Commun. Math. Phys. 149, 263–278 (1992)

Communications in
Mathematical
Physics

Solving the KP Hierarchy by Gauge Transformations

Ling-Lie Chau[1], **J.C. Shaw**[2], **and H.C. Yen**[3]

[1] Department of Physics, University of California, Davis, California 95616, and Los Alamos National Laboratory, Los Alamos, NM 87545, USA

[2] Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, R.O.C.

[3] Department of Physics, National Tsing Hua University, Hsinchu, Taiwan, R.O.C.

Received July 1, 1991; in revised form January 10, 1992

Abstract. We show that it is convenient to use "gauge" transformations (sometimes called explicit Bäcklund transformations) to generate new solutions for the KP hierarchy. Two particular kinds of gauge transformation operators, constructed out of the initial wave functions, are of fundamental importance in this approach. Through such gauge transformations, a very simple formula for the tau-function is obtained, encompassing and unifying all kinds of existing solutions. The corresponding free fermion representation and Baker functions for the new τ function can also be constructed.

1. Introduction

There are several different ways to formulate the mathematical problem of the KP hierarchy equations. For our purpose it is most convenient to adopt the pseudo-differential operator formalism developed by Sato and his school [1–5]. By the KP hierarchy we mean a particular infinite set of coupled nonlinear equations for u_i ($i = 2, 3, \ldots$), where each $u_i = u_i(x_1, x_2, x_3, \ldots)$ depends on one "spatial" variable x_1 and infinitely many "time" variables $x_2, x_3, \ldots$. These coupled equations are to be generated in the following way [2].

Let Λ denote the pseudo-differential operator

$$\Lambda \equiv \partial + u_2\partial^{-1} + u_3\partial^{-2} + u_4\partial^{-3} + \ldots, \tag{1.1}$$

where $\partial \equiv \partial/\partial x_1$, and ∂^{-1} is a suitable inverse of ∂, obeying the generalized Leibniz rule

$$\partial^{-n} \circ f(x_1) = \sum_{l=0}^{\infty} (-1)^l \frac{(n+l-1)!}{l!(n-1)!} f^{(l)}(x_1)\partial^{-n-l}, \quad (n > 0) . \tag{1.2}$$

For an operator multiplication we put a "$\circ$" in between, e.g., $\partial \circ f \equiv \partial f + f\partial \circ$. Now let

$$B_n \equiv [\Lambda^n]_+ , \tag{1.3}$$

where the symbol $[\Omega]_+$ denotes the differential operator part of a pseudo-differential operator Ω. Later on we will often symbolically represent B_n as

$$B_n = \sum_{j=0}^{n} b_{nj}\partial^j \quad \text{with} \quad b_{nn} = 1, \quad b_{nn-1} = 0\,, \tag{1.4}$$

where each b_{nj} is a specific functional of $u_2, u_3, \ldots, u_{n-j}$ and their spatial derivatives.

Now we impose a generalized Lax equation on Λ:

$$\frac{\partial \Lambda}{\partial x_n} = [B_n, \Lambda], \quad (n = 2, 3, \ldots)\,, \tag{1.5}$$

which, upon expansion into Laurent series in ∂^{-1}, gives rise to an infinite set of equations of the form [3]

$$\frac{\partial u_i}{\partial x_n} = \begin{array}{l}\text{a functional of } u_2, u_3, \ldots, u_{i+n-1} \text{ and their}\\ \text{spatial derivatives, } (i, n = 2, 3, \ldots)\end{array} \,. \tag{1.6}$$

This infinite set of equations is called the KP hierarchy.

However, it is easy to derive from (1.3) and (1.5) the following infinite set of operator equations:

$$\frac{\partial B_m}{\partial x_n} - \frac{\partial B_n}{\partial x_m} + [B_m, B_n] = 0 \quad (m, n = 2, 3, \ldots)\,, \tag{1.7}$$

which is called the Zakharov–Shabat (ZS) equation [6]. Although (1.7) is a consequence of (1.5), it turns out [2, 4] that the whole set of equations in (1.7) is exactly equivalent to the whole set of equations in (1.6). Hence the KP hierarchy can be alternatively represented by the ZS equation (1.7).

If we can find a set of functions $\{u_2, u_3, \ldots\}$ and hence a corresponding set of differential operators $\{B_2, B_3, \ldots\}$ satisfying (1.7), then we have a solution to the KP hierarchy. But it has been shown (see, for example, ref. 3) that any such set of $\{u_2, u_3, \ldots\}$ can be generated from a single function $\tau(x)$, the so-called tau-function, such that

$$u_2 = \frac{\partial^2}{\partial x_1^2} \log \tau\,, \tag{1.8a}$$

$$u_3 = \frac{1}{2}\left[\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^3}{\partial x_1^3}\right] \log \tau\,, \tag{1.8b}$$

etc. Thus we will alternatively represent a solution to the KP hierarchy by its corresponding tau-function.

The purpose of this paper is to carefully study the ZS Eq. (1.7) as well as its associated linear system

$$\frac{\partial}{\partial x_n} \phi(x_1, x_2, \ldots) = B_n \phi(x_1, x_2, \ldots)\,, \quad (n = 2, 3, \ldots)\,, \tag{1.9}$$

aiming at the establishment of a general constructive procedure for generating a new solution to the KP hierarchy. In contrast to some other existing approaches

[2, 3], our method in the following will need not concern the Lax Eq. (1.5) itself, [and consequently nor its associated eigenvalue equation

$$\Lambda w(x;\lambda) = \lambda w(x;\lambda)] . \tag{1.10}$$

What we call the eigenfunction, $w(x, \lambda)$, is commonly called the Baker function [7]. The reason to concentrate solely on the ZS equation is that it represents a zero-curvature condition, which enables us to make a direct extension to the present case of a gauge transformation method [8–11] that has been successfully applied to a broad class of $1 + 1$ dimensional nonlinear evolution equations.

The basic idea of the gauge transformation method is the following. Suppose $\{B_n^{(0)}, n = 2, 3, \ldots\}$ already satisfies (1.7). Let

$$B_n^{(1)} \equiv \Psi \circ B_n^{(0)} \circ \Psi^{-1} + \frac{\partial \Psi}{\partial x_n} \circ \Psi^{-1} , \tag{1.11}$$

where $\Psi = \Psi(x_1, x_2, \ldots)$ is any reasonable pseudo-differential operator. Then $\{B_n^{(1)}\}$ will necessarily satisfy (1.7) also. Note that although $B_n^{(0)}$ are differential operators, the right-hand side of Eq. (1.11) will in general not be a purely differential operator. But $\{B_n^{(1)}\}$ represents a valid new solution to the KP hierarchy only if all $B_n^{(1)}$, as defined by (1.11), happen to be purely differential operators. Through some educated guesses we have managed to find two particular constructions of the desired operator Ψ that will make all $B_n^{(1)}$ of (1.11) purely differential operators. [Some special cases of these transformations have been considered in ref. 4 and for the KdV system in ref. 11.] By repeatedly applying these two kinds of gauge transformations on any given input solution, one can generate all sorts of new solutions to the KP hierarchy.

