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Brief Paper

Block-Decoupling Linear Multivariable Systems: Necessary and Sufficient Conditions*

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Key Words—Block decoupling; linear multivariable systems; internal stability.

Abstract—We study block decoupling of linear multivariable systems under unity-feedback configuration. We consider plants which have unstable pole-zero coincidences and give necessary and sufficient conditions for the existence of the block decoupling controllers. The conditions are particularly simple for plants with only simple unstable pole-zero coincidences. An illustrative example is given. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Necessary and sufficient conditions for the existence of block decoupling controllers for linear multivariable systems under unity feedback configuration have been given in Linnemann and Wang (1993). The approach there is to find conditions under which there is an open-loop block decoupling precompensator that does not introduce unstable pole-zero cancellations. It is well known that a sufficient, but not necessary, condition for existence of (stabilizing) block decoupling controllers is that the plant has no unstable pole-zero coincidences. To find a necessary condition, we thus only need to consider plants in which unstable pole-zero coincidences do occur.

When there is an unstable pole-zero coincidence, closed-loop stability and block decoupling requirements introduce potentially conflictory interpolation conditions on the input–output transfer matrix (or equivalently the sensitivity matrix). Our approach is to derive conditions under which the interpolation equations arise from stability requirement have a solution under the block decoupling constraints. The conditions so obtained are particularly simple when the unstable coincidences are all simple.

We consider first the simple case in which the plant is square and the unstable pole-zero coincidences are all simple (Lin and Wu, 1996). We then generalize the conditions to the case in which the plant has a second-order unstable pole-zero coincidence. Finally, it is shown that for a general rectangular full normal row rank plant P, the existence of a block decoupling controller is equivalent to the existence of a block decoupling controller for a square plant derived from P.

The following notations are used. $\mathbb{C}$:= the field of complex numbers. $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$; $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$. $\mathbb{R}[s]$:= the ring of polynomials in s with real coefficients; $\mathbb{R}(s)$:= the field of rational functions in s with real coefficients; $\mathbb{R}_p(s)$ ($\mathbb{R}_{po}(s)$) := the set of proper (strictly proper, resp.) rational functions in s with real coefficients. For $H(s) \in \mathbb{R}(s)^{n\times m}$, $\mathscr{Z}[H]$:= the set of all zeros of H in $\mathbb{C}$, $\mathscr{P}[H]$:= the set of all poles of H in $\mathbb{C}$, $\mathscr{Z}_+[H] := \mathscr{Z}[H] \cap \mathbb{C}_+$, and $\mathscr{P}_+[H] := \mathscr{P}[H] \cap \mathbb{C}_+$. Dynamic interpretations of poles and zeros of transfer matrices can be found in Callier and Desoer (1982). $\mathbf{S}$:= the set of all proper rational functions with no poles in $\mathbb{C}_+$. A proper transfer matrix $P(s) \in \mathbb{R}_p(s)^{n\times m}$ is stable if and only if $\mathscr{P}[P] \subset \mathbb{C}_-$. A list of positive integers $(n_1, n_2, \ldots, n_k)$ satisfying $\sum_{i=1}^{k} n_i = n$ is said to be a *partition* of n. We use $\mathrm{diag}\{H_i\}_{i=1}^{k}$ to denote the block diagonal rational matrix with H_i as its ith block diagonal entry, where $H_i \in \mathbb{R}(s)^{n_i\times n_i}$. For $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{C}^{p\times q}$ the Kronecker product $A \otimes B$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq};$$

for $A = [a_1 \ \cdots \ a_n] \in \mathbb{C}^{m\times n}$, $\mathrm{vec}(A) := [a_1^{\mathrm{T}} \ \cdots \ a_n^{\mathrm{T}}]^{\mathrm{T}}$ (Horn and Johnson, 1991).

2. UNITY-FEEDBACK SYSTEMS

Consider the unity-feedback system $S(P, C)$ shown in Fig. 1, where $P \in \mathbb{R}_{po}(s)^{n\times m}$ is the plant, $C \in \mathbb{R}_p(s)^{m\times n}$ is the controller, (u_1, u_2) is the input

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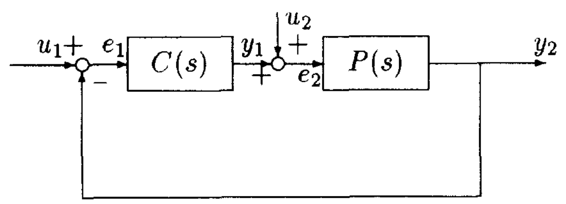

Fig. 1. Unity-feedback system $S(P, C)$.

