t D_1V_{i,m-\tau+1}(r-t, y)\\
&\le c_0 + \alpha E_t DL(r-t) + \alpha E_t D_1V_{i,m-\tau+1}(r-t, y+\Delta) = D_1G_{i,m-\tau+2}(r, y+\Delta).
\end{aligned}
$$

This implies that $r_{i,m-\tau+2}(y) \ge r_{i,m-\tau+2}(y+\Delta)$. Furthermore, as $D_1V_{i,m-\tau+1}(x-\Delta, y+\Delta) = D_1V_{i,m-\tau+1}(x, y)$ by Lemma 2,

$$
\begin{aligned}
D_1G_{i,m-\tau+2}(r, y) &= c_0 + \alpha E_t DL(r-t) + \alpha E_t D_1V_{i,m-\tau+1}(r-\Delta-t, y+\Delta) \ge c_0 + \alpha E_t DL(r-\Delta-t)\\
&\quad + \alpha E_t D_1V_{i,m-\tau+1}(r-\Delta-t, y+\Delta) = D_1G_{i,m-\tau+2}(r-\Delta, y+\Delta).
\end{aligned}
$$

As $D_1G_{i,m-\tau+2}(r, y) < 0$ for $r < r_{i,m-\tau+2}(y)$, it follows that $D_1G_{i,m-\tau+2}(r-\Delta, y+\Delta) < 0$ for $r < r_{i,m-\tau+2}(y)$ or $r-\Delta < r_{i,m-\tau+2}(y) - \Delta$. This implies that $r_{i,m-\tau+2}(y+\Delta) \ge r_{i,m-\tau+2}(y) - \Delta$. □

PROOF OF LEMMA 3: For the nonnegativity of the mixed second derivative of $V_{i,m-\tau+2}(x, y)$, the proof is similar to that of Lemma 2. We show that the other part of this lemma is true. By Theorem 3, $r_{i,m-\tau+2}(y+\Delta) + \Delta \ge r_{i,m-\tau+2}(y)$. It follows from (18) and Lemma 2 (and observations mentioned in the text) that (i) for $x \le r_{i,m-\tau+2}(y)$ [and thus $x-\Delta \le r_{i,m-\tau+2}(y+\Delta)$], $D_1V_{i,m-\tau+2}(x, y) = -c_0 = D_1V_{i,m-\tau+2}(x-\Delta, y+\Delta)$ and

$$
\begin{aligned}
D_2V_{i,m-\tau+2}(x, y) &= \alpha E_t D_2V_{i,m-\tau+1}(r_{i,m-\tau+2}(y)-t, y)\\
&= \alpha E_t D_2V_{i,m-\tau+1}(r_{i,m-\tau+2}(y)-\Delta-t, y+\Delta) \le \alpha E_t D_2V_{i,m-\tau+1}(r_{i,m-\tau+2}(y+\Delta)-t, y+\Delta)\\
&= D_2V_{i,m-\tau+2}(x-\Delta, y+\Delta);
\end{aligned}
$$

(ii) for $x \ge r_{i,m-\tau+2}(y+\Delta) + \Delta$,

$$
\begin{aligned}
D_1V_{i,m-\tau+2}(x, y) &= \alpha E_t DL(x-t) + \alpha E_t D_1V_{i,m-\tau+1}(x-t, y)\\
&= \alpha E_t DL(x-t) + \alpha E_t D_1V_{i,m-\tau+1}(x-\Delta-t, y+\Delta)\\
&\ge \alpha E_t DL(x-\Delta-t) + \alpha E_t D_1V_{i,m-\tau+1}(x-\Delta-t, y+\Delta) = D_1V_{i,m-\tau+2}(x-\Delta, y+\Delta)
\end{aligned}
$$

and

$$
\begin{aligned}
D_2V_{i,m-\tau+2}(x, y) &= \alpha E_t D_2V_{i,m-\tau+1}(x-t, y)\\
&= \alpha E_t D_2V_{i,m-\tau+1}(x-\Delta-t, y+\Delta)\\
&= D_2V_{i,m-\tau+2}(x-\Delta, y+\Delta);
\end{aligned}
$$

(iii) finally, for $x \in (r_{i,m-\tau+2}(y), r_{i,m-\tau+2}(y+\Delta)+\Delta)$, $D_1V_{i,m-\tau+2}(x, y) \ge -c_0 = D_1V_{i,m-\tau+2}(x-\Delta, y+\Delta)$, and

$$
\begin{aligned}
D_2V_{i,m-\tau+2}(x, y) &= \alpha E_t D_2V_{i,m-\tau+1}(x-t, y) = \alpha E_t D_2V_{i,m-\tau+1}(x-\Delta-t, y+\Delta)\\
&\le \alpha E_t D_2V_{i,m-\tau+1}(r_{i,m-\tau+2}(y+\Delta)-t, y+\Delta) = D_2V_{i,m-\tau+2}(x-\Delta, y+\Delta).
\end{aligned}
$$