Such a procedure is very elementary both conceptually and computationally, yet is powerful enough to encompass all kinds of solutions that have been known so far. For example, both the Wronskian solutions [12–14] and the Nakamura determinant solutions [15, 16] can now be derived in a completely unified fashion, while originally they were separately discovered through some kind of conjectures. See Sect. IV.

In the next section, we describe how these two kinds of gauge transformation operators are constructed. In Sect. III, we analyze the results of successive applications of such gauge transformations and obtain an exceedingly simple formula for the tau-function of a general new solution, which represents the main achievement of our work. Comparisons with some other methods are discussed in Sect. IV. In Sect. V, we discuss some relations to the Baker function and give an interpretation in terms of free fermion representations.

II. Gauge Transformation Operators

Suppose $\phi^{(0)}$ is a known wave function for $\{B_n^{(0)}\}$, i.e., a known solution to the equation

$$\frac{\partial \phi^{(0)}}{\partial x_n} = B_n^{(0)} \phi^{(0)} \quad (n = 2, 3, \ldots) . \tag{2.1}$$

Now let

$$\Psi_D \equiv \partial - \phi_x^{(0)}/\phi^{(0)} , \tag{2.2}$$

which can also be written as

$$\Psi_D \equiv \phi^{(0)} \circ \partial \circ \phi^{(0)-1} . \tag{2.3}$$

[This operator was called Crum transformation in ref. 11.] Then, by using (2.1), it is easy to show that the right-hand side of (1.11) is indeed a differential operator. So $\{B_n^{(1)}\}$ will also satisfy the ZS equation (1.7) and represent a new solution to the KP hierarchy.

Furthermore, for the lowest few n, we find explicitly

$$\begin{aligned} B_2^{(1)} &= \partial^2 + 2[u_2^{(0)} + (\log \phi^{(0)})_{xx}] \\ &\equiv \partial^2 + 2u_2^{(1)} , \end{aligned} \tag{2.4}$$

[1]

$$\begin{aligned} B_3^{(1)} &= \partial^3 + 3[u_2^{(0)} + (\log \phi^{(0)})_{xx}]\partial + 3[u_3^{(0)} + u_{2x}^{(0)} \\ &\quad + \phi_x^{(0)} \phi_{xx}^{(0)}/\phi^{(0)^2} - \phi_x^{(0)^3}/\phi^{(0)3}] + 3[u_2^{(0)} + (\log \phi^{(0)})_{xx}]_x \\ &\equiv \partial^3 + 3u_2^{(1)}\partial + 3u_3^{(1)} + 3u_{2x}^{(1)} , \end{aligned} \tag{2.5}$$

etc., which means,

$$u_2^{(1)} = u_2^{(0)} + (\log \phi^{(0)})_{xx} , \tag{2.6a}$$

$$u_3^{(1)} = u_3^{(0)} + \frac{1}{2}\left[\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^3}{\partial x_1^3}\right] \log \phi^{(0)} , \tag{2.6b}$$

etc. Comparing with Eqs. (1.8), we see that under the gauge transformation

$$B_n^{(0)} \xrightarrow{\Psi_D} B_n^{(1)} , \tag{2.7}$$

where Ψ_D is given by (2.2), the tau-function is transformed according to

$$\tau^{(0)} \xrightarrow{\Psi_D} \tau^{(1)} = \phi^{(0)} \tau^{(0)} . \tag{2.8}$$

In addition, the wave function $\phi^{(1)}$ for the new solution $\{B_n^{(1)}\}$ can be taken simply as

$$\phi^{(1)} = \Psi_D \phi_1^{(0)} , \tag{2.9}$$

where $\phi_1^{(0)}$ is an arbitrary wave function for $\{B_n^{(0)}\}$ different from $\phi^{(0)}$, because then

$$\begin{aligned} \frac{\partial \phi^{(1)}}{\partial x_n} &= \frac{\partial \Psi_D}{\partial x_n} \phi_1^{(0)} + \Psi_D \frac{\partial \phi_1^{(0)}}{\partial x_n} \\ &= \left(\frac{\partial \Psi_D}{\partial x_n} \circ \Psi_D^{-1} + \Psi_D \circ B_n^{(0)} \circ \Psi_D^{-1} \right) \circ \Psi_D \phi_1^{(0)} = B_n^{(1)} \phi^{(1)} . \end{aligned} \tag{2.10}$$

Having described our first construction of the gauge transformation operator, (2.2), we now turn to our second construction. This however will call for a "conjugate" linear system, to be discussed in the following.

[1] To be economic with notations, we will use the abbreviations f_x for $\frac{\partial f}{\partial x_1}$

From a given $\{B_n\}$, Eq. (1.4), we define its conjugate $\{\bar{B}_n\}$ by

$$\bar{B}_n \equiv \sum_{j=0}^{n} (-1)^{j+1}\partial^j \circ b_{nj}\,, \tag{2.11}$$

and call the following equation the conjugate linear system:

$$\frac{\partial \bar{\phi}}{\partial x_n} = \bar{B}_n \bar{\phi}, \quad (n = 2, 3, \ldots)\,. \tag{2.12}$$

$\bar{\phi}$ is then called the conjugate wave function. It can be directly verified that the ZS Eq. (1.7) can be equivalently replaced by

$$\frac{\partial \bar{B}_m}{\partial x_n} - \frac{\partial \bar{B}_n}{\partial x_m} + [\bar{B}_m, \bar{B}_n] = 0 \quad (m, n = 2, 3, \ldots)\,. \tag{2.13}$$

Now the construction of our second gauge transformation operator: let

$$\Psi_I \equiv (\partial + \bar{\phi}_x^{(0)}/\bar{\phi}^{(0)})^{-1}\,, \tag{2.14}$$

which can also be written as

$$\Psi_I = (\bar{\phi}^{(0)-1} \circ \partial \circ \bar{\phi}^{(0)})^{-1} = \bar{\phi}^{(0)-1} \circ \partial^{-1} \circ \bar{\phi}^{(0)}\,. \tag{2.15}$$

Now define $B_n^{(1)}$ by

$$B_n^{(1)} \equiv \Psi_I \circ B_n^{(0)} \circ \Psi_I^{-1} + \frac{\partial \Psi_I}{\partial x_n} \circ \Psi_I^{-1}\,. \tag{2.16}$$

Again we can show that the right-hand side of (2.16) is in fact a purely differential operator, which hence represents another new solution to the KP hierarchy.

