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D-Stability Analysis for Discrete Uncertain Time-Delay Systems

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Abstract—Two cases of the robust D-stability criterion are derived for discrete uncertain systems with multiple time delays. One is a direct test and the other is a boundary test. These cases provide the sufficient conditions under which all solutions of the characteristic equation remain inside the specific disk $D(\alpha, r)$ in the presence of parametric uncertainties.

Keywords—D-stability, Multiple time delays, D-pole placement.

1. INTRODUCTION

The problem of pole assignment in linear system theory has been discussed by many authors and solved in various ways. However, locations of poles vary and cannot be fixed because of parametric uncertainties that originate from different sources, e.g., identification errors, aging of devices, variation of operating points, etc. Therefore, placing all poles in a specific region rather than assigning them to precise locations may be satisfactory in practical applications. One such specific region for discrete systems is a disk $D(\alpha, r)$ centered at $D(\alpha, 0)$ with radius r, where $|\alpha| + r < 1$. The assignment of all poles of a system in the specific disk $D(\alpha, r)$ shown in Figure 1 is known as a D-pole placement problem [1]. This subject has received much attention in the literature [1–3].

The problem of stabilization of time-delay systems has been explored over the years, primarily because the delay is often encountered in various engineering systems, e.g., chemical process—steel smelting and refinery—or in long transmission lines, in pneumatic, hydraulic, or electrical networks. Its occurrence may frequently result in undesirable system responses. Consequently, the problem of stability analysis of time-delay systems is one of the main concerns of the researchers who would like to inspect the properties of such systems. Numerous reports in regard to this subject have been published [4–6].

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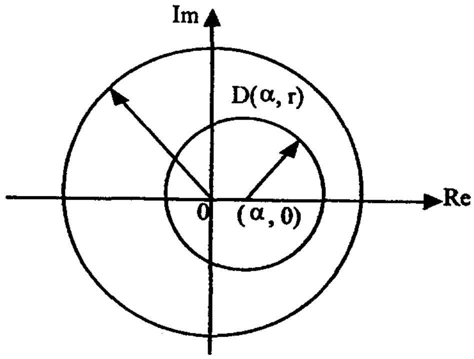

Figure 1. A disk $D(\alpha, r)$ centered at $(\alpha, 0)$ with radius r.

The introduction of time-delay factor complicates the D-pole placement problem, since the number of poles of a system will increase due to time delays. The D-stability problem for discrete time-delay systems has been discussed by Lee *et al.* [2] and their result is extended to include multiple time-delay systems by Su and Shyr [3]. However, the criteria proposed by Lee *et al.* [2] and Su and Shyr [3] are too conservative. In order to improve their results, two cases of the robust D-stability criterion in terms of complex stability radius are proposed for discrete uncertain systems with multiple time delays. One is a direct test (i.e., check $d_1 < d_s$) and the other is a boundary test. The robust D-stability is first checked by the direct test. If it fails, resort to the boundary test.

2. ROBUST D-STABILITY ANALYSIS

Consider a discrete uncertain system, with multiple time delays, described by the following difference equation:

$$X(k+1) = AX(k) + \Delta AX(k) + \sum_{i=1}^{n} A_{di} X(k - h_i) + \sum_{i=1}^{n} \Delta A_{di} X(k - h_i), \tag{1}$$

in which $X(k) \in R^m$ and h_i, $i = 1, 2, \ldots, n$, are positive integer numbers; A and A_{di} are constant matrices with proper dimensions. Also, ΔA and ΔA_{di} denote the parametric uncertainties with the following upper norm-bounds:

$$\|\Delta A\| \le \beta, \tag{2}$$

$$\|\Delta A_{di}\| \le \eta_i, \qquad i = 1, 2, \ldots, n, \tag{3}$$

where β and η_i are given constants.

Before proceeding to the main result, some useful concepts are given in the following.

DEFINITION 1. *A system is said to be $D(\alpha, r)$-stable if all poles of the system are within the specific disk $D(\alpha, r)$ centered at $(\alpha, 0)$ with radius r. Namely, all the solutions of its characteristic equation satisfy $|(z - \alpha)/r| < 1$, in which $r > 0$ and $|\alpha| + r < 1$.*

DEFINITION 2. *Let all eigenvalues of the matrix A be inside the unit circle of the complex plane, then the positive value*

$$\rho(A) = \left\{ \max_{0 \le \theta \le 2\pi} \left\{ \left\| \left[e^{j\theta} I - A \right]^{-1} \right\| \right\} \right\}^{-1} \tag{4}$$

is said to be a complex stability radius of the matrix A.