and (y_1, y_2) is the output. Let $u := [u_1^{\mathrm{T}}\ u_2^{\mathrm{T}}]^{\mathrm{T}}$ and $y := [y_1^{\mathrm{T}}\ y_2^{\mathrm{T}}]^{\mathrm{T}}$. The closed-loop transfer matrix $H_{yu} \in \mathbb{R}_p(s)^{(m+n)\times(m+n)}$ is given by

$$H_{yu} = \begin{bmatrix} H_{y_1u_1} & H_{y_1u_2} \\ H_{y_2u_1} & H_{y_2u_2} \end{bmatrix} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix}. \quad (1)$$

We say that the system $S(P, C)$ is (internally) stable and C is a stabilizing controller for P if H_{yu} is stable; the system is *block decoupled with* respect to the partition $(n_1, \ldots, n_k)$ of n and C is a *block decoupling controller* for P if C stabilizes P and the I/O map $H_{y_2u_1}$ is nonsingular and block diagonal, i.e.,

$$H_{y_2u_1} = \begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_k \end{bmatrix} := \mathrm{diag}\{H_i\}_{i=1}^k, \quad (2)$$

where $H_i \in \mathbb{R}_p(s)^{n_i\times n_i}$. We assume that $m \geq n$ and P has full normal rank n throughout.

Let $Q = C(I + PC)^{-1} \in \mathbb{R}_p(s)^{m\times n}$. In terms of Q, the closed-loop transfer matrix H_{yu} becomes

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & (I-PQ)P \end{bmatrix}. \quad (3)$$

Since the plant P is strictly proper, $Q = C(I + PC)^{-1} \in \mathbb{R}_p(s)^{m\times n}$ if and only if $C = Q(I - PQ)^{-1} \in \mathbb{R}_p(s)^{m\times n}$ (Callier and Desoer, 1982).

3. SQUARE PLANT CASE

We will consider first the case where the plant is a square rational matrix. Since $P \in \mathbb{R}_{po}(s)^{n\times n}$ has full normal rank by assumption, P^{-1} exists.

It follows from equation (3) that stability of $S(P, C)$ requires the stability of the four block entries of H_{yu}. The following result says that if the diagonal entries of equation (3) are stable then the only unstable poles that may appear in the off-diagonal entries are those that are both poles and (transmission) zeros of P.

Lemma 3.1 (Lin, 1995). For the system $S(P, C)$ with $P \in \mathbb{R}_{po}(s)^{n\times n}$ and H_{yu} given in equation (3), if Q and $(I - PQ)$ P are stable then $\mathscr{P}_+[PQ] \subset (\mathscr{P}_+[P] \cap \mathscr{Z}_+[P])$ and $\mathscr{P}_+[QP] \subset (\mathscr{P}_+[P] \cap \mathscr{Z}_+[P])$.

3.1. *Necessary and sufficient conditions*

We consider plants in which coincidences of $\mathbb{C}_+$ pole-zero do occur. To simplify derivations we consider first the case where there is only one simple $\mathbb{C}_+$-coincidence. Given the plant $P \in R_{po}(s)^{n\times n}$ with $P^{-1} \in \mathbb{R}(s)^{n\times n}$. Write

$$P(s) = \frac{R}{s-\lambda} + U(s) \text{ and } P(s)^{-1} = \frac{T}{s-\lambda} + V(s), \quad (4)$$

where $\lambda \in \mathbb{C}_+$, $R, T \in \mathbb{C}^{n\times n}$, $U(s) \in \mathbb{R}_{po}(s)^{n\times n}$ and $V(s) \in \mathbb{R}(s)^{n\times n}$ are analytic at λ and $\mathscr{P}_+[U] \cap \mathscr{P}_+[V] = \emptyset$. The plant P has an $\mathbb{C}_+$-coincidence at $s = \lambda$.

Consider the system $S(P, C)$ shown in Fig. 1 with $P \in \mathbb{R}_{po}(s)^{n\times n}$. Suppose for some stabilizing controller C the resulting I/O map $H_{y_2u_1} =: H$ is block diagonal with respect to the partition $(n_1, \ldots, n_k)$,* that is, C is a block decoupling controller for P. Write $H = \mathrm{diag}\{H_i\}_{i=1}^k$ where $H_i \in \mathbb{R}_p(s)^{n_i\times n_i}$ is stable. With $Q = C(I + PC)^{-1}$ we have $H = PQ$. Internal stability of $S(P, C)$ implies that Q, $(I - PQ)$ P and QP are all stable, in particular, they are all analytic at λ.

Let us examine the consequences of these requirements. Since $Q = P^{-1}H$, Q is analytic at λ if and only if

$$P^{-1}H = \left[\frac{T}{s-\lambda} + V(s)\right]H(s) \quad \text{is analytic at } \lambda.$$

Since V and H are analytic at λ, Q is analytic at λ if and only if

$$TH(\lambda) = 0. \quad (5)$$

Write $T = [T_1 \ \cdots \ T_k]$, where $T_i \in \mathbb{C}^{n\times n_i}$. Since H is block diagonal, equation (5) is equivalent to

$$T_iH_i(\lambda) = 0, \quad i = 1, \ldots, k. \quad (6)$$

Similarly, $(I - PQ)$ P is analytic at λ if and only if

$$(I - H)P = [I - H(s)]\left[\frac{R}{s-\lambda} + U(s)\right]$$

is analytic at λ.