From (2.14) and (2.16), we find explicitly

$$B_2^{(1)} = \partial^2 + 2u_2^{(0)} + 2(\log \bar{\phi}^{(0)})_{xx}\,, \tag{2.17}$$

that is,

$$\tau^{(0)} \xrightarrow{\Psi_I} \tau^{(1)} = \bar{\phi}^{(0)}\tau^{(0)}\,. \tag{2.18}$$

Furthermore, the new wave function $\phi^{(1)}$ corresponding to $\{B_n^{(1)}\}$ of (2.16) can be taken to be

$$\phi^{(1)} = \Psi_I \phi^{(0)} = (\partial + \bar{\phi}_x^{(0)}/\bar{\phi}^{(0)})^{-1}\phi^{(0)}\,. \tag{2.19}$$

It can be shown that the right-hand side of (2.19) evaluates to

$$\left(\partial + \frac{\bar{\phi}_x^{(0)}}{\bar{\phi}^{(0)}}\right)^{-1} \phi^{(0)} = \frac{1}{\bar{\phi}^{(0)}}\left[C + \int_{x_c}^{x} \bar{\phi}^{(0)}\phi^{(0)}\,dx_1\right], \tag{2.20}$$

where C is some arbitrary function of $x_2, x_3, \ldots$, satisfying the following boundary conditions [15] at $x_1 = x_c$ (x_c is any fixed constant):

$$\frac{\partial C}{\partial x_2} = [\phi_x^{(0)}\bar{\phi}^{(0)} - \phi^{(0)}\bar{\phi}_x^{(0)}]_{x_1 = x_c}\,, \tag{2.21a}$$

$$\frac{\partial C}{\partial x_3} = [\phi_{xx}^{(0)}\bar{\phi}^{(0)} - \phi_x^{(0)}\bar{\phi}_x^{(0)} + \phi^{(0)}\bar{\phi}_{xx}^{(0)} + 3u_2^{(0)}\phi^{(0)}\bar{\phi}^{(0)}]_{x_1 = x_c}\,, \tag{2.21b}$$

etc. These boundary conditions follow from the requirement that the new wave function $\phi^{(1)}$ of (2.19) must satisfy

$$\frac{\partial \phi^{(1)}}{\partial x_n} = B_n^{(1)} \phi^{(1)} \quad (n = 2, 3, \dots) . \tag{2.22}$$

In particular, if we choose $x_c = \infty$ and assume that $\phi^{(0)}$, $\bar{\phi}^{(0)}$, $u_n^{(0)}$ and all their spatial derivatives vanish at $x_c = \infty$, then C can be taken to be any numerical constant. We will on some future occasions choose specialized values for such constants, and write, for example,

$$\left(\partial + \frac{\bar{\phi}_{jx}}{\bar{\phi}_j} \right)^{-1} \phi_i = \frac{-1}{\bar{\phi}_j} \left[\delta_{ij} + \int_x^{\infty} \bar{\phi}_j \phi_i \, dx_1 \right] , \tag{2.23}$$

where $\{\phi_1, \phi_2, \dots\}$ is a prechosen set of wave functions for $\{B_n^{(0)}\}$, and $\{\bar{\phi}_1, \bar{\phi}_2, \dots\}$ another prechosen set of conjugate wave functions for $\{\bar{B}_n^{(0)}\}$. However, in the following and throughout the paper, we will generally denote

$$\frac{1}{\phi_i} \left[C_{ij} + \int_{x_c}^{x} \phi_i \bar{\phi}_j \, dx_1 \right] \quad \text{simply as} \quad \frac{1}{\phi_i} \int^{x} \phi_i \bar{\phi}_j \tag{2.24}$$

for notational simplicity.

To summarize, we have found two particular kinds of gauge transformation operators (called from now on the differential type Ψ_D and the integral type Ψ_I,) which seem to be of fundamental importance for generating new solutions to the KP hierarchy.

Naturally, the gauge transformation operation can be repeatedly applied, and will be schematically represented as

$$\begin{array}{ccccccc}
B_n^{(0)} & \xrightarrow{\Psi^{(1)}} & B_n^{(1)} & \xrightarrow{\Psi^{(2)}} & B_n^{(2)} \longrightarrow \dots & \xrightarrow{\Psi^{(N)}} & B_n^{(N)} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\phi^{(0)} & & \phi^{(1)} & & \phi^{(2)} & & \phi^{(N)} , \\
\bar{\phi}^{(0)} & & \bar{\phi}^{(1)} & & \bar{\phi}^{(2)} & & \bar{\phi}^{(N)} \\
\tau^{(0)} & & \tau^{(1)} & & \tau^{(2)} & & \tau^{(N)}
\end{array} \tag{2.25}$$

where the wave functions at each stage satisfy

$$\frac{\partial \phi^{(i)}}{\partial x_n} = B_n^{(i)} \phi^{(i)} , \qquad \frac{\partial \bar{\phi}^{(i)}}{\partial x_n} = \bar{B}_n^{(i)} \bar{\phi}^{(i)} , \tag{2.26}$$

$$\phi^{(i)} = \Psi^{(i)} \phi^{(i-1)} , \qquad \bar{\phi}^{(i)} = \bar{\Psi}^{(i)} \bar{\phi}^{(i-1)} , \tag{2.27}$$

and each $\Psi^{(i)}$ can be of either differential or integral type ($\Psi_D^{(i)}$ or $\Psi_I^{(i)}$). When $\Psi^{(i)}$ is constructed out of a particular wave function $\phi_p^{(i-1)}$ (or $\bar{\phi}_p^{(i-1)}$), then

$$\tau^{(i)} = \phi_p^{(i-1)} \tau^{(i-1)} \quad (\text{or } \tau^{(i)} = \bar{\phi}_p^{(i-1)} \tau^{(i-1)}) . \tag{2.28}$$

Also, if $\Psi^{(i)} = \Psi_D[\phi_p^{(i-1)}]$, then $\bar{\Psi}^{(i)} = \Psi_I[\phi_p^{(i-1)}]$, and if $\Psi^{(i)} = \Psi_I[\bar{\phi}_p^{(i-1)}]$, then $\bar{\Psi}^{(i)} = \Psi_D[\bar{\phi}_p^{(i-1)}]$.