REMARK 1. The value $\rho(A)$ depends on the choice of norm. For instance, if the Euclidean norm is used, then it is easy to show that

$$\rho(A) = \min_{0 \le \theta \le 2\pi} \left\{ \underline{\sigma} \left[e^{j\theta} I - A \right] \right\}, \tag{5}$$

in which $\underline{\sigma}(\cdot)$ is the minimal singular value of matrix $(\cdot)$.

LEMMA 1 [7]. *Let all eigenvalues of the matrix M be inside the unit disk of the complex plane. All the eigenvalues of all matrices $M+\Delta M$ are inside the unit disk if and only if $\|\Delta M\| < \rho(M)$.*

LEMMA 2 [8]. *Let a matrix $E(z) \in \Re_\infty^{m\times n}$ with $\Re_\infty^{m\times n}$ denoting the set of $m \times n$ matrices whose elements are proper stable rational functions, then*

$$\sup_{z\in\Omega}\|E(z)\| = \sup_{|z|\geq 1}\|E(z)\| = \sup_{\theta\in[0,2\pi]}\left\|E\left(e^{j\theta}\right)\right\|, \tag{6}$$

where $\Omega \equiv \{z = re^{j\theta},\ \theta \in [0,2\pi],\ r \geq 1\}$. Since $E(z)$ is analytic for $z \in \Omega$, this norm is well defined.

After reviewing the above definition and lemma, we are in the position to derive the robust D-stability criterion in terms of complex stability radius for a discrete uncertain multiple time-delay system.

THEOREM 1.

(I) *Suppose that all the eigenvalues of A are within the specific disk $D(\alpha,r)$. System (1) is robustly $D(\alpha,r)$-stable (with $|\alpha| < r$), if*

$$\frac{1}{r}\left(\beta+\sum_{i=1}^{n}\left(\|A_{di}\|+\eta_i\right)(r-|\alpha|)^{-h_i}\right) \equiv d_1 < d_s \equiv \rho\left(\frac{A-\alpha I}{r}\right). \tag{7}$$

(II) *If $d_1 \geq d_s$ and the function*

$$h(g) \equiv \frac{1}{r}\left[\beta+\left\|\sum_{i=1}^{n}A_{di}(rg+\alpha)^{-h_i}\right\|+\sum_{i=1}^{n}\eta_i\,(r-|\alpha|)^{-h_i}\right] \tag{8}$$

lies outside the interval $[d_s,d_1]$, where $|\alpha| < r$ and g take the values in the bounded region $U_1 = \{\delta \mid 1 \leq |\delta| \leq d_{1r}\}$ with $d_{1r} = \|(A-\alpha I)/r\| + d_1$, then system (1) is robustly $D(\alpha,r)$-stable.

PROOF. See the Appendix.

REMARK 2. Case (I) of Theorem 1 gives a nice algebraic condition to test the robust D-stability of system (1) at the cost of conservativeness. It is therefore reasonable to check the D-stability with Case (I), and then if it fails, resort to Case (II). Thus, Cases (I) and (II) complement each other.

REMARK 3. It is easy to see that the D-stability criterion in Theorem 1 will get a less conservative result than the criteria proposed by Su and Shyr [3].

However, for a practical application, it is difficult to examine Case (II) of Theorem 1. The following 'boundary test' may be helpful in examining Case (II) of Theorem 1.

COROLLARY 1. *If $d_1 \geq d_s$ and the following inequality holds:*

$$h(g) \equiv \frac{1}{r}\left[\beta+\left\|\sum_{i=1}^{n}A_{di}(rg+\alpha)^{-h_i}\right\|+\sum_{i=1}^{n}\eta_i(r-|\alpha|)^{-h_i}\right] < d_s, \tag{9}$$

where $|\alpha| < r$ and $g = e^{j\theta}$ for $\theta \in [0,2\pi]$, then system (1) is robustly $D(\alpha,r)$-stable.

PROOF. The matrix $\sum_{i=1}^{n}A_{di}(rg+\alpha)^{-h_i}$ of which all poles of the elements have the modulus $|g| = |\alpha|/r < 1$ belongs to $\Re_\infty^{m\times n}$. Consequently, based on Lemma 2, the function $h(g)$ in (21) takes on its supremum in the range given by $g = e^{j\theta}$ for $\theta \in [0,2\pi]$. Therefore, if inequality (9) holds, $h(g)$ lies outside the interval $[d_s,d_1]$ for all $g \in U_1$ and then system (1) is robustly $D(\alpha,r)$-stable (according to Case (II) of Theorem 1). This completes the proof.