In summary, we have shown that $D_1V_{i,m-\tau+2}(x, y) \geq D_1V_{i,m-\tau+2}(x - \Delta, y + \Delta)$ and $D_2V_{i,m-\tau+2}(x, y) \leq D_2V_{i,m-\tau+2}(x - \Delta, y + \Delta)$. □

PROOF OF THEOREM 4: It follows from Corollary and (6) that

$$D_2J_{i,0}(r - \Delta, z) = c_1 + \alpha E_t D_2V_{i-1,m-1}(r - \Delta - t, z) \leq c_1 + \alpha E_t D_2V_{i-1,m-1}(r - t, z) = D_2J_{i,0}(r, z).$$

This implies that $z_i(r) \leq z_i(r - \Delta)$. Moreover, by the corollary

$$\begin{aligned} D_2J_{i,0}(r, z) &= c_1 + \alpha E_t D_2V_{i-1,m-1}(r - t, z) \leq c_1 + \alpha E_t D_2V_{i-1,m-1}(r - \Delta - t, z + \Delta) \\ &= D_2J_{i,0}(r - \Delta, z + \Delta). \end{aligned}$$

Since $D_2J_{i,0}(r, z) \geq 0$ for $z \geq z_i(r)$, $D_2J_{i,0}(r - \Delta, z + \Delta) \geq 0$ for $z \geq z_i(r)$ or equivalently $z + \Delta \geq z_i(r) + \Delta$. This implies that $z_i(r - \Delta) \leq z_i(r) + \Delta$. □

PROOF OF THEOREM 5: We divide the proof in three parts. Parts 1 and 2 show that (14) and (15), respectively, hold for $n = i$, and Part 3 establishes that $D_1V_{i+1,0}(x, 0) = D_1V_{i,0}(x, 0)$ for $x \leq R_{i+1} = R_i$. Thus, the argument can be repeated and (14) and (15) hold for all $n \geq i$.

Part 1. (If $\tau = m$, the proof starts in the next paragraph.) It follows from (9) that

$$D_1G_{i-1,1}(r, 0) = c_0 + \alpha E_t DL(r - t) + \alpha E_t D_1V_{i-1,0}(r - t, 0),$$

$$D_1G_{i,1}(r, 0) = c_0 + \alpha E_t DL(r - t) + \alpha E_t D_1V_{i,0}(r - t, 0).$$

If $D_1V_{i,0}(x, 0) = D_1V_{i-1,0}(x, 0)$ for $x \leq R_i$, then $D_1G_{i,1}(r, 0) = D_1G_{i-1,1}(r, 0)$ for $x \leq R_i$. As $R_i \geq r_{i-1,m-\tau}$, $R_i \geq r_{i-1,1}$ (by Theorem 1). Since $r_{i-1,1}$ minimizes $G_{i-1,1}(r, 0)$, it follows that it also minimizes $G_{i,1}(r, 0)$, i.e., $r_{i,1} = r_{i-1,1}$. In addition, for $x > r_{i-1,1}$, $D_1V_{i-1,1}(x, 0) = \alpha E_t DL(x - t) + \alpha E_t D_1V_{i-1,0}(x - t, 0)$ and $D_1V_{i,1}(x, 0) = \alpha E_t DL(x - t) + \alpha E_t D_1V_{i,0}(x - t, 0)$. Hence, $D_1V_{i,1}(x, 0) = D_1V_{i-1,1}(x, 0)$ for $x \in (r_{i-1,1}, R_i]$. Also, $D_1V_{i,1}(x, 0) = D_1V_{i-1,1}(x, 0) = -c_0$ for $x \leq r_{i-1,1}$. As a result, $D_1V_{i,1}(x, 0) = D_1V_{i-1,1}(x, 0)$ for $x \leq R_i$. Similarly, we can show that $r_{i,j} = r_{i-1,j}$ for $j = 2, \ldots, m - \tau$, and

$$D_1V_{i,j}(x, 0) = D_1V_{i-1,j}(x, 0), \qquad j = 2, \ldots, m - \tau, \qquad \text{for } x \leq R_i. \tag{19}$$

In addition, if $R_i \geq r_{i-1,m-1}(Z_i) + Z_i$, $R_i \geq r_{i-1,m-\tau+1}(Z_i) + Z_i = r_{i-1,m-\tau+1}(0)$ (by Theorems 1 and 2). By using the same reasoning (for showing $r_{i,1} = r_{i-1,1}$), it follows from (17) and (19) that $r_{i,m-\tau+1}(0) = r_{i-1,m-\tau+1}(0)$, and thus $r_{i,m-\tau+1}(y) = r_{i-1,m-\tau+1}(y)$. Moreover, as $D_1V_{i,m-\tau}(x, 0) = D_1V_{i-1,m-\tau}(x, 0)$ for $x \leq R_i$, it can be seen from (4) that

$$D_1V_{i,m-\tau+1}(x, y) = D_1V_{i-1,m-\tau+1}(x, y) \qquad \text{for } x \leq R_i - y, \tag{20}$$

$$D_2V_{i,m-\tau+1}(x, y) = D_2V_{i-1,m-\tau+1}(x, y) \qquad \text{for } y \leq R_i - x. \tag{21}$$