Note. The symbol $\phi^{(i-1)}$ $[\bar{\phi}^{(i-1)}]$ actually denotes any one of infinitely many possible $(i-1)^{\text{th}}$ stage (conjugate) wave functions. The $\phi^{(i-1)}$ $[\bar{\phi}^{(i-1)}]$ used in (2.27) can be any such wave function except $\phi_p^{(i-1)}$ $[\bar{\phi}_p^{(i-1)}]$, since the latter will make $\phi^{(i)}$ $[\bar{\phi}^{(i)}]$ vanish identically.

III. Successive Applications of Gauge Transformations

Let us examine the net results of successive applications of such gauge transformations in a specific example. This will eventually enable us to write down a very neat general formula of the final tau-function $\tau^{(N)}$ when the gauge transformation operation has been applied N times on the initial solution $\tau^{(0)}$. For the discussion in this section, we adopt the following special notations to reduce some possible confusions. $\{\phi_1, \phi_2, \ldots\}$ will denote the wave functions for $B_n^{(0)}$ and $\{\bar{\phi}_1, \bar{\phi}_2, \ldots\}$ the corresponding conjugate wave functions. After successive gauge transformations $B_n^{(0)} \to B_n^{(1)} \to B_n^{(2)} \to \ldots, \{\alpha_i, \bar{\alpha}_j\}$ will denote the (conjugate) wave functions for $B_n^{(1)}$; $\{\beta_i, \bar{\beta}_j\}$ for $B_n^{(2)}$; $\{\gamma_i, \bar{\gamma}_j\}$ for $B_n^{(3)}$, etc. We exhibit a calculation with three Ψ_D and one Ψ_I transformations as an illustration.

$$B_n^{(0)} \xrightarrow{\Psi_D(\phi_1)} B_n^{(1)} \xrightarrow{\Psi_D(\alpha_2)} B_n^{(2)} \xrightarrow{\Psi_D(\beta_3)} B_n^{(3)} \xrightarrow{\Psi_I(\bar{\gamma}_1)} B_n^{(4)}$$

$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
ϕ_1	α_2	β_3	$\bar{\gamma}_1$
ϕ_2	α_3	$\bar{\beta}_1$	
ϕ_3	$\bar{\alpha}_1$		
$\bar{\phi}_1$			

After the first gauge transformation,

$$\tau^{(0)} \to \tau^{(1)} = \phi_1 \tau^{(0)}, \tag{3.1}$$

and the wave function α_i is given by [see (2.9)]

$$\alpha_i \equiv \Psi_D(\phi_1)\phi_i = \phi_1 \left(\frac{\phi_i}{\phi_1}\right)_x \quad (i = 2, 3), \tag{3.2}$$

while the conjugate wave function $\bar{\alpha}_1$ is given by [see (2.19) and (2.24)]

$$\bar{\alpha}_1 \equiv \bar{\Psi}_I(\phi_1)\bar{\phi}_1 = \left(\partial + \frac{\phi_{1x}}{\phi_1}\right)^{-1} \bar{\phi}_1 = \frac{1}{\phi_1} \int^x \phi_1 \bar{\phi}_1 \,. \tag{3.3}$$

After the second gauge transformation,

$$\tau^{(1)} \to \tau^{(2)} = \alpha_2 \tau^{(1)}, \tag{3.4}$$

$$\beta_3 \equiv \Psi_D(\alpha_2)\alpha_3 = \alpha_2 \left(\frac{\alpha_3}{\alpha_2}\right)_x, \tag{3.5}$$

$$\bar{\beta}_1 \equiv \bar{\Psi}_I(\alpha_2)\bar{\alpha}_1 = \frac{1}{\alpha_2} \int^x \alpha_2 \bar{\alpha}_1 \,. \tag{3.6}$$

After the third gauge transformation,

$$\tau^{(2)} \to \tau^{(3)} = \beta_3 \tau^{(2)} , \tag{3.7}$$

$$\bar{\gamma}_1 \equiv \bar{\Psi}_I(\beta_3)\bar{\beta}_1 = \frac{1}{\beta_3} \int^x \beta_3 \bar{\beta}_1 . \tag{3.8}$$

After the fourth gauge transformation,

$$\tau^{(3)} \to \tau^{(4)} = \bar{\gamma}_1 \tau^{(3)} . \tag{3.9}$$

Using (3.1)–(3.7), we have

$$\begin{aligned} \tau^{(2)} &= \alpha_2 \phi_1 \tau^{(0)} = (\phi_1 \phi_{2x} - \phi_2 \phi_{1x}) \tau^{(0)} \\ &\equiv W(\phi_1, \phi_2) \tau^{(0)} , \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \tau^{(3)} &= \beta_3 \alpha_2 \phi_1 \tau^{(0)} = (\alpha_2 \alpha_{3x} - \alpha_3 \alpha_{2x}) \phi_1 \tau^{(0)} \\ &= (\phi_1 \phi_{2x} \phi_{3xx} - \phi_2 \phi_{1x} \phi_{3xx} - \phi_3 \phi_{2x} \phi_{1xx} \\ &\quad - \phi_1 \phi_{3x} \phi_{2xx} + \phi_3 \phi_{1x} \phi_{2xx} + \phi_2 \phi_{3x} \phi_{1xx}) \tau^{(0)} \\ &\equiv W(\phi_1, \phi_2, \phi_3) \tau^{(0)} . \end{aligned} \tag{3.11}$$

We have used the notation $W(\phi_1, \phi_2, \ldots, \phi_N)$ to denote the Wronksian of $\phi_1, \phi_2, \ldots, \phi_N$. Substituting (3.2) and (3.3) into (3.6) and using integration by parts, we get

$$\bar{\beta}_1 = \frac{1}{\alpha_2} \left[\left(\frac{\phi_2}{\phi_1} \right) \int^x \phi_1 \bar{\phi}_1 - \int^x \phi_2 \bar{\phi}_1 \right] . \tag{3.12}$$

Similarly, from (3.5), (3.6), and (3.8), we have

$$\bar{\gamma}_1 = \frac{1}{\beta_3} \left[\left(\frac{\alpha_3}{\alpha_2} \right) \int^x \alpha_2 \bar{\alpha}_1 - \int^x \alpha_3 \bar{\alpha}_1 \right] , \tag{3.13}$$

which can be further reduced to

$$\begin{aligned} \bar{\gamma}_1 = \frac{1}{\beta_3 \alpha_2} \Bigg\{ & \phi_1 \left(\frac{\phi_3}{\phi_1} \right)_x \left[\left(\frac{\phi_2}{\phi_1} \right) \int^x \phi_1 \bar{\phi}_1 - \int^x \phi_2 \bar{\phi}_1 \right] \\ & - \phi_1 \left(\frac{\phi_2}{\phi_1} \right)_x \left[\left(\frac{\phi_3}{\phi_1} \right) \int^x \phi_1 \bar{\phi}_1 - \int^x \phi_3 \bar{\phi}_1 \right] \Bigg\} . \end{aligned} \tag{3.14}$$

