the associated characteristic set. Hence

$$S_{i_0,\cdots,i_{j+1}} \le (\tau S_{i_0,\cdots,i_j} + 2u + 2v)/2^n + 2v.$$

Let

$$T_j = \max_{i_0,\cdots,i_j} S_{i_0,\cdots,i_j}.$$

Then

$$T_{j+1} \le (\tau T_j + 2u + 2v)/2^n + 2v.$$

The sequence T_j is bounded if $\tau < 2^n$. Since the variance of X conditioned on the codewords $i_0, \cdots, i_j$ is bounded by T_j^2, the theorem, therefore, holds. □

Remark: If $V_i = U_i = 0$, then the equal-partition coder-estimator sequence converges in the quadratic mean if the inequality on τ holds.

X. Conclusion

In this paper, a new class of state estimation problems with communication bandwidth constraints is proposed. These problems couple the issue of estimation with the issue of information communication. Although the estimation problem investigated here is by itself quite simple, it serves to illustrate the complexity and the intricacy of these finite communication bandwidth problems. Extension of this work to more sophisticated estimation problems and feedback control problems will be reported in subsequent papers.

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Nonovershooting and Monotone Nondecreasing Step Responses of a Third-Order SISO Linear System

Shir-Kuan Lin and Chang-Jia Fang

***Abstract*— This paper presents the necessary and sufficient conditions for a third-order single-input/single-output linear system to have a nonovershooting (or monotone nondecreasing) step response. If the transfer function of an overall system has real poles, a necessary and sufficient condition is found for the nonovershooting (or monotone nondecreasing) step response. In the case of complex poles, one sufficient condition and two necessary conditions are obtained. The resulting conditions are all in terms of the coefficients of the numerator of the transfer function. Simple calculations can be used to check a system for the nonovershooting (or monotone nondecreasing) step response. Another feature is that the conditions in terms of pole-zero configurations can be easily derived from the present results.**

***Index Terms*—Linear system, PID controller, step response.**

I. Introduction

The controller design for a third-order linear system has been drawing the attention of many researchers for several decades [1]–[6] because a conventional dynamic plant controlled by a proportional-integral-derivative (PID) controller turns out to be a third-order system. It was pointed out [7] that not only poles but also zeros significantly characterize the step response of a transfer function. Recently, the focus is on the pole-zero relations for the step response without overshoot and undershoot. Note that a step response has no undershoot in the whole history if and only if it is a monotone nondecreasing step response. The number of undershoot times (or local extrema) in the step response has been widely discussed for a strictly proper transfer function with only real poles and real zeros [8]–[11]. A special case of this theme is the initial undershoot [12], which is actually an old result [11], [13]–[15]. Incidentally, another kind of old result for the monotone nondecreasing step response was also repeatedly reported in the recent works, which will be explained in the following paragraph. For a single-input/single-output (SISO) discrete-time system, the linear programming approach formulated by the l^1 theory [16] or simple coefficient relations [17] can be used to design a minimum overshoot controller.

On the other hand, many works [18]–[24] were devoted to finding explicit conditions for a nonovershooting and a monotone nondecreasing step response. The condition proposed in [18] is in terms of the

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state transition matrix, so it has limited usefulness. The other works relate the pole-zero configurations to the step response [19]–[24]. Zemanian [19] summarized the earlier results and presented some new sufficient conditions of the pole-zero configurations for the monotone nondecreasing step response. Additionally, he pointed out a synthesis technique in which the convolution of two transfer functions with monotone nondecreasing step responses still have a monotone nondecreasing step response. Zemanian's work [19] already encompasses the results of Jayasuriya and Song [20] and Rachid [24], which considered only a special minimum phase transfer function with real poles and real zeros. Jayasuriya and Song [21], [22] extended some of these results to the nonovershooting step response and presented an additional synthesis theory showing that the convolution of a transfer function with a monotone nondecreasing step response and another transfer function with no overshoot has a nonovershooting step response. Although the resulting sufficient conditions in [21] and [22] apply only to third-order or lower order systems, they can be extended to higher order systems by the synthesis theory. Kobayashi [23] presented an additional sufficient condition for a third-order SISO system with three distinct real poles that has a nonovershooting step response. Although this condition covers a wide class of zeros, it is still not a necessary and sufficient condition.