3. EXAMPLE

Consider a discrete uncertain multiple time-delay system:

$$\begin{aligned} X(k+1) = &\begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} X(k) + \begin{bmatrix} 0.0168 & 0 \\ 0.01 & -0.05 \end{bmatrix} X(k-1) + \begin{bmatrix} -0.001 & 0 \\ 0.01 & 0.001 \end{bmatrix} X(k-2) \\ &+ \Delta A X(k) + \Delta A_{d1} X(k-1) + \Delta A_{d2} X(k-2), \end{aligned} \tag{10}$$

in which

$$\|\Delta A\| \le 0.4749, \qquad \|\Delta A_{d1}\| \le 0.0186, \quad \text{and} \quad \|\Delta A_{d2}\| \le 0.016. \tag{11}$$

(see footnote[1]).

The purpose is to inspect whether system (10) satisfies the following time-domain specifications:

(i) overshoot $\le 15\%$, or equivalently, damping ratio $\xi \ge 0.5$;
(ii) rise time ≤ 4.17 s, or equivalently, natural frequency $\omega_n \ge 0.6$;
(iii) settling time $\le$ 43.65 s, or equivalently, all poles less than 0.9 (the sampling interval $T = 1$ s).

These constraints (i)–(iii) may be interpreted as pole locations inside the specified disk $D(0.2,\ 0.7)$ (see [9]).

SOLUTION. According to (2), (3), and (11), the norm-bounds of parametric uncertainties are given as $\beta = 0.4749$, $\eta_1 = 0.0186$, and $\eta_2 = 0.016$. From (7), we have

$$d_1 = \frac{1}{r}\left(\beta + \sum_{i=1}^{2}\left(\|A_{di}\| + \eta_i\right)(r - |\alpha|)^{-h_i}\right) = 1.2264 > d_s = \rho\left(\frac{A - \alpha I}{r}\right) = 1. \tag{12}$$

Therefore, the inequality (7) is not satisfied. We now proceed to Corollary 1.

The simulation of the function $h(g)$ in (9), where $g = e^{j\theta}$ for $\theta \in [0, 2\pi]$, is depicted in Figure 2. This figure reveals that $h(g) < d_s = 1$. Therefore, according to Corollary 1, we can conclude that system (10) is $D(0.2,\ 0.7)$-stable.

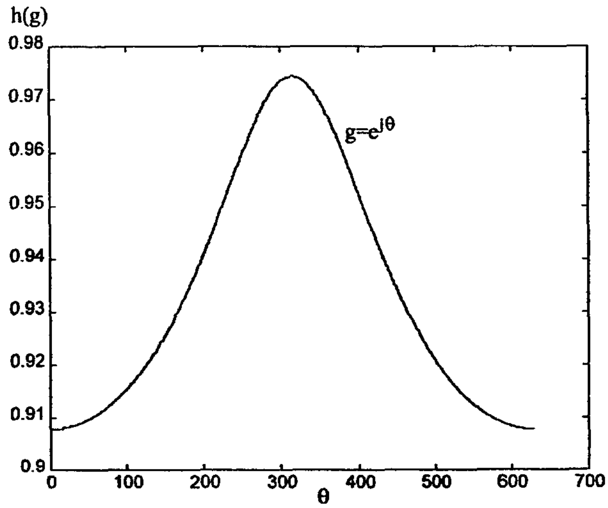

Figure 2.

In order to verify this result, a set of parametric uncertainties which satisfies the norm-bound conditions (11) is chosen as follows:

$$\Delta A = \begin{bmatrix} 0.141 & -0.13 \\ 0.16 & 0.44 \end{bmatrix}, \qquad \Delta A_{d1} = \begin{bmatrix} 0.012 & 0.012 \\ 0 & 0.01 \end{bmatrix}, \qquad \Delta A_{d2} = \begin{bmatrix} 0.011 & 0 \\ 0.009 & 0.01 \end{bmatrix}. \tag{13}$$

[1]The Euclidean norm is considered in this example.

By the computer simulation shown in Figure 3, we find that all poles ($0.5299 \pm j0.1218$, $-0.0585 \pm j0.164$, $0.019 \pm j0.1091$) of the system lie inside the specific disk $D(0.2, 0.7)$. Therefore, the multiple time-delay system (10) meets the time-domain specifications (i)–(iii) in the presence of parametric uncertainties as depicted in (13). This justifies our result.