Finally, we obtain

$$\begin{aligned} \tau^{(4)}(DDDI) &= \bar{\gamma}_1 \beta_3 \alpha_2 \phi_1 \tau^{(0)} \\ &= \begin{vmatrix} \int^x \phi_1 \bar{\phi}_1 & \int^x \phi_2 \bar{\phi}_1 & \int^x \phi_3 \bar{\phi}_1 \\ \phi_1 & \phi_2 & \phi_3 \\ \phi_{1x} & \phi_{2x} & \phi_{3x} \end{vmatrix} \cdot \tau^{(0)} . \end{aligned} \tag{3.15}$$

It can be verified by separate computations that the following three different orderings of the three Ψ_D and one Ψ_I transformations all lead to the same expression for the final $\tau^{(4)}$ as given by (3.15):

$$B_n^{(0)} \xrightarrow{\Psi_D(\phi_1)} B_n^{(1)} \xrightarrow{\Psi_D(\alpha_2)} B_n^{(2)} \xrightarrow{\Psi_I(\bar{\beta}_1)} B_n^{(3)} \xrightarrow{\Psi_D(\gamma_3)} B_n^{(4)}$$

$$B_n^{(0)} \xrightarrow{\Psi_D(\phi_1)} B_n^{(1)} \xrightarrow{\Psi_I(\bar{\alpha}_1)} B_n^{(2)} \xrightarrow{\Psi_D(\beta_2)} B_n^{(3)} \xrightarrow{\Psi_D(\gamma_3)} B_n^{(4)}$$

$$B_n^{(0)} \xrightarrow{\Psi_I(\bar{\phi}_1)} B_n^{(1)} \xrightarrow{\Psi_D(\alpha_1)} B_n^{(2)} \xrightarrow{\Psi_D(\beta_2)} B_n^{(3)} \xrightarrow{\Psi_D(\gamma_3)} B_n^{(4)} .$$

Thus $\tau^{(4)}(DDDI) = \tau^{(4)}(DDID) = \tau^{(4)}(DIDD) = \tau^{(4)}(IDDD) = (3.15)$. This property of ordering-independence is completely general, and exactly corresponds to the well-known permutability [17, 18] of Bäcklund transformations for the 1 + 1 dimensional integrable nonlinear equations. We note that results such as (3.10) and (3.11) are well known for the KdV system [11], but the expression like (3.15) seems to be completely new.

In general, consider the N-step transformation

$$B_n^{(0)} \xrightarrow{\Psi^{(1)}} B_n^{(1)} \xrightarrow{\Psi^{(2)}} B_n^{(2)} \longrightarrow \dots \xrightarrow{\Psi^{(N)}} B_n^{(N)} . \tag{3.16}$$

It can be shown by straightforward computation that if among the N gauge transformation operators $\Psi^{(1)}, \Psi^{(2)}, \dots, \Psi^{(N)}$, s of them are of Ψ_D type and $r(= N - s)$ of them are of Ψ_I type relative to the linear system (as opposed to the conjugate linear system), then the final tau-function $\tau^{(N)}$ is given by the following formula:

(i) if $s \geqq r$,

$$\tau^{(N)} = \begin{vmatrix} \int^x \phi_1\bar{\phi}_1 & \int^x \phi_2\bar{\phi}_1 & \cdots & \int^x \phi_s\bar{\phi}_1 \\ \vdots & \vdots & & \vdots \\ \int^x \phi_1\bar{\phi}_r & \int^x \phi_2\bar{\phi}_r & \cdots & \int^x \phi_s\bar{\phi}_r \\ \phi_1 & \phi_2 & \cdots & \phi_s \\ \phi_{1x} & \phi_{2x} & \cdots & \phi_{sx} \\ \phi_{1xx} & \phi_{2xx} & \cdots & \phi_{sxx} \\ \vdots & \vdots & & \vdots \end{vmatrix} \cdot \tau^{(0)} , \tag{3.17a}$$

(ii) if $r \geqq s$,

$$\tau^{(N)} = \begin{vmatrix} \int^x \bar{\phi}_1\phi_1 & \int^x \bar{\phi}_2\phi_1 & \cdots & \int^x \bar{\phi}_r\phi_1 \\ \vdots & \vdots & & \vdots \\ \int^x \bar{\phi}_1\phi_s & \int^x \bar{\phi}_2\phi_s & \cdots & \int^x \bar{\phi}_r\phi_s \\ \bar{\phi}_1 & \bar{\phi}_2 & \cdots & \bar{\phi}_r \\ \bar{\phi}_{1x} & \bar{\phi}_{2x} & \cdots & \bar{\phi}_{rx} \\ \bar{\phi}_{1xx} & \bar{\phi}_{2xx} & \cdots & \bar{\phi}_{rxx} \\ \vdots & \vdots & & \vdots \end{vmatrix} \cdot \tau^{(0)} , \tag{3.17b}$$

where $\{\phi_1, \phi_2, \ldots, \phi_s\}$ is an arbitrarily chosen set of wave functions corresponding to the initial solution $B_n^{(0)}$, and $\{\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_r\}$ is another arbitrarily chosen set of conjugate wave functions corresponding to the same initial solution, while the symbol $\int^x \phi_i \bar{\phi}_j$ has been explained in (2.20), (2.21), and (2.24).

IV. Comparisons with Results from Some Other Methods

To recapitulate, starting from a given solution $\{B_n^{(0)}\}$ of the KP hierarchy, we first solve for its wave functions $\{\phi_i\}$ and its conjugate wave functions $\{\bar{\phi}_i\}$. Then the net result of applying N successive transformations on $\tau^{(0)}$ is the expression $\tau^{(N)}$ given by (3.17). Several interesting particular cases will be discussed in this section.

IV.a. The Wronskian Solutions. When $r = 0$ $(N = s)$,

$$\tau^{(s)} = W(\phi_1, \phi_2, \ldots \phi_s)\tau^{(0)}\,, \tag{4.1}$$

and when $s = 0$ $(N = r)$,

$$\tau^{(r)} = W(\bar{\phi}_1, \bar{\phi}_2, \ldots \bar{\phi}_s)\tau^{(0)}\,. \tag{4.2}$$

These are the well-known Wronskian solutions [12, 13] for the KP hierarchy. Our result here has effectively provided a simple derivation of the Wronskian solutions. When $u_n^{(0)} = 0$ $(n = 2, 3, \ldots)$ is taken, solutions of various types can be obtained [14] straightforwardly from the Wronskain formula (4.1). Here we will note only the familiar (exponential type) N-soliton solution and Pöppe's rational solutions [19].

