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SYMMETRY BREAKING AND EXISTENCE OF MANY POSITIVE NONSYMMETRIC SOLUTIONS FOR SEMI-LINEAR ELLIPTIC EQUATIONS ON FINITE CYLINDRICAL DOMAINS

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1. INTRODUCTION

Let ω be a bounded smooth domain in R^n, $n \le 1$, $a > o$, and $\Omega_a = (-a, a) \times \omega$ be a finite cylindrical domain in $R^{n+1} = \{(x, y): x \in R^1 \text{ and } y \in R^n\}$. Taking the length a as a bifurcation parameter, we consider the symmetry-breaking problem of the following semilinear elliptic equation with mixed-type boundary conditions.

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_a \\ u = 0 & \text{on } [-a, a] \times \partial\omega \\ \dfrac{\partial u}{\partial x} = 0 & \text{on } \{-a, a\} \times \omega. \end{cases} \tag{1.1}$$

This problem was posed by Professor H. Berestycki and studied by Lin [1]. In this paper, we shall study the problem when $n = 1$.

We consider the equation

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_a = (-a, a) \times (0, b) \\ u = 0 & \text{on } [-a, a] \times \{0, b\} \\ \dfrac{\partial u}{\partial x} = 0 & \text{on } \{-a, a\} \times (0, b), \end{cases} \tag{1.2}$$

where f satisfies the following conditions:

(H-0) $f \in C^1(R^1)$, $f(u) > 0$ for $u > 0$.

(H-1) there exists $\sigma > 0$ such that $uf'(u) \ge (1 + \sigma) f(u)$.

For example, $f(u) = u^p$, $p > 1$, satisfies (H-0) and (H-1).

Definition 1.1. A solution $u \in C^2(\bar{\Omega}_a)$ of (1.2) is said to be symmetric (with respect to the x-axis, i.e. $u = u(y)$) when it satisfies

$$\begin{cases} u'' + f(u) = 0 & \text{in } (0, b) \\ u(0) = u(b) = 0, \end{cases} \tag{1.3}$$

otherwise, u is called an asymmetric solution.

Remark 1.2. Assume conditions (H-0), (H-1) are satisfied. Then the nontrivial solution of (1.3) is unique and exists for all $b > 0$. The existence result can be found in Lin [2], and the uniqueness is proven in Remark 2.3 below.

From Remark 1.2, we can let $b = 1$. Our main results are as follows.

THEOREM 1.3. Assume conditions (H-0), (H-1) are satisfied. Let u_0 be the nontrivial symmetric solution. Then there must exist an increasing sequence $a_k \to \infty$, as $k \to \infty$, such that (1.2) possesses symmetry-breaking from u_0 at a_k.

In addition, if $f(u)$ satisfies
(H-2) $f \in C^2(R^1)$,
then the bifurcation in Theorem 1.3 is global. We express the theorem precisely below.

THEOREM 1.4. Assume condition (H-0) ~ (H-2) are satisfied. Let u_0 be the nontrivial symmetric solution. Then there must exist an increasing sequence $a_k \to \infty$, as $k \to \infty$, such that (a_k, u_0) is a bifurcation point of a global, unbounded branch of solutions of (1.2), and the branches must be globally separated.

In Lin [1], using variational method, Lin proved that an increasing sequence $a_k \to \infty$, as $k \to \infty$ exists, such that (1.1) has at least $k + 1$ solutions for $a > a_k$, for all $n \geq 1$, provided that (H-0), (H-1) are satisfied and f is sub-critical. We use the bifurcation method to prove that there are at least $2k + 1$ solutions for $a > a_k$, $n = 1$, and provide a global bifurcation diagram. Our result follows.

THEOREM 1.5. Assume f satisfies (H-0) ~ (H-2) and
(H-3) $\lim_{u \to \infty} f(u)/u^p = C > 0$ for $1 < p < \infty$.
Then there exists an increasing sequence $a_k \to \infty$, as $k \to \infty$, such that (1.2) has at least $2k + 1$ solutions for $a > a_k$.