Since U is analytic at λ, $(I - PQ)$ P is analytic at λ if and only if

$$H(\lambda)R = R. \quad (7)$$

Write $R^{\mathrm{T}} = [R_1^{\mathrm{T}} \ \cdots \ R_k^{\mathrm{T}}]$ where $R_i \in \mathbb{C}^{n_i\times n}$. Since H is block diagonal, equation (7) is equivalent to

$$H_i(\lambda)R_i = R_i, \quad i = 1, \ldots, k. \quad (8)$$

Conditions (6) and (8) together imply that

$$T_iR_i = 0, \quad i = 1, \ldots, k. \quad (9)$$

*We consider block decoupling with respect to a fixed but arbitrary partition $(n_1, \ldots, n_k)$ throughout.

Assume that both Q and $(I - PQ)P$ are analytic at λ and write

$$QP = \left[\frac{T}{s-\lambda} + V(s)\right] H(s) \left[\frac{R}{s-\lambda} + U(s)\right],$$
$$= \frac{TH(s)R}{(s-\lambda)^2} + \frac{TH(s)U(s)}{s-\lambda}$$
$$+ \frac{V(s)H(s)R}{s-\lambda} + V(s)H(s)U(s).$$

By expansion,

$$QP = \frac{TH(\lambda)R}{(s-\lambda)^2} + \frac{TH'(\lambda)R + TH(\lambda)U(\lambda) + V(\lambda)H(\lambda)R}{s-\lambda} + A(s) \quad (10)$$

for some $A(s)$ analytic at λ. Thus QP is analytic at λ if and only if there exists a block diagonal $H'(\lambda) \in \mathbb{C}^{n\times n}$ such that

$$TH'(\lambda)R + V(\lambda)R = 0. \quad (11)$$

where we have used equations (5) and (7).

It remains to see under what conditions (on T, R and $V(\lambda)$) it is possible to find block diagonal $H'(\lambda)$ that satisfies equation (11). For this we need the following lemma.

Lemma 3.2. Given $A_i \in \mathbb{C}^{m\times n_i}$, $B_i \in \mathbb{C}^{p_i\times q}$, $D \in \mathbb{C}^{m\times q}$. The matrix equation $\sum_{i=1}^{k} A_i X_i B_i = D$ is equivalent to the matrix–vector equation

$$[B_1^{\mathrm{T}} \otimes A_1 \ \cdots \ B_k^{\mathrm{T}} \otimes A_k] \begin{bmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_k) \end{bmatrix} = \mathrm{vec}(D).$$

where $A \otimes B$ is the Kronecker product of A and B.

Rewrite equation (11) as

$$\sum_{i=1}^{k} T_i H_i'(\lambda) R_i = -V(\lambda)R. \quad (12)$$

By Lemma 3.2 there exists $\{H_i'(\lambda)\}_{i=1}^{k}$ satisfying equation (12) if and only if

$$\mathrm{vec}(V(\lambda)R) \in \mathrm{Range}(R_1^{\mathrm{T}} \otimes T_1 \ \cdots \ R_k^{\mathrm{T}} \otimes T_k]) \quad (13)$$

or equivalently

$$\mathrm{rank}[R_1^{\mathrm{T}} \otimes T_1 \cdots R_k^{\mathrm{T}} \otimes T_k]$$
$$= \mathrm{rank}[R_1^{\mathrm{T}} \otimes T_1 \ \cdots \ R_k^{\mathrm{T}} \otimes T_k \ \mathrm{vec}(V(\lambda)R)].$$

Thus equations (9) and (13) are necessary conditions for the existence of a block decoupling controller for P.

To show that equations (9) and (13) together are sufficient conditions for existence of block decoupling controllers for the plant in equation (4), we have to construct a controller which stabilizes P and results in a block diagonal I/O map $H_{y_2u_1}$. We do this by showing that if equations (9) and (13) are satisfied, then it is possible to find a block diagonal I/O map H so that $P^{-1}H$, $(I - H)P$ and $P^{-1}HP$ are all stable. The resulting block decoupling controller will then be $C = P^{-1}H(I - H)^{-1}$.