Since $u_n^{(0)} = 0$, we have $B_n^{(0)} = \partial^n$ and we may choose $\tau^{(0)} = 1$. The most general wave function $\phi_i(x)$ can then be expressed as

$$\phi_i(x) = \int dk\, W_i(k) e^{\xi(x,k)}\,, \tag{4.3}$$

where

$$\xi(x, k) \equiv x_1 k + x_2 k^2 + x_3 k^3 + \ldots\,, \tag{4.4}$$

and $W_i(k)$ is an arbitrary function (or distribution) of k, while the path of integration in the complex k-plane is also arbitrary, as long as the right-hand side of (4.3) produces a well-defined function $\phi_i(x)$. As an example, the choice

$$\phi(x) = \int_{-\infty}^{\infty} ds\, e^{\xi(x, -is)} \tag{4.5}$$

would lead to a solution generalizing the Airy function solution [20, 21] of the KP equation.

To derive the N-soliton solution through the Wronskian formula, we may take

$$W_i(k) = \delta(k - q_i) + d_i \delta(k - p_i), \quad (i = 1, 2, \ldots N) \tag{4.6}$$

and hence

$$\phi_i(x) = e^{\xi(x, q_i)} + d_i e^{\xi(x, p_i)}\,, \tag{4.7}$$

where $q_i \neq p_j$ for all i and j, and $q_i \neq q_j$ for $i \neq j$. We will just compute $\tau^{(2)}$ as an illustration,

$$\tau^{(2)} = W(\phi_1, \phi_2) = e^{\xi(x,q_1)+\xi(x,q_2)} \cdot \begin{vmatrix} 1 + d_1 e^{\xi_1} & 1 + d_2 e^{\xi_2} \\ q_1 + d_1 p_1 e^{\xi_1} & q_2 + d_2 p_2 e^{\xi_2} \end{vmatrix}, \tag{4.8}$$

where

$$\xi_i \equiv \xi(x, p_i) - \xi(x, q_i) . \tag{4.9}$$

Equation (4.8) can be rewritten as

$$\tau^{(2)} = e^{\xi(x,q_1)+\xi(x,q_2)} \cdot (q_2 - q_1) \cdot [1 + a_1 e^{\xi_1} + a_2 e^{\xi_2} + C_{12} a_1 a_2 e^{\xi_1+\xi_2}] \tag{4.10}$$

with

$$a_1 = d_1 \left(\frac{p_1 - q_2}{q_1 - q_2} \right), \quad a_2 = d_2 \left(\frac{p_2 - q_1}{q_2 - q_1} \right), \tag{4.11}$$

and

$$C_{12} = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} . \tag{4.12}$$

Thus up to an allowable factor $e^{\xi(x,q_1)+\xi(x,q_2)} \cdot (q_2 - q_1)$, this tau-function coincides with the standard tau-function of the 2-soliton solution as given in ref. 2. Obviously this procedure will also work for higher N-soliton solutions.

Some algebraic type solutions of the KP hierarchy can also be easily obtained through the Wronskian formula if $\phi_i(x)$ are appropriately chosen. For example, we may choose

$$W_i(k) = - e^{\eta_i k} \delta'(k - q_i) \tag{4.13}$$

for the weighting function in (4.3), where η_i is an arbitrary constant. Then we have

$$\begin{aligned} \phi_i(x) &= \frac{\partial}{\partial q_i} [e^{\xi(x,q_i)+\eta_i q_i}] \\ &= (\eta_i + x_1 + 2q_i x_2 + 3q_i^2 x_3 + \ldots) e^{\xi(x,q_i)+\eta_i q_i} . \end{aligned} \tag{4.14}$$

Again, we compute $\tau^{(2)}$ as an illustration.

$$\begin{aligned} \tau^{(2)} &= W(\phi_1, \phi_2) \\ &= \exp[\xi(x, q_1) + \xi(x, q_2) + \eta_1 q_1 + \eta_2 q_2] \cdot (q_2 - q_1) \end{aligned}$$

$$\cdot \begin{vmatrix} x_1 + 2q_1 x_2 + \ldots + \left(\eta_1 - \dfrac{1}{q_2 - q_1} \right) & \dfrac{-1}{q_1 - q_2} \\ \dfrac{-1}{q_2 - q_1} & x_1 + 2q_2 x_2 + \ldots + \left(\eta_2 + \dfrac{1}{q_2 - q_1} \right) \end{vmatrix} . \tag{4.15}$$

Thus up to an allowable factor $\exp[\xi(x, q_1) + \xi(x, q_2) + \eta_1 q_1 + \eta_2 q_2] \cdot (q_2 - q_1)$, this tau-function is the same as that for the $N = 2$ case of Pöppe's rational solutions

[19], which are generally given by

$$\tau = \det(F_{ij})_{N \times N} , \tag{4.16}$$

with

$$F_{ij} = \begin{cases} x_1 + 2q_i x_2 + 3q_i^2 x_3 + \ldots + \eta_i', & i = j , \\ \dfrac{-1}{q_i - q_j}, & i \neq j . \end{cases} \tag{4.17}$$

IV.b. The Nakamura Determinant Solutions. Now, we come back to the general formula (3.17) and consider the special cases in which $r = s$ ($N = 2s$). Then we have

$$\tau^{(2s)} = \det(h_{ij})_{s \times s} \cdot \tau^{(0)} , \tag{4.18}$$

where

$$h_{ij} = \int^x \phi_i \bar{\phi}_j = C_{ij} + \int_{x_c}^x \phi_i \bar{\phi}_j \, dx_1 . \tag{4.19}$$