The paper is organized as follows. In Section 2 we study the linear eigenvalue problems and give the Proof of Theorem 1.3. In Section 3 we prove Theorems 1.4 and 1.5 and give the bifurcation diagram.

2. LOCAL BIFURCATION

We begin with the linearized eigenvalue problems of (1.2) at u_0. It is clear that the linearized eigenvalue of (1.2) at u_0 is

$$\begin{cases} \Delta W + f(u_0(y))W = -\mu W & \text{in } (-a, a) \times (0, 1) \\ W = 0 & \text{on } [-a, a] \times \{0, 1\} \\ \dfrac{\partial W}{\partial x} = 0 & \text{on } \{-a, a\} \times (0, 1). \end{cases} \tag{2.1}$$

By separation of variables, (2.1) is equivalent to the problems

$$\begin{cases} \phi'' + v\phi = 0 & \text{in } (-a, a) \\ \phi'(-a) = \phi'(a) = 0 \end{cases} \tag{2.2}$$

and

$$\begin{cases} -\psi'' - f'(u_0)\psi = (\mu - \nu)\psi & \text{in } (0, 1) \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{2.3}$$

with

$$W(x, y) = \phi(x)\psi(y). \tag{2.4}$$

It is clear that the eigenvalues in (2.2) are given by $\nu_k(a) = -(k\pi/2a)^2$, $k = 0, 1, 2, \ldots$ and the associated eigenfunctions are

$$\phi_k(x) = \begin{cases} \sin(k\pi/2a)x & \text{if } k \text{ is odd} \\ \cos(k\pi/2a)x & \text{if } k \text{ is even.} \end{cases}$$

Let $\tau = \mu - \nu$, then there is a sequence $\{\tau_1, \tau_2, \tau_3, \ldots\}$ of eigenvalues for (2.3) that satisfies $\tau_1 < \tau_2 < \tau_3, \ldots$, where each distinct eigenvalue is multiplicity one. Therefore, μ is an eigenvalue of (2.1) if an only if $\tau_l = \mu - \nu_k$ for some k and l. Hence, we denote

$$\mu_{k,l} = \mu_{k,l}(a) = \tau_l + \mu_k = \tau_l + (k\pi/2a)^2,$$

where $k = 0, 1, 2, \ldots$ and $l = 1, 2, 3, \ldots$.

LEMMA 2.1. For any symmetric positive solution u_0 we have $\tau_1 < 0$.

Proof. It is well known that τ_1 can be characterized by

$$\tau_1 = \inf\left\{Q(v)\Big/\int_0^1 v^2 : v \in H_0^1(0, 1)\backslash\{O\}\right\},$$

where

$$Q(v) = \int_0^1 |\nabla v|^2 - f'(u_0)v^2.$$

Since u_0 is a solution of (1.3), we have

$$\int_0^1 |u_0'|^2 = \int_0^1 u_0 f(u_0).$$

Hence, by (H-1), we have

$$Q(u_0) = \int_0^1 |u_0'|^2 - f'(u_0)u_0^2 = \int_0^1 u_0 f(u_0) - f'(u_0)u_0^2$$
$$\leq -\sigma\int_0^1 u_0 f(u_0) = -\sigma\int_0^1 |u_0'|^2.$$

Therefore, we have $\tau_1 < 0$. ■

By modifying an argument in Ni–Nussbaum [3], we can prove the following lemma.

LEMMA 2.2. For any symmetric positive solution u_0, we have $\tau_2 > 0$.