Likewise, we can show from (20), (21), (3), and (9) that, for all y such that $y + r_{i-1,j}(y) \leq R_i$, $r_{i,j}(y) = r_{i-1,j}(y)$, $j = m - \tau + 2, \ldots, m - 1$, and

$$D_1V_{i,j}(x, y) = D_1V_{i-1,j}(x, y) \qquad \text{for } x \leq R_i - y, \qquad j = m - \tau + 2, \ldots, m - 1, \tag{22}$$

$$D_2V_{i,j}(x, y) = D_2V_{i-1,j}(x, y) \qquad \text{for } y \leq R_i - x, \qquad j = m - \tau + 2, \ldots, m - 1. \tag{23}$$

Part 2. From (6), $D_2J_{i,0}(r, z) = c_1 + \alpha E_t D_2V_{i-1,m-1}(r - t, z)$ and $D_2J_{i+1,0}(r, z) = c_1 + \alpha E_t D_2V_{i,m-1}(r - t, z)$. Hence, It follows from (23) that $D_2J_{i+1,0}(r, z) = D_2J_{i,0}(r, z)$ for all $z \leq R_i - r$. Also, for $r \leq R_i$, $z_i(r) \leq R_i - r$ (by Theorem 4). As $z_i(r)$, $r \leq R_i$, minimizes $J_{i,0}(r, z)$, it follows that it also minimizes $J_{i+1,0}(r, z)$, i.e.,

$$z_{i+1}(r) = z_i(r), \qquad r \leq R_i. \tag{24}$$

Since R_i is the minimum value of r for which $z_i(r) = 0$, it follows from (24) that R_i is also the minimum value of r for which $z_{i+1}(r) = 0$, i.e., $R_{i+1} = R_i$.

Next, we show that $r_{i+1,0} = r_{i,0}$ (as a result, $Z_{i+1} = Z_i$). From (5), $J_i(r) = J_{i,0}(r, z_i(r)) = c_1 z_i(r) + \alpha E_t V_{i-1,m-1}(r - t, z_i(r))$. Thus, for $r \le R_i$,

$$\begin{aligned} DJ_i(r) &= D_1 J_{i,0}(r, z_i(r)) + D_2 J_{i,0}(r, z_i(r)) Dz_i(r) \\ &= D_1 J_{i,0}(r, z_i(r)) + 0 = \alpha E_t D_1 V_{i-1,m-1}(r - t, z_i(r)). \end{aligned}$$

The second equality is due to the fact that, for $r \le R_i$, $D_2 J_{i,0}(r, z_i(r)) = 0$ (note that, for all $r < R_i$, $z_i(r) > 0$). Similarly, for $r \le R_{i+1}$, $DJ_{i+1}(r) = \alpha E_t D_1 V_{i,m-1}(r - t, z_{i+1}(r))$. It follows from (22) and (24) that for $r \le R_i = R_{i+1}$ [thus $r + z_i(r) \le R_i$],

$$\begin{aligned} DJ_i(r) &= \alpha E_t D_1 V_{i,m-1}(r - t, z_i(r)) \\ &= \alpha E_t D_1 V_{i,m-1}(r - t, z_{i+1}(r)) = DJ_{i+1}(r). \end{aligned} \tag{25}$$

In addition, as we see from (8), $DG_{i,0}(r) = c_0 + \alpha E_t DL(r - t) + DJ_i(r)$ and $DG_{i+1,0}(r) = c_0 + \alpha E_t DL(r - t) + DJ_{i+1}(r)$. Hence, $DG_{i+1,0}(r) = DG_{i,0}(r)$ for $r \le R_i$. As $r_{i,0}$ ($r_{i,0} < R_i$) minimizes $G_{i,0}(r)$, it follows that it also minimizes $G_{i+1,0}(r)$, i.e., $r_{i+1,0} = r_{i,0}$.

Part 3. For $x \le r_{i,0}$, $D_1 V_{i+1,0}(x, 0) = D_1 V_{i,0}(x, 0) = -c_0$. For $x \in (r_{i,0}, R_i]$, it follows from (8), (11), and (25) that

$$\begin{aligned} D_1 V_{i,0}(x, 0) &= \alpha E_t DL(x - t) + DJ_i(x) \\ &= \alpha E_t DL(x - t) + DJ_{i+1}(x) = D_1 V_{i+1,0}(x, 0). \end{aligned}$$

As a result, $D_1 V_{i+1,0}(x, 0) = D_1 V_{i,0}(x, 0)$ for all $x \le R_{i+1} = R_i$. □

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