Note that the stability of $P^{-1}H$, $(I - H)P$ and $P^{-1}HP$ is equivalent to that the block diagonal stable $H = \mathrm{diag}\{H_i\}_{i=1}^{k}$ satisfies a set of interpolation condition at the poles and zeros of P. If the plant has no $\mathbb{C}_+$-coincidence, that is, if $\mathscr{P}_+[P] \cap \mathscr{Z}_+[P] = \emptyset$, then the interpolation equations on H (block diagonal or not) can be properly setup and solved by standard polynomial interpolation techniques. For example, if α and $\beta \in \mathbb{C}_+$, $\alpha \neq \beta$, are, respectively, a (simple) pole and zero of P, then we could set the interpolation equations as

$$H(\alpha) = I \quad \text{and} \quad H(\beta) = 0.$$

With $\mathbb{C}_+$-coincidences, the interpolation conditions for the pole and the zero occur at the same point and unless certain conditions (on the plant) are satisfied, the interpolation equations may have no solution due to the block diagonal structure of H imposed by the block decoupling requirement. Conditions (9) and (13) ensure that a solvable interpolation equation can be setup.

Lemma 3.3. Suppose $p \geq q$. If $A \in \mathbb{C}^{p\times q}$ and $B \in \mathbb{C}^{q\times p}$ are such that $AB = 0$ then there exists $C \in \mathbb{C}^{q\times q}$ such that

$$AC = 0 \quad \text{and} \quad CB = B. \quad (14)$$

Proof. If $A = 0$, $C = I$ satisfies equation (14). Suppose $A \neq 0$. There exists $E \in \mathbb{C}^{q\times q}$ nonsingular such that $AE = [\bar{A} \ 0]$, where $\bar{A}$ is full column rank, $\bar{A} \in \mathbb{C}^{p\times\tau}$, $\tau \leq q$. Let $C = E\Sigma E^{-1}$ where

$$\Sigma = \begin{bmatrix} 0_\tau & 0 \\ 0 & I_{q-\tau} \end{bmatrix},$$

$$AC = (AE)\cdot\Sigma E^{-1} = [\bar{A} \ 0] \begin{bmatrix} 0_\tau & 0 \\ 0 & I_{q-\tau} \end{bmatrix} E^{-1} = 0.$$

And it is obvious that $AB = 0$ if and only if the matrix $\bar{B} := E^{-1}B$ has the form $[0^{\mathrm{T}} \ \bar{B}_2^{\mathrm{T}}]^{\mathrm{T}}$. Hence,

$$(I - C)B = (I - E\Sigma E^{-1})B = E(I - \Sigma)(E^{-1}B)$$
$$= E\begin{bmatrix} I_\tau & 0 \\ 0 & 0_{q-\tau} \end{bmatrix}\begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix} = 0,$$

thus $CB = B$. □

By condition (9) and Lemma 3.3, there exist $\Phi_i \in \mathbb{C}^{n_i\times n_i}$, $i = 1, \ldots, k$, that satisfy

$$T_i\Phi_i = 0 \quad \text{and} \quad \Phi_i R_i = R_i. \quad (15)$$

Now, $P^{-1}H$ and $(I - H)\,P$ is analytic at λ if and only if $H = \text{diag}\{H_i\}_{i=1}^{k}$ satisfies the interpolation condition

$$H_i(\lambda) = \Phi_i, \quad i = 1, \ldots, k. \tag{16}$$

With equation (13) satisfied, it is possible to find Ψ_i, $i = 1, \ldots, k$ so that if we set

$$H_i'(\lambda) = \Psi_i, \tag{17}$$

then $P^{-1}HP$ is analytic at λ. Thus equations (16) and (17) constitute the interpolation equations on H at λ so as to ensure that $P^{-1}H$, $(I - H)P$ and $P^{-1}HP$ are analytic at λ. Equations (16) and (17) together with interpolation equations on H at other poles and zeros of P to ensure stability of $P^{-1}H$ and $(I - H)P$ can be solved by standard polynomial interpolation techniques (Stoer and Bulirsch, 1992) to obtain a stable block diagonal rational matrix H so that $P^{-1}H$ and $(I - H)P$ are stable. By lemma 3.1, the only possible unstable poles of $P^{-1}HP$ is the one at $s = \lambda$. But by satisfying equation (17), $P^{-1}HP$ is analytic at λ. Thus the closed-loop system is stable with block diagonal $H_{y_2u_1} = H$. We thus have established the following theorem.

Theorem 3.4. For the plant $P(s)$ together with its inverse $P(s)^{-1}$ given in equation (4), there exists a block decoupling controller with respect to the partition $(n_1, \ldots, n_k)$ if and only if the conditions (9) and (13) hold.

Remark. If $n_i = 1$, $\forall i$, condition (13) reduces to $V(\lambda)\,R = 0$ (Lin, 1995).