Proof. Let $u(r, d)$ be the solution of the initial-value problem

$$\begin{cases} u''(r) + f(u(r)) = 0 & \text{for } r > 0 \\ u(0) = 0, \quad u'(0) = d > 0. \end{cases} \tag{2.5}$$

Since $u'(r, d) = d - \int_0^r f(u(s, d))\, ds$, (H-0) and (H-1) implies there are $0 < \eta(d) < R(d)$ such that

$$u'(\eta(d), d) = 0$$

and

$$u'(r, d) > 0 \qquad \text{if } r < \eta(d)$$

$$u'(r, d) < 0 \qquad \text{if } r \in (\eta(d), R(d));$$

i.e. $R(d)$ be the first zero of $u(r, d)$. Set $\Phi(r, d) = \partial u(r, d)/\partial d$, Φ then satisfies

$$\begin{cases} \Phi'' + f'(u)\Phi = 0 \\ \Phi(0) = 0, \qquad \Phi'(0) = 1. \end{cases} \tag{2.6}$$

We will claim that Φ has exactly one zero in $(0, R(d))$. First, we prove that Φ has at most one zero. Suppose Φ has zeros in $(0, R(d))$. Denote the first zero of Φ by $\xi(d)$.

Let

$$X = ru'(r, d)$$

$$Y = u'(r, d).$$

We then have

$$\frac{d}{dr}(X'\Phi - \Phi'X) = -2f(u)\Phi \tag{2.7}$$

and

$$\frac{d}{dr}(Y'\Phi - \Phi'Y) = 0. \tag{2.8}$$

From (2.7) we have

$$X'\Phi - \Phi'X|_0^\xi = \int_0^\xi -2f(u)\Phi,$$

which implies

$$X'(\xi)\Phi(\xi) - \Phi'(\xi)X(\xi) - X'(0)\Phi(0) + \Phi'(0)X(0) < 0.$$

Since $\Phi(\xi) = \Phi(0) = X(0) = 0$ and $\Phi'(\xi) \leq 0$. We thus have $\Phi(\xi) < 0$ and $X(\xi) < 0$ (note: $X(\xi) < 0$ implies $\eta(d) < \xi(d)$). If there exists a point $\zeta \in (\xi(d), R(d))$ such that $\Psi'(\zeta) = 0$, let ζ be the first point. By (2.8), we have

$$Y'\Phi - \Phi'Y|_\xi^\zeta = 0,$$

which implies

$$Y'(\zeta)\Phi(\zeta) - \Phi'(\zeta)Y(\zeta) + \Phi'(\xi)Y(\xi) - Y'(\xi)\Phi(\xi) = 0.$$

Since $\Phi'(\zeta) = 0$, $\Phi(\xi) = 0$, $\Phi(\zeta) < 0$, $\Phi'(\xi) < 0$, $Y(\xi) < 0$ and $Y'(\zeta) = u''(\zeta) = -f(u(\zeta)) < 0$, the left-hand side is negative, a contradiction. Therefore, Φ has at most one zero in $(0, R(d))$.

Now, compare these two equations:

$$\Phi''(r, d) + f'(u(r, d))\Phi(r, d) = 0 \qquad \text{in } (0, R(d))$$

$$\psi''(r) + f'(u(r))\psi + \tau\psi = 0 \qquad \text{in } (0, R(d))$$

Since $\tau_1 < 0$, by the Sturm comparison principle, Φ must have at least one zero in $(0, R(d))$. Hence, Φ has exactly one zero in $(0, R(d))$. Now, if $\tau_2 \le 0$, by the Sturm comparison principle again, Φ must have at least two zeros in $(0, R(d)]$, a contradiction, so $\tau_2 > 0$. ■

Remark 2.3. In Lemma 2.2, Φ has exactly one zero in $(0, R(d))$, so we have $\Phi(R(d), d) < 0$. By $u(R(d), d) \equiv 0$ for all $d > 0$, we have $R'(d) = -\Phi(R(d), d)/u'(R(d), d)$, so $R'(d) < 0$ for all d. Hence, there is a unique solution for (1.3) (see Ni–Nussbaum [3]).

LEMMA 2.4. There exists an increasing sequence $a_1, a_2, \ldots$, such that $\mu_{n,1}(a_n) = 0$ and $\mu_{n,1}$ is a simple eigenvalue.