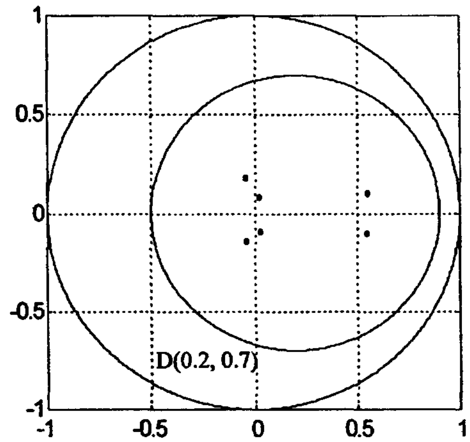

Figure 3.

4. CONCLUSION

Two cases of the robust D-stability criterion are proposed for discrete uncertain systems with multiple time delays. One is a direct test (i.e., check $d_1 < d_s$) and the other is a boundary test. The robust D-stability of system (1) is first checked by the direct test. If it fails, resort to the boundary test, as illustrated in the example.

APPENDIX

PROOF OF THEOREM 1.

CASE I. From (7), the following inequality (14) can be achieved:

$$\frac{1}{r}\left[\|\Delta A\| + \sum_{i=1}^{n}\left(\|A_{di}\| + \|\Delta A_{di}\|\right)|rg+\alpha|^{-h_i}\right] < \rho\left(\frac{A-\alpha I}{r}\right), \quad \text{for } |g| \geq 1, \tag{14}$$

$$\Rightarrow \left\|\frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}\left(A_{di} + \Delta A_{di}\right)(rg+\alpha)^{-h_i}\right]\right\| < \rho\left(\frac{A-\alpha I}{r}\right), \quad \text{for } |g| \geq 1. \tag{15}$$

Hence, from Lemma 1,

$$\left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left(\Delta A + \sum_{i=1}^{n}\left(A_{di}+\Delta A_{di}\right)(rg+\alpha)^{-h_i}\right)\right\}\right| < 1, \quad \text{for } |g| \geq 1. \tag{16}$$

This implies that

$$|g| \neq \left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left(\Delta A + \sum_{i=1}^{n}\left(A_{di}+\Delta A_{di}\right)(rg+\alpha)^{-h_i}\right)\right\}\right|, \quad \text{for } |g| \geq 1. \tag{17}$$

In view of (17), we can see that the solutions of the characteristic equation (of system (1))

$$\det\left\{zI - \left[A + \Delta A + \sum_{i=1}^{n}\left(A_{di}+\Delta A_{di}\right)z^{-h_i}\right]\right\} = 0, \tag{18}$$

or equivalently (with $z = rg + \alpha$),

$$\det\left\{gI - \left[\frac{A - \alpha I}{r} + \frac{1}{r}\left(\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right)\right]\right\} = 0 \tag{19}$$

satisfy $|g| < 1$ (i.e., $|(z-\alpha)/r| < 1$). Therefore, system (1) is robustly $D(\alpha, r)$-stable. This completes the proof of Case I.

Case II. If system (1) is not $D(\alpha, r)$-stable, then there exists a solution g of the characteristic equation (19) satisfying

$$|g| = \left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\}\right| \geq 1. \tag{20}$$

Based on Lemma 1 and (20), we can get the following inequality:

$$\begin{aligned} d_s = \rho\left(\frac{A-\alpha I}{r}\right) &\leq \left\|\frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\| \\ &\leq \frac{1}{r}\left[\|\Delta A\| + \left\|\sum_{i=1}^{n} A_{di}(rg+\alpha)^{-h_i}\right\| + \sum_{i=1}^{n}\|\Delta A_{di}\|\;|rg+\alpha|^{-h_i}\right] = h(g) \quad (21) \\ &\leq \frac{1}{r}\left[\beta + \sum_{i=1}^{n}(\|A_{di}\| + \eta_i)(r - |\alpha|)^{-h_i}\right] = d_1, \qquad \text{for } |g| \geq 1. \end{aligned}$$

Moreover, according to (20), we have

$$\begin{aligned} 1 \leq |g| &= \left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\}\right| \\ &\leq \left\|\frac{A-\alpha I}{r}\right\| + \frac{1}{r}\left(\beta + \sum_{i=1}^{n}(\|A_{di}\| + \eta_i)(r-|\alpha|)^{-h_i}\right) = d_{1r}. \end{aligned} \tag{22}$$

This implies that if system (1) is not $D(\alpha, r)$-stable, then all the unstable poles of this system must be within the bounded region $U_1 = \{\delta \mid 1 \leq |\delta| \leq d_{1r}\}$. Hence, if inequality (21) is not true (i.e., $h(g)$ lies outside the interval $[d_s, d_1]$) for all $g \in U_1$, then system (1) is robustly $D(\alpha, r)$-stable. This completes the proof of Case II.

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