Let's consider a more general case where many (simple) $\mathbb{C}_+$-coincidences occur. Let

$$\begin{aligned} P(s) &= \sum_{j=1}^{M} \frac{R^j}{s - \lambda_j} + U(s), \\ P(s)^{-1} &= \sum_{l=1}^{M} \frac{T^l}{s - \lambda_l} + V(s), \end{aligned} \tag{18}$$

where $\lambda_j \in \mathbb{C}_+$ are distinct, $R^j, T^l \in \mathbb{C}^{n \times n}$, $U(s) \in \mathbb{R}_{po}(s)^{n \times n}$ and $V(s) \in \mathbb{R}(s)^{n \times n}$ are analytic at $\{\lambda_j\}_{j=1}^{M}$ and $\mathscr{P}_+[U] \cap \mathscr{P}_+[V] = \emptyset$. The plant has M simple $\mathbb{C}_+$-coincidences at $\{\lambda_j\}_{j=1}^{M}$. Let $T^l = [T_1^l \; \cdots \; T_k^l]$ and $R^{l^T} = [R_1^{l^T} \; \cdots \; R_k^{l^T}]$, $T_i^l \in \mathbb{C}^{n \times n_i}$, $R_i^l \in \mathbb{C}^{n_i \times n}$, $l = 1, \ldots, M$.

Again if there exists a block decoupling controller for P, then there exists H stable, proper, block diagonal such that Q, $(I - PQ)P$ and QP are stable. Since Q and $(I - PQ)P$ are analytic at $\{\lambda_l\}_{l=1}^{M}$, it follows that

$$T_i^l R_i^l = 0, \quad i = 1, \ldots, k, \quad l = 1, \ldots, M. \tag{19}$$

Furthermore, QP is analytic at $\{\lambda_j\}_{j=1}^{M}$ implies

$$T^j H'(\lambda_j) R^j + \sum_{l=1, l \neq j}^{M} \frac{T^l R^j}{\lambda_j - \lambda_l} + V(\lambda_j) R^j = 0, \quad j = 1, \ldots, M. \tag{20}$$

Let

$$C_j = \sum_{l=1, l \neq j}^{M} \frac{T^l R^j}{\lambda_j - \lambda_l} + V(\lambda_j) R^j, \quad j = 1, \ldots, M. \tag{21}$$

Since H is block diagonal, equation (20) can be rewritten as

$$\sum_{i=1}^{k} T_i^j H_i'(\lambda_j) R_i^j + C_j = 0, \quad j = 1, \ldots, M. \tag{22}$$

By Lemma 3.2, each of the equations in equation (20) has a block diagonal solution $H'(\lambda_j)$ if and only if

$$\text{vec}(C_j) \in \text{Range}([R_1^{j^T} \otimes T_1^j \; \cdots \; R_k^{j^T} \otimes T_k^j]), \quad j = 1, \ldots, M. \tag{23}$$

Thus equations (19) and (23) are necessary conditions for the existence of a block decoupling controller for P; the conditions also ensure that solvable interpolation equations on H can be set up and thus a block decoupling controller exists.

Theorem 3.5. For the plant $P(s)$ together with its inverse $P^{-1}(s)$ given in equation (18) there exists a block decoupling controller with respect to the partition $(n_1, \ldots, n_k)$ if and only if

$$T_i^l R_i^l = 0, \quad i = 1, \ldots, k, \quad l = 1, \ldots, M \tag{24}$$

and

$$\text{vec}(C_j) \in \text{Range}[R_1^{j^T} \otimes T_1^j \; \cdots \; R_k^{j^T} \otimes T_k^j]), \quad j = 1, \ldots, M \tag{25}$$

where C_j is defined in equation (21). □

Example 3.6. Consider the plant

$$P(s) = \begin{bmatrix} \dfrac{s}{(s-1)(s+1)} & \dfrac{-1}{(s-1)(s+1)} & \dfrac{s}{(s-1)(s+1)(s+2)} \\ \dfrac{-1}{s+1} & \dfrac{1}{s+1} & \dfrac{-s}{(s+1)(s+2)} \\ 0 & 0 & \dfrac{s-1}{(s+1)(s+2)} \end{bmatrix}.$$

By computation,

$$P(s)^{-1} = \begin{bmatrix} s+1 & \dfrac{s+1}{s-1} & 0 \\ s+1 & \dfrac{s(s+1)}{s-1} & \dfrac{s(s+1)}{s-1} \\ 0 & 0 & \dfrac{(s+1)(s+2)}{s-1} \end{bmatrix},$$

the plant has a simple $\mathbb{C}_+$-coincidence at $s = 1$. Write

$$P(s) = \frac{R}{s-1} + U(s)$$

and

$$P(s)^{-1} = \frac{T}{s-1} + V(s),$$

where

$$R = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$U(s) = \begin{bmatrix} \frac{1/2}{s+1} & \frac{1/2}{s+1} & \frac{1/2}{s+1} - \frac{2/3}{s+2} \\ \frac{-1}{s+1} & \frac{1}{s+1} & \frac{-s}{(s+1)(s+2)} \\ 0 & 0 & \frac{s-1}{(s+1)(s+2)} \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

and

$$V(s) = \begin{bmatrix} s+1 & 1 & 0 \\ s+1 & s+2 & s+2 \\ 0 & 0 & s+4 \end{bmatrix}.$$