Proof.

$$\mu_{k,l}(a) = \tau_l + \nu_k = \tau_l + (k\pi/2a)^2.$$

So $\mu_{k,l}(a) = 0$, unless $\tau_1 \le 0$. By Lemmas 2.1, 2.2, only $\tau_1 < 0$, so

$$\mu_{k,1}(a) = 0 \qquad \text{if and only if } a = k\pi/(2\sqrt{-\tau_1}).$$

Let $a_n = n\pi/(2\sqrt{-\tau_1})$, $n = 1, 2, \ldots$, where τ_1 is the first eigenvalue of (2.3). We have

$$\mu_{k,l}(a_n) = 0 \qquad \text{if } k = n \text{ and } n = 1$$

$$\mu_{k,l}(a_n) \ne 0 \qquad \text{otherwise.}$$

Since there are no generalized eigenfunctions (by the symmetry of the operator, see Healey–Kielhöfer [4]), the proof of Lemma 2.3 is complete. ■

We now prove Theorem 1.3.

Proof. We first let $x = \sqrt{\lambda}t$, $\lambda = a^2$, $v(t, y) = u(x, y) - u_0(x, y)$. (1.2) is then equivalent to

$$\begin{cases} \dfrac{1}{\lambda}\dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial y^2} + f(u_0 + v) - f(u_0) = 0 & \text{in } \Omega_1 = (-1, 1) \times (0, 1) \\ v = 0 & \text{on } [-1, 1] \times \{0, 1\} \\ \dfrac{\partial v}{\partial t} = 0 & \text{on } \{-1, 1\} \times (0, 1). \end{cases} \tag{2.9}$$

Let $F(\lambda, v)$: $R^+ \times X \to Y$, be defined by

$$F(\lambda, v) = \frac{1}{\lambda}\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial y^2} + f(u_0 + v) - f(u_0),$$

where

$$X = \left\{v \in C^{2,\alpha}(\bar{\Omega}_1): v = 0 \text{ on } [-1, 1] \times \{0, 1\}, \frac{\partial v}{\partial t} = 0 \text{ on } \{-1, 1\} \times (0, 1)\right\},$$

$$Y = C^{0,\alpha}(\bar{\Omega}_1).$$

$C^{k,\alpha}(\bar{\Omega}_1)$ is the usual space of all k-times differentiable functions u in $\bar{\Omega}_1$ such that u and its derivatives are Hölder-continuous with exponent α. We will use the bifurcation result from the simple eigenvalues in Crandall–Rabinowitz [5], we reproduce the theorem as follows.

THEOREM 2.5 (Crandall–Rabinowitz). Let X, Y be Banach space, V a neighborhood of O in X and $F\colon (-1, 1) \times V \to Y$ have the properties:

(a) $F(t, 0) = 0$ for $|t| < 1$;

(b) the partial derivatives F_t, F_x and F_{tx} exist and are continuous;

(c) $N(F_x(0, 0))$ and $Y/R(F_x(0, 0))$ are one-dimensional;

(d) $F_{tx}(0, 0)x_0 \notin R(F_x(0, 0))$, where $N(F_x(0, 0)) = \text{span}\{x_0\}$.

If Z is any complement of $N(F_x(0, 0))$ in X, then there is a neighborhood U of $(0, 0)$ in $R \times X$, and interval $(-a, a)$, and continuous functions $\phi\colon (-a, a) \to R$, $\psi\colon (-a, a) \to Z$ such that $\phi(0) = 0$, $\psi(0) = 0$ and $F^{-1}(0) \cap U = \{(\phi(\alpha), \alpha x_0 + \alpha\psi(\alpha))\colon |\alpha| < a\} \cup \{(t, 0)\colon (t, 0) \in U\}$.

We check whether F satisfies all properties of Theorem 2.4.

(a) $F(\lambda, 0) = 0$ for all $\lambda > 0$.

(b) The smoothness of f ensures that F_λ, F_v, $F_{\lambda v}$ exist and are continuous.