This is exactly the Nakamura determinant solution [15]. In particular, if we choose $x_c = \infty$ and $C_{ij} = -\delta_{ij}$, then $\tau^{(2s)}$ of (4.18) coincides, as shown in refs. 16 and 20, with the tau-function that would be obtained via the dressing method [9], i.e., by solving the Gel'fand–Levitan–Marchenko equation

$$K(x, z) + F(x, z) + \int_x^\infty K(x, s) F(s, z) \, ds = 0 , \tag{4.20}$$

with

$$F(x, z) \equiv \sum_{i=1}^{s} \phi_i(x) \bar{\phi}_i(z) . \tag{4.21}$$

IV.c. The Character Polynomials. It is also straightforward to make a connection between our formula (3.17) and the character polynomials $\chi_Y(x)$, which have been shown [2] to be tau-functions for the KP hierarchy. Starting from $u_n^{(0)} = 0$ and $\tau^{(0)} = 1$, we Taylor expand the wave function $e^{\xi(x, k)}$ around $k = 0$,

$$e^{\xi(x, k)} \equiv \sum_{l=0}^{\infty} p_l(x) k^l , \tag{4.22}$$

and pick the l^{th} coefficient as the wave function ϕ_l,

$$\phi_l(x) = p_l(x) . \tag{4.23}$$

We note that

$$p_0 = 1, \quad p_1 = x_1, \quad p_2 = \frac{x_1^2}{2} + x_2, \quad p_3 = \frac{x_1^3}{6} + x_1 x_2 + x_3 , \tag{4.24}$$

etc., and the relation

$$\frac{\partial}{\partial x_1} p_l = p_{l-1} \quad (l \geqq 1) . \tag{4.25}$$

Similarly, we Taylor expand the conjugate wave function $e^{\xi(-x, -k)}$ around $k = 0$,

$$e^{\xi(-x, -k)} \equiv \sum_{l=0}^{\infty} q_l(x) k^l , \tag{4.26}$$

and pick the l^{th} coefficient as the conjugate wave function $\bar{\phi}_l$,

$$\bar{\phi}_l(x) = q_l(x) . \tag{4.27}$$

We note that

$$q_0 = 1, \quad q_1 = x_1, \quad q_2 = \frac{x_1^2}{2} - x_2, \quad q_3 = \frac{x_1^3}{6} - x_1 x_2 + x_3 , \tag{4.28}$$

etc., and

$$\frac{\partial}{\partial x_1} q_l = q_{l-1} \quad (l \geqq 1) , \tag{4.29}$$

$$q_l(x) = (-1)^l p_l(-x) . \tag{4.30}$$

Now we compute

$$C_{ij} + \int_0^x p_i(x) q_j(x) \, dx_1 , \tag{4.31}$$

where C_{ij} are polynomials in $x_2, x_3, \ldots$, so chosen as to satisfy the boundary conditions (2.21) at $x_c = 0$. The result of this computation is that

$$C_{ij} + \int_0^x p_i(x) q_j(x) dx_1 = \chi_{-(j+1),i}(x) , \tag{4.32}$$

where $\chi_{-(j+1),i}$ is the character polynomial $\chi_Y(x)$ with the Young diagram $Y = (i+1, \underbrace{1, 1, \ldots, 1}_{j})$; see ref. 2 for more details about $\chi_Y(x)$.

More generally, let us reassign

$$\phi_i \equiv p_{n_i} \quad (i = 1, 2, \ldots s), \quad n_i \geqq 0 , \tag{4.33}$$

$$\bar{\phi}_j \equiv q_{-m_j-1} \quad (j = 1, 2, \ldots r), \quad m_j \geqq 0 , \tag{4.34}$$

and get

$$C_{ij} + \int_0^x \phi_i(x) \bar{\phi}_j(x) dx_1 = \chi_{m_j n_i}(x) \tag{4.35}$$

according to the formula (4.32). Then Eq. (3.17) becomes, using (4.25) and (4.29), (i) if $s \geqq r$,

$$\tau^{(N)} = \begin{vmatrix} \chi_{m_1 n_1} & \chi_{m_1 n_2} & \cdots & \chi_{m_1 n_s} \\ \vdots & \vdots & & \vdots \\ \chi_{m_r n_1} & \chi_{m_r n_2} & \cdots & \chi_{m_r n_s} \\ p_{n_1} & p_{n_2} & \cdots & p_{n_s} \\ \vdots & \vdots & & \vdots \\ p_{n_1 - s + r + 1} & p_{n_2 - s + r + 1} & \cdots & p_{n_s - s + r + 1} \end{vmatrix} \tag{4.36a}$$

(ii) if $r \geqq s$,

$$\tau^{(N)} = \begin{vmatrix} \chi_{m_1 n_1} & \chi_{m_2 n_1} & \cdots & \chi_{m_r n_1} \\ \vdots & \vdots & & \vdots \\ \chi_{m_1 n_s} & \chi_{m_2 n_s} & \cdots & \chi_{m_r n_s} \\ q_{-m_1-1} & q_{-m_2-1} & \cdots & q_{-m_r-1} \\ \vdots & \vdots & & \vdots \\ q_{-m_1-r+s} & q_{-m_2-r+s} & \cdots & q_{-m_r-r+s} \end{vmatrix} \tag{4.36b}$$

These expressions of $\tau^{(N)}$ are identical (up to a sign) to those of $\chi_Y(x)$ with a general Young diagram Y obtained by *DKJM* [2] via a Fock space approach.

V. Free Fermion Representations of the New τ Functions and Relations to Baker Functions

Now we wish to make some remarks on the relevance of the eigenvalue Eq. (1.10) in our approach. The full linear system corresponding to the Lax Eq. (1.5) is given by

$$\Lambda w(x, \lambda) = \lambda w(x, \lambda) , \tag{5.1}$$

$$\frac{\partial}{\partial x_n} w(x, \lambda) = B_n w(x, \lambda) . \tag{5.2}$$

We also call the Baker function $w(x, \lambda)$ the eigenfunction of the KP hierarchy, which is uniquely determined up to a normalization factor $f(\lambda)$. In this paper we reserve the terminology "wave function" solely for the solution $\phi(x)$ of Eq. (1.9), which is the same as (5.2). Therefore, $\phi(x)$ and $w(x, \lambda)$ are generally related to each other by

$$\phi(x) = \int_\Gamma g(\lambda) w(x, \lambda)\, d\lambda , \tag{5.3}$$

where $g(\lambda)$ is an arbitrary distribution and Γ an arbitrary path of integration.

Under the gauge transformation Ψ that we have been considering, the pseudo-differential operator Λ of (5.1) simply transforms as

$$\Lambda^{(0)} \rightarrow \Lambda^{(1)} = \Psi \circ \Lambda^{(0)} \circ \Psi^{-1} , \tag{5.4}$$

and our $B_n^{(1)}$ in (1.11) are still correctly given by (1.3), i.e.,

$$B_n^{(1)} = [\Lambda^{(1)n}]_+ \quad (n = 2, 3, \ldots) . \tag{5.5}$$

However, as far as solution-generation is concerned, we have not found it necessary to use the eigenfunction $w(x, \lambda)$. Though our approach for solving the KP hierarchy has been based entirely on the Z-S equations, without directly referring to the eigenvalue equation (5.1), our method gives automatically new solutions for $w(x, \lambda)$. From a new tau-function $\tau^{(1)}(x)$ by the gauge transformation method, the corresponding eigenfunction $w^{(1)}(x, \lambda)$ is automatically given by [2, 3]

$$w^{(1)}(x, \lambda) = \frac{\tau^{(1)}(x_l - l^{-1}\lambda^{-l})}{\tau^{(1)}(x)} e^{\xi(x, \lambda)} , \tag{5.6}$$

where $\xi(x, \lambda)$ has been defined in (4.4).