For each of the partitions (1, 1, 1), (2, 1) and (1, 2), condition (9) is satisfied. Since $V(1)\ R \neq 0$, there exists no block decoupling controller for P with respect to the partition (1, 1, 1), that is, there exists no decoupling controller. To check condition (13) for the partition (2, 1), compute

$$[R_1^{\mathrm{T}} \otimes T_1 \;\; R_2^{\mathrm{T}} \otimes T_2]^{\mathrm{T}}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\operatorname{vec}(V(1)R)^{\mathrm{T}} = [1 \;\; 1 \;\; 0 \;\; -1 \;\; 0 \;\; \tfrac{1}{3} \;\; \tfrac{1}{3} \;\; 0].$$

We find that $\operatorname{vec}(V(1)R)$ belongs to the range of $[R_1^{\mathrm{T}} \otimes T_1 \;\; R_2^{\mathrm{T}} \otimes T_2]$, thus condition (13) is satisfied. Thus, a block decoupling controller with respect to the partition (2, 1) exists. To check condition (13) for the partition (1, 2), compute

$$[R_1^{\mathrm{T}} \otimes T_1 \;\; R_2^{\mathrm{T}} \otimes T_2] = 0 \quad \text{and} \quad \operatorname{vec}(V(1)R) \neq 0.$$

Hence condition (13) does not hold and hence there exists no block decoupling controller with respect to the partition (1, 2).

4. SECOND-ORDER $\mathbb{C}_+$-COINCIDENCE

We generalize the result to the case where the plant has second order $\mathbb{C}_+$-coincidences. To simplify derivations we consider the case where there is only one second-order $\mathbb{C}_+$-coincidence. Given the plant $P \in \mathbb{R}_{po}(s)^{n\times n}$ with $P^{-1} \in \mathbb{R}(s)^{n\times n}$. Write

$$P(s) = \frac{R^1}{s-\lambda} + \frac{R^2}{(s-\lambda)^2} + U(s)$$

and (26)

$$P(s)^{-1} = \frac{T^1}{s-\lambda} + \frac{T^2}{(s-\lambda)^2} + V(s),$$

where $\lambda \in \mathbb{C}_+$, $R^i, T^i \in \mathbb{C}^{n\times n}$, $i = 1, 2$. $U(s) \in \mathbb{R}_{po}(s)^{n\times n}$ and $V(s) \in \mathbb{R}(s)^{n\times n}$ are analytic at λ. $\mathscr{P}_+[U] \cap \mathscr{P}_+[V] = \emptyset$. Write $T^i = [T_1^i \cdots T_k^i]$ and $R^{i^{\mathrm{T}}} = [R_1^{i^{\mathrm{T}}} \cdots R_k^{i^{\mathrm{T}}}]$, where $T_j^i \in \mathbb{C}^{n\times n_j}$ and $R_j^i \in \mathbb{C}^{n_j\times n}$.

Suppose there exists a block decoupling controller C so that the system $S(P, C)$ is stable and the I/O map $H_{y_2u_1} := H$ is block diagonal, then the transfer matrices $P^{-1}H$, $(I - H)P$ and $P^{-1}HP$ are stable. In particular, $P^{-1}H$, $(I - H)P$ and $P^{-1}HP$ are analytic at λ.

Now, $P^{-1}H$ is analytic at λ if and only if

$$P^{-1}H = \frac{T^1H(s)}{s-\lambda} + \frac{T^2H(s)}{(s-\lambda)^2} + V(s)H(s)$$

is analytic at λ.

Since V and H are analytic at λ and H is block diagonal, $P^{-1}H$ is analytic at λ if and only if

$$T_j^2 H_j(\lambda) = 0, \quad j = 1, \ldots, k \tag{27}$$

and

$$T_j^2 H_j'(\lambda) + T_j^1 H_j(\lambda) = 0, \quad j = 1, \ldots, k. \tag{28}$$

Similarly, $(I - H)P$ is analytic at λ if and only if

$$(I - H(s))\left[\frac{R^1}{s-\lambda} + \frac{R^2}{(s-\lambda)^2} + U(s)\right]$$

is analytic at λ.

Since U and H are analytic at λ and H is block diagonal, $(I - H)P$ is analytic at λ is equivalent to

$$H_j(\lambda)R_j^2 = R_j^2, \quad j = 1, \ldots, k \tag{29}$$

and

$$H_j'(\lambda)R_j^2 + H_j(\lambda)R_j^1 = R_j^1, \quad j = 1, \ldots, k. \tag{30}$$

The matrix $P^{-1}HP$ is analytic at λ if and only if

$$P^{-1}HP = \left[\frac{T^1}{s-\lambda} + \frac{T^2}{(s-\lambda)^2} + V\right] \times H\left[\frac{R^1}{s-\lambda} + \frac{R^2}{(s-\lambda)^2} + U\right]$$

is analytic at λ.