(c) $F_v(\bar\lambda, 0)h = (1/\lambda)(\partial^2 h/\partial t^2) + (\partial^2 h/\partial y^2) + f'(u_0)h$, for all $h \in X$, where $\bar\lambda = a_n^2$ is a simple eigenvalue, $n = 1, 2, \ldots$, and since the dimension of $N(F_v(\bar\lambda, 0))$ is equal to the co-dimensions of $R(F_v(\bar\lambda, 0))$ (see Remark 3.5), $N(F_v(\bar\lambda, 0))$ and $Y/R(F_v(\bar\lambda, 0))$ are one-dimensional.

(d) Let $N(F_v(\bar\lambda, 0)) = \text{span}\langle\bar v\rangle$. If $w \in R(F_v(\bar\lambda, 0))$, then there exists $v_0 \in X$, such that $(1/\bar\lambda)(\partial^2 v_0/\partial t^2) + (\partial^2 v_0/\partial y^2) + f'(u_0)v_0 = w$, and since $(1/\bar\lambda)(\partial^2 \bar v/\partial t^2) + (\partial^2 \bar v/\partial y^2) + f'(u_0)\bar v = 0$, $\int_{\Omega_1} \bar v w = 0$. Hence, $F_{\lambda v}(\bar\lambda, 0)\bar v \notin R(F_v(\bar\lambda))$ if $\int_{\Omega_1} \bar v F_{\lambda v}(\bar\lambda, 0)\bar v \neq 0$.

$$
\begin{aligned}
F_{\lambda v}(\bar\lambda, 0)\bar v &= \frac{-1}{\bar\lambda^2}\frac{\partial^2 \bar v}{\partial t^2} \\
&= \frac{-1}{\bar\lambda^2}\frac{\partial^2}{\partial t^2}\bar\phi(x)\bar\psi(y) \qquad \text{[by (2.4)]} \\
&= \frac{-1}{\bar\lambda^2}\bar\psi(y)\left(\frac{dx}{dt}\right)^2\frac{\partial^2}{\partial x^2}\bar\phi(x) \\
&= \frac{1}{\bar\lambda}\bar\psi(y)\bar\nu\bar\phi(x) \qquad \text{[by (2.2)]} \\
&= \frac{\bar\nu}{\bar\lambda}\bar v.
\end{aligned}
$$

So,

$$
\int_\Omega \bar v F_{\lambda v}(\bar\lambda, 0) = \int_{\Omega_1} \bar v \frac{\bar\nu}{\bar\lambda}\bar v = \frac{\bar\nu}{\bar\lambda}\int_{\Omega_1} |\bar v|^2 \neq 0,
$$

where $\bar\nu = -(k\pi/2a)^2$ for $k = 1, 2, \ldots$. Hence, by Theorem 2.4, it will bifurcate at $\bar\lambda = a_n^2$ for all $n = 1, 2, \ldots$; i.e. problem (1.2) will bifurcate from u_0 at $\bar\lambda = a_n^2$. ■

3. GLOBAL BIFURCATION

We prove Theorem 1.4 by modifying an argument in Healey–Kielhöfer [4]. We only sketch the difference here. If v is a solution of (2.9), then v has an even 4-periodic extension on R^1; see, for example, Lin [1] or Healey–Kielhöfer [4]. Therefore, we define the function spaces

$$
C_T^{k,\alpha}(\Omega_\infty) \equiv \{u \in C^{k,\alpha}(\Omega_\infty)\colon u \text{ has period } T \text{ in } t\} \text{ where } \Omega_\infty = R \times (0, 1),
$$

and the Banach spaces

$$D = C_4^{2,\alpha}(\Omega_\infty) \qquad \text{with Hölder norm } \|\cdot\|_{2,\alpha}$$

and

$$E = C_4^{2,\alpha}(\Omega_\infty) \qquad \text{with Hölder norm } \|\cdot\|_{0,\alpha}.$$

Let $G(\lambda, v)$: $R^+ \times D \to E$, be defined by

$$g(\lambda, v) = \frac{1}{\lambda}\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial y^2} + f(u_0 + v) - f(u_0).$$

Clearly, the smoothness of f ($f \in C^2(R)$) ensures that G is at least twice continuously Frechet-differentiable. We use the global result in Kielhöfer [6], which is a generalization of Rabinowitz's result [7]. First, we give some definitions.