Finally, we shall reexpress the main result of this paper in the free fermion language. As is well known [2], every tau function in this language is a vacuum expectation value of the form

$$\tau^{(0)}(x) = \langle \text{vac} | e^{H(x)} g^{(0)} | \text{vac} \rangle \,, \tag{5.7}$$

where

$$H(x) \equiv \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty} x_n : \psi_l \psi_{l+n}^* : \,, \tag{5.8}$$

and $g^{(0)}$ is some chosen element of the Clifford group generated by the free fermion operators ψ_l and ψ_l^*. Our result, Eq. (2.8), states that once a known tau function $\tau^{(0)}(x)$ and a corresponding wave function $\phi^{(0)}(x)$ have been given, then a new tau function $\tau^{(1)}(x)$ can be simply taken to be

$$\tau^{(1)} = \phi^{(0)} \tau^{(0)} \,.$$

In particular, we may choose for the present discussion $\phi^{(0)}$ to be the Baker function $w^{(0)}(x, \lambda)$. Through the relation between the Baker function and the tau function, viz.,

$$w^{(0)}(x, \lambda) = \frac{\tau^{(0)}(x_l - l^{-1}\lambda^{-l})}{\tau^{(0)}(x)} e^{\xi(x, \lambda)} \,, \tag{5.9}$$

we have

$$\tau^{(1)}(x, \lambda) = \tau^{(0)}(x_l - l^{-1}\lambda^{-l}) e^{\xi(x, \lambda)} \tag{5.10}$$

as the new tau function, which can also be expressed as

$$\tau^{(1)}(x, \lambda) = e^{\xi(x, \lambda)} e^{-\xi(x, \tilde{\partial})} \langle \text{vac} | e^{H(x)} g^{(0)} | \text{vac} \rangle \,, \tag{5.11}$$

with

$$\tilde{\partial} \equiv \left(\frac{\partial}{\partial x_1}, \frac{1}{2} \frac{\partial}{\partial x_2}, \frac{1}{3} \frac{\partial}{\partial x_3}, \ldots \right).$$

A simple calculation [2] shows that (5.11) can be written as

$$\tau^{(1)}(x, \lambda) = \langle 1 | e^{H(x)} \psi(\lambda) g^{(0)} | \text{vac} \rangle \,, \tag{5.12}$$

where $\psi(\lambda) \equiv \sum_{n=-\infty}^{\infty} \psi_n \lambda^n$, and $\langle 1 | \equiv \langle \text{vac} | \psi_0^*$. In other words, the effect of our differential type $\Psi_D(w^{(0)}(x, \lambda))$ gauge transformation on a tau function $\tau^{(0)}(x)$ is equivalent to inserting a fermion operator $\psi(\lambda)$ in front of $g^{(0)}$ in the vacuum expectation value and simultaneously changing the bra state from $\langle \text{vac} |$ to the next highest weight state $\langle 1 |$. For an independent check, one can directly verify that (5.12) is indeed a tau function by using the bilinear identity given in ref. [2].

Similarly, for integral type $\Psi_I(\bar{w}^{(0)}(x, \lambda))$ gauge transformation (2.18), we find

$$\bar{\tau}^{(1)}(x, \lambda) = \langle -1 | e^{H(x)} \psi^*(\lambda) g^{(0)} | \text{vac} \rangle \,, \tag{5.13}$$

where

$$\psi^*(\lambda) \equiv \sum_{n=-\infty}^{\infty} \psi_n^* \lambda^{-n}$$

and

$$\langle -1| \equiv \langle \mathrm{vac}|\psi_{-1} .$$

Acknowledgements. We thank Professor A. Schwarz for enlightening discussions and for pointing out the connections of our new τ functions to the free fermion representations. This work is partly supported by the US Department of Energy.

References

1. Sato, M.: RIMS Kokyuroku **439**, 30 (1981)
2. Date, E., Kashiwara, M., Jimbo, M., Miwa, T.: Nonlinear Integrable Systems-Classical Theory and Quantum Theory. Jimbo, M., Miwa, T. (eds.) p. 39, Singapore: World Scientific 1983
3. Ohta, Y., Satzuma, J., Takahashi, D., Tokihiro, T.: Progr. Theor. Phys. [Suppl.] **94**, 210 (1988)
4. Mulase, M.: Adv. Math. **54**, 57 (1984)
5. Shiota, T.: Invent. Math. **83**, 333 (1986)
6. Zakharov, V.E., Shabat, A.B.: Funct. Anal. Appl. **8**, 226 (1974)
7. Baker, H.F.: Note on the foregoing paper: Commutative ordinary differential operators. Burchnall, J.L., Chaundy, T.W. (eds.) Proc. R. Soc. London (A) **118**, 584–593 (1928); Segal, G., Wilson, G.: Publ. Math. IHES **61**, 5 (1985)
8. Zakharov, V.E., Mikhailov, A.V.: Sov. Phys. JETP **47**, 1017 (1978)
9. Novikov, S., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E.: Theory of Solitons New York: Consultants Bureau 1984
10. Chau, L.-L., Shaw, J.C., Yen, H.C.: J. Math. Phys. **32**, 1737 (1991)
11. Adler, M., Moser, J.: Commun. Math. Phys. **61**, 1 (1978)
12. Satsuma, J.: Phys. Soc. Jpn Lett. **46**, 359 (1979)
13. Freeman, N.C., Nimmo, J.J.C.: Phys. Lett. **95A**, 1 (1983)
14. Hirota, R., Ohta, Y., Satsuma, J.: Progr. Theor. Phys. [Suppl.] **94**, 59 (1988)
15. Nakamura, A.: J. Phys. Soc. Jpn **58**, 412 (1989)
16. Miyake, S., Ohta, Y., Satsuma, J.: J. Phys. Soc. Jpn **59**, 48 (1990)
17. See, for example, Bäcklund Transformations. Lecture Notes in Math. vol. **515**, Miura, R.M. (ed.) Berlin, Heidelberg, New York: Springer 1976
18. See, for example, Rogers, C., Shadwick, W.F.: Bäcklund transformations and their applications. New York: Academic Press 1982
19. Pöppe, C.: Inverse Problems **5**, 613 (1989)
20. Johnson, R.S., Thompson, S.: Phys. Lett. **66A**, 279 (1978)
21. Nakamura, A.: J. Phys. Soc. Jpn **51**, 19 (1982)

Communicated by S.-T. Yau