With $P^{-1}H$ and $(I - H)P$ analytic at λ, $P^{-1}HP$ is analytic at λ if and only if

$$\frac{1}{2!}T^2H''(\lambda)R^2 + T^1H'(\lambda)R^2 + T^2H'(\lambda)R^1 + T^1H(\lambda)R^1 + V(\lambda)R^2 = 0, \tag{31}$$

$$\frac{1}{3!}T^2H'''(\lambda)R^2 + \frac{1}{2!}[T^1H''(\lambda)R^2 + T^2H''(\lambda)R^1] + T^1H'(\lambda)R^1 + V'(\lambda)R^2 + V(\lambda)R^1 = 0. \tag{32}$$

Rewriting equations (31) and (32) to get

$$\frac{1}{2!}\sum_{i=1}^{k} T_i^2H_i''(\lambda)R_i^2 + \sum_{i=1}^{k}[T_i^1H_i'(\lambda)R_i^2 + T_i^2H_i'(\lambda)R_i^1] + \sum_{i=1}^{k} T_i^1H_i(\lambda)R_i^1 + V(\lambda)R^2 = 0, \tag{33}$$

$$(R_i^{2^{\mathrm{T}}} \otimes I)\mathrm{vec}(H_i') + (R_i^{1^{\mathrm{T}}} \otimes I)\mathrm{vec}(H_i) = \mathrm{vec}(R_i^1), \quad i = 1, \dots, k, \tag{38}$$

$$\sum_{i=1}^{k}(R_i^{2^{\mathrm{T}}} \otimes T_i^2)\mathrm{vec}\left(\frac{1}{2!}H_i''(\lambda)\right) + \sum_{i=1}^{k}[(R_i^{2^{\mathrm{T}}} \otimes T_i^1) + (R_i^{1^{\mathrm{T}}} \otimes T_i^2)]\mathrm{vec}(H_i'(\lambda)) + \sum_{i=1}^{k}(R_i^{1^{\mathrm{T}}} \otimes T_i^1)\mathrm{vec}(H_i(\lambda)) + \mathrm{vec}(V(\lambda)R^2) = 0, \tag{39}$$

$$\sum_{i=1}^{k}(R_i^{2^{\mathrm{T}}} \otimes T_i^2)\mathrm{vec}\left(\frac{1}{3!}H_i'''(\lambda)\right) + \sum_{i=1}^{k}[(R_i^{2^{\mathrm{T}}} \otimes T_i^1) + (R_i^{1^{\mathrm{T}}} \otimes T_i^2)]\mathrm{vec}\left(\frac{1}{2!}H_i''(\lambda)\right) + \sum_{i=1}^{k}(R_i^{1^{\mathrm{T}}} \otimes T_i^1)\mathrm{vec}(H_i'(\lambda)) + \mathrm{vec}(V'(\lambda)R^2 + V(\lambda)R^1) = 0. \tag{40}$$

Equations (35)–(40) have a solution if and only if

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{col}\{\mathrm{vec}(R_i^2)\}_1^k \\ \mathrm{col}\{\mathrm{vec}(R_i^1)\}_1^k \\ -\mathrm{vec}(V(\lambda)R^2) \\ -\mathrm{vec}(V'(\lambda)R^2 + V(\lambda)R^1) \end{bmatrix} \text{ belongs to the range space of}$$

$$\begin{bmatrix} \mathrm{diag}\{I \otimes T_i^2\} & 0 & 0 & 0 \\ \mathrm{diag}\{I \otimes T_i^1\} & \mathrm{diag}\{I \otimes T_i^2\} & 0 & 0 \\ \mathrm{diag}\{R_i^{2^{\mathrm{T}}} \otimes I\} & 0 & 0 & 0 \\ \mathrm{diag}\{R_i^{1^{\mathrm{T}}} \otimes I\} & \mathrm{diag}\{R_i^{2^{\mathrm{T}}} \otimes I\} & 0 & 0 \\ \mathrm{row}\{R_i^{1^{\mathrm{T}}} \otimes T_i^1\}_1^k & \mathrm{row}\{R_i^{1^{\mathrm{T}}} \otimes T_i^2 + R_i^{2^{\mathrm{T}}} \otimes T_i^1\}_1^k & \mathrm{row}\{R_i^{2^{\mathrm{T}}} \otimes T_i^2\}_1^k & 0 \\ 0 & \mathrm{row}\{R_i^{1^{\mathrm{T}}} \otimes T_i^1\}_1^k & \mathrm{row}\{R_i^{1^{\mathrm{T}}} \otimes T_i^2 + R_i^{2^{\mathrm{T}}} \otimes T_i^1\}_1^k & \mathrm{row}\{R_i^{2^{\mathrm{T}}} \otimes T_i^2\}_1^k \end{bmatrix} \tag{41}$$

$$\frac{1}{3!}\sum_{i=1}^{k} T_i^2H_i'''(\lambda)R_i^2 + \frac{1}{2!}\sum_{i=1}^{k}[T_i^1H_i''(\lambda)R_i^2 + T_i^2H_i''(\lambda)R_i^1] + \sum_{i=1}^{k} T_i^1H_i'(\lambda)R_i^1 + V'(\lambda)R^2 + V(\lambda)R^1 = 0. \tag{34}$$