Definition 3.1. Let $D \subset E$ be both separable Banach spaces. A linear operator $A: D \to E$ is called admissible if it satisfies

(a) A is a Fredholm operator of index zero.

(b) There exists $c > 0$, $\varepsilon > 0$ such that the spectrum $\sigma(A)$ of A in the strip $S_A = (-\infty, c) \times (-i\varepsilon, i\varepsilon)$ consists of finitely many eigenvalues of finite (algebraic) multiplicity.

Definition 3.2. Let $\Omega \subset D$ be a bounded domain. A map $G: \Omega \to E$ is called admissible if $G \in C^2(\Omega, E)$. Its Frechet-derivative $DG(u) = G'(u)$ is admissible in the sense of Definition 3.1 for all $u \in \Omega$. G is proper, i.e. the inverse image in $\bar{\Omega}$ of any compact set in E is compact in D.

Note. The definition of "crossing number" is more complicated, so for simplicity, we omit it here. For details, see Kielhöfer [6].

THEOREM 3.3 (Kielhöfer). Let $G: R^+ \times D \to E$ be a C^2-map satisfying the following conditions:

(a) G is proper on any bounded and closed domain in $R^+ \times D$.

(b) $G(\lambda, \cdot)$ is admissible for any $\lambda \in R^+$.

Assume that $G(\lambda, 0) = 0$ for all $\lambda \in R^+$, that at some $\lambda_0 \in R^+$, the operator $A(\lambda) = G_u(\lambda, 0)$ has an eigenvalue of zero, and that $A(\lambda)$ for $0 < |\lambda - \lambda_0| < \delta$ has no eigenvalue zero. If $A(\lambda)$ has an odd crossing number $\chi(\lambda_0)$ at $\lambda = \lambda_0$ (i.e. $A(\bar{\lambda})$ has an odd crossing number $\chi(0)$ at $\bar{\lambda} = 0$ for $\bar{\lambda} = \lambda - \lambda_0$; see Kielhöfer [6]), then $(\lambda_0, 0)$ is an (isolated) bifurcation point for $G(\lambda, u) = 0$. Call

$$NS = \mathrm{cl}\{(\lambda, 0) \in R^+ \times D,\ G(\lambda, u) = 0,\ u \neq 0\}$$

the closure of the nontrivial solution set. Then the component $NS_{(\lambda_0, 0)}$ of NS connected to the bifurcation point (λ_0) is either unbounded in $R^+ \times D$, or $NS_{(\lambda_0, 0)}$ meets a different bifurcation point $(\lambda_1, 0)$.

LEMMA 3.4. For each $(\lambda, v) \in R^+ \times D$, the linear operator $G_v(\lambda, v)$: $D \to E$ is a Fredholm operator of index zero, and the spectrum of $-[G_v(\lambda, v)]$ in the strip $\{z \in \mathbf{C}: \mathrm{Re}\, z \leq 1, |\mathrm{Im}\, z| \leq 1\}$ consists of finitely many eigenvalues of finite (algebraic) multiplicity.

Proof.

$$G_v(\lambda, v)h = \frac{1}{\lambda}\frac{\partial^2 h}{\partial t^2} + \frac{\partial^2 h}{\partial y^2} + f'(u_0 + v)h \qquad \text{for all } h \in D.$$

So $G_v(\lambda, v)$ is a uniformly elliptic operator for each $(\lambda, v) \in R^+ \times D$. By periodicity we then have the following Schaulder estimate

$$\|h\|_{2,\alpha} \leq C(\|h\|_{0,\alpha} + \|G_v(\lambda, v)h\|_{0,\alpha}) \qquad \text{for all } (\lambda, v) \in R^+ \times D,\ h \in D, \tag{3.1}$$

where C is independent of h (see Gilbarg–Trudinger [8]). Since the embedding of D into E is compact, the estimate (3.1) implies that $G_v(\lambda, v)$ is a semi-Fredholm operator for each (λ, v) (see Grisvard [9, Lemma 4.4.1.1]). In a Hilbert-space setting ($L_4^2(\Omega_\infty) = \{u \in L^2(\Omega_\infty)$: u has period 4 in $t\}$ and $H_4^2(\Omega_\infty) = \{u \in H^2(\Omega_\infty)$: u has period 4 in $t\}$) the symmetry of $G_v(\lambda, v)$ implies that the co-dimension of $R(G_v(\lambda, v))$ is equal to the dimension of $N(G_v(\lambda, v))$ i.e. $G_v(\lambda, v)$ is a Fredholm operator of index zero. By standard regularity theory, $G_v(\lambda, v)$ is also a Fredholm operator of index zero in our Hölder-space setting.

Establishing the second assertion duplicates Lemma 2.1 in Healey–Kielhöfer [4], so we omit it. ■

Remark 3.5. By periodicity, the norms with respect to Ω_1 and Ω_∞ are identical, so we have the same Schoulder estimate. Therefore, the dimension of $N(G_v(\bar{\lambda}, 0))$ is equal to the co-dimension of $R(G_v(\bar{\lambda}, 0))$.

LEMMA 3.6. The mapping $G: R^+ \times D \to E$ is proper on any bounded closed domain; i.e. $G^{-1}((K) \cap B$ is compact in $R^+ \times D$ whenever $K \subset E$ is compact and $B \subset R^+ \times D$ is bounded and closed.

Proof. Decompose G as the sum

$$G(\lambda, v) = A(\lambda)v + F(v),$$

where

$$A(\lambda)v = \frac{1}{\lambda}\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial y^2}$$

$$F(v) = f(u_0 + v) - f(u_0).$$

Let $G(\lambda_n, v_n) = f_n$, where $f_n \to f$ in E and $\{(\lambda_n, v_n)\} \subset B$ is bounded and closed in $R^+ \times D$. Without loss of generality, we note that $\{\lambda_n\}$ converges to λ in R^+ and, by compact embedding, $\{v_n\}$ converges to v in the Banach space $C_4^{1,\alpha}(\Omega_\infty)$. This implies that

$$F(u_n) \to F(u) \qquad \text{in } E$$

$$\|[A(\lambda_n) - A(\lambda)]v_n\|_{0,\alpha} \leq \varepsilon\|v_n\|_{2,\alpha} \leq \varepsilon M \qquad \text{for all } n \geq N(\varepsilon).$$

Hence, the estimate means that

$$A(\lambda_n)v_n - A(\lambda)v_n \to 0 \qquad \text{in } E.$$

Let $\mu \in R$ be such that $A(\lambda) - \mu I: D \to E$ is bijective [i.e. μ is in the resolvent set of $A(\lambda)$]. Then,

$$A(\lambda)v_n - A(\lambda_n)v_n - F(v_n) - \mu v_n + f_n = [A(\lambda) - \mu I]v_n,$$

which converges to $-F(v) - \mu v + f$ in E. Finally, the convergence of $\{[A(\lambda) - \mu I]v_n\}$ in E is equivalent to the convergence of $\{v_n\}$ in D.