Rewriting the matrix equations (27)–(30), (33) and (34), we have

$$(I^{\mathrm{T}} \otimes T_i^2)\mathrm{vec}(H_i) = 0, \quad i = 1, \dots, k, \tag{35}$$

$$(I^{\mathrm{T}} \otimes T_i^2)\mathrm{vec}(H_i') + (I^{\mathrm{T}} \otimes T_i^1)\mathrm{vec}(H_i) = 0, \quad i = 1, \dots, k, \tag{36}$$

$$(R_i^{2^{\mathrm{T}}} \otimes I)\mathrm{vec}(H_i) = \mathrm{vec}(R_i^2), \quad i = 1, \dots, k, \tag{37}$$

where $\mathrm{col}\{R_i\}_1^k$ is defined as $[R_i^{\mathrm{T}} \cdots R_k^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathrm{row}\{R\}_1^k$ is defined as $[R_1 \cdots R_k]$. Thus, equation (41) is a necessary condition for existence of block decoupling controllers. Also if equation (41) is satisfied, then solvable interpolation equations can be set up for the block diagonal I/O map H and hence a block decoupling controller can be obtained for P. We state this result as follows.

Theorem 4.1. For the plant P with a second-order $\mathbb{C}_+$-coincidence given in equation (26), there exists a block decoupling controller with respect to the partition $(n_1, \dots, n_k)$ if and only if the condition (41) holds.

Remark. The extension to many second-order $\mathbb{C}_+$-coincidences case in straightforward. The same approach can be used to derive conditions for high-order coincidence case, however, the conditions are expected to be much more complicated.

5. RECTANGULAR PLANT CASE

We now consider the feedback system $S(P, C)$ with rectangular plant $P \in \mathbb{R}_{po}(s)^{n\times m}$ shown in Fig. 1. Since $m \geq n$ and P is assumed to be full normal row rank, there exists a unimodular matrix $U \in \mathbf{S}^{m\times m}$ such that

$$PU = [\bar{P}\ 0], \tag{42}$$

where $\bar{P} \in \mathbb{R}_{po}(s)^{n\times n}$ (Lin and Hsieh, 1993). Define $U^{-1}C = [C_1^T\ C_2^T]^T$, where $C_1 \in \mathbb{R}_p(s)^{n\times n}$, $C_2 \in \mathbb{R}_p(s)^{(m-n)\times n}$. The closed-loop transfer matrix of $S(P, C)$ becomes

$$H_{yu} = \begin{bmatrix} U\cdot U^{-1}C(I + PU\cdot U^{-1}C)^{-1} & -U\cdot U^{-1}C(I + PU\cdot U^{-1}C)^{-1}PU\cdot U^{-1} \\ PU\cdot U^{-1}C(I + PU\cdot U^{-1}C)^{-1} & (I + PU\cdot U^{-1}C)^{-1}PU\cdot U^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_1(I + \bar{P}C_1)^{-1} \\ C_2(I + \bar{P}C_1)^{-1} \end{bmatrix} & -\begin{bmatrix} C_1(I + \bar{P}C_1)^{-1}\bar{P} & 0 \\ C_2(I + \bar{P}C_1)^{-1}\bar{P} & 0 \end{bmatrix} \\ \bar{P}C_1(I + \bar{P}C_1)^{-1} & [(I + \bar{P}C_1)^{-1}\bar{P}\ \ 0] \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U^{-1} \end{bmatrix}. \tag{43}$$

From equation (43) we note that the I/O map

$$H_{y_2u_1} = PC(I + PC)^{-1} = \bar{P}C_1(I + \bar{P}C_1)^{-1}.$$

Since U and U^{-1} are both stable and proper, it follows from equation (43) that if C is a block decoupling controller for P, then C_1 is a block decoupling controller for $\bar{P}$. On the other hand, if $\bar{C}_1$ is a block decoupling controller for $\bar{P}$, then

$$C = U\begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix}$$

where $\bar{C}_2 \in \mathbf{S}^{(m-n)\times n}$, is a block decoupling controller for P. Thus there exists a block decoupling controller for P if and only if there exists a block decoupling controller for $\bar{P}$. Since $\bar{P} \in \mathbb{R}_{po}(s)^{n\times n}$, the necessary and sufficient conditions given in the previous sections apply.

6. CONCLUSIONS

We have derived necessary and sufficient conditions for existence of block decoupling controllers for linear multivariable system under unity-feedback configuration. The conditions are particularly simple for plants with only simple unstable pole-zero coincidences.

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