Now define

$$D_n = \left\{u \in C^{2,\alpha}(\Omega_\infty)\colon u \text{ is } \frac{4}{n} \text{ periodic, odd in } t = -1 + \frac{1}{n} \text{ and even in } t = -1 + \frac{2}{n}\right\}$$

$$E_n = \left\{u \in C^{0,\alpha}(\Omega_\infty)\colon u \text{ is } \frac{4}{n} \text{ periodic, odd in } t = -1 + \frac{1}{n} \text{ and even in } t = -1 + \frac{2}{n}\right\}.$$

Consider $G\colon R^+ \times D_n \to E_n$, the restriction of $G(\lambda, \cdot)$ to D_n (see Healey–Kielhöfer [4]). By Theorem 1.3, we have $G_v(\lambda, 0)$ has an odd crossing number $\chi(\bar{\lambda})$, where $\bar{\lambda} = \lambda_1, \lambda_2, \ldots$ and all properties of G remain valid for its restriction, so all conditions of Theorem 3.3 are fulfilled. ■

Therefore, we may summarize the following.

THEOREM 3.7. Assume that (H-0)–(H-2) are satisfied. $(\lambda_n, 0)$ is then a bifurcation point of the global branch $\Sigma^n \subset R^+ \times D_n$ of nontrivial solutions (subject to Rabinowitz alternative; i.e. either unbounded in $R^+ \times D_n$ or meeting a different bifurcation point $(\lambda_m, 0)$) of $G(\lambda, 0)$.

Now each eigenfunction v_n is positive or negative on the open rectangle $\Omega_m = (-1 - 1/n, -1 + 1/n) \times (0, 1)$. By the proof of Theorem 3.1 in Healey–Kielhöfer [4], it will preserve the sign in Ω_n along the branch, so we have the following theorem.

THEOREM 3.8. Assume that (H-0)–(H-2) are satisfied. $(\lambda_n, 0)$ is then a bifurcation point of a global, unbounded branch of solutions of $G(\lambda, v) = 0$ having precisely the nodal configuration of v_n along the entire continuum. Therefore, continuua emanating from different nodal configurations are globally separated.

Theorem 1.4 has thus been proven. To prove Theorem 1.5, as well as Theorem 3.8, we need the following result from Lin [1].

THEOREM 3.9 (Lin). Assume condition (H-3) is satisfied, there then must exist a constant $C > 0$ such that for any $a > 0$ and any positive solution u_a of (1.2), we have

$$\|u_a\|_\infty = \max\{|u_a(x)|\colon x \in \Omega_a\} \leq C.$$

Proof of Theorem 1.5. Recall the scaling $x = \sqrt{\lambda}t$, $\lambda = a^2$, $v(t, y) = u(x, y) - u_0(x, y)$. So by Theorems 3.8 and 3.9, Fig. 1 represents the bifurcation diagram for (1.2). Therefore, we have also proven Theorem 1.5 here. ■

Remark 3.10. It is of interest to obtain results similar to Theorems 1.3–1.5 in the case in which $n \geq 2$.

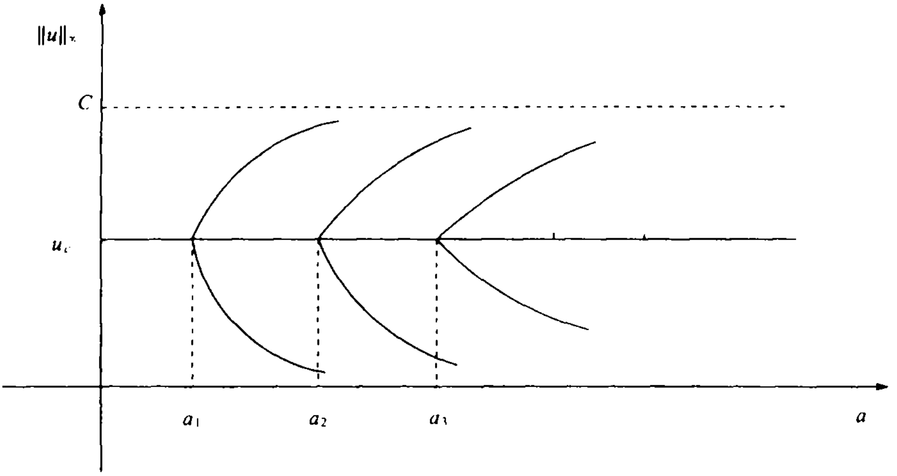

Fig. 1.

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