In this paper, we are confined to a third-order SISO linear system and attempt to derive a necessary and sufficient condition for a nonovershooting (or monotone nondecreasing) step response. The approach is different from those of the earlier works in forming the conditions. The resulting conditions in this paper are all in terms of the coefficients of the numerator of the transfer function. Therefore, the results are not restricted to minimum phase systems. The main concept is to divide third-order systems into five types according to their poles and then to study the error history of the step response for the nonovershooting step response and the output velocity for the monotone nondecreasing step response. For each type of pole, the error and the output velocity are similar functions with different coefficients. Thus, the same theory can be used to derive conditions for both types of step responses. That means our approach reduces two problems to a single one.

An explicitly necessary and sufficient condition can be obtained for a system with real poles that has a nonovershooting (or monotone nondecreasing) step response. However, if the system has a pair of complex poles, such a closed-form condition is still not found. As a consolation, one sufficient and two necessary conditions are obtained for the case of complex poles.

This paper is organized as follows. Section II deals with the nonovershooting step response, while Section III discusses the monotone nondecreasing step response. The common theorems used to derive the theory in these two sections are presented in the Appendix. However, their proofs are omitted due to the limitation of space. An example in Section IV will show that the pole-zero configurations can be easily derived from the present results.

II. Nonovershooting Step Response

We consider a third-order SISO linear system with a nonstrictly proper input–output transfer function of

$$\frac{Y(s)}{R(s)} = K\frac{cs^3+bs^2+as+1}{ps^3+qs^2+rs+1} \tag{1}$$

where K is set to one without loss of generality, since it is the amplitude ratio. The system with the order of the numerator greater than that of the denominator is excluded, for its initial response to a step input is an impulse.

We are concerned with the performance of the unit step response, i.e., $R(s) = 1/s$. The following discussions are divided into two parts: 1) three negative real poles

$$ps^3+qs^2+rs+1 = (T_1s+1)(T_2s+1)(T_3s+1) \tag{2}$$

with $T_1 \geq T_2 \geq T_3 > 0$ and 2) a negative real pole and a pair of complex poles

$$ps^3+qs^2+rs+1 = (T_1s+1)\left[\left(\frac{s}{\omega}\right)^2 + 2\zeta\left(\frac{s}{\omega}\right)+1\right] \tag{3}$$

where T_1 and ω are both strictly positive real and $0 < \zeta < 1$. Note that $\zeta = 0$ is excluded, because it provides a nondecaying sinusoidal vibration.

Define the error of the step response as $\varepsilon(t) = 1 - y(t)$, where $y(t) = \mathcal{L}^{-1}Y(s)$. The step response of the transfer function (1) has no overshoot if and only if $\varepsilon(t) \geq 0$ for all $t \geq 0$. Taking the inverse Laplace transform of (1) with (2), we obtain the following results.

- *Type A:* For three distinct real poles, $T_1 > T_2 > T_3 > 0$
$$\varepsilon(t) = \alpha_1 e^{-t/T_1} + \alpha_2 e^{-t/T_2} + \alpha_3 e^{-t/T_3}. \tag{4}$$
- *Type B:* For a real double pole less than the other real pole, $T_1 > T_2 = T_3 > 0$
$$\varepsilon(t) = \alpha_1 e^{-t/T_1} + (\alpha_4+\alpha_5 t)e^{-t/T_2}. \tag{5}$$
- *Type C:* For a real double pole greater than the other real pole, $T_1 = T_2 > T_3 > 0$
$$\varepsilon(t) = (\alpha_6+\alpha_7 t)e^{-t/T_1} + \alpha_3 e^{-t/T_3}. \tag{6}$$
- *Type D:* For a real triple pole, $T_1 = T_2 = T_3 > 0$
$$\varepsilon(t) = (\alpha_8+\alpha_9 t+\alpha_{10}t^2)e^{-t/T_1}. \tag{7}$$

In (4)–(7), the coefficients are $\alpha_1 = (T_1^3 - aT_1^2 + T_1 b - c)/[T_1(T_1 - T_2)(T_1 - T_3)]$, $\alpha_2 = (-T_2^3 + aT_2^2 - T_2 b + c)/[T_2(T_1 - T_2)(T_2 - T_3)]$, $\alpha_3 = (T_3^3 - aT_3^2 + T_3 b - c)/[T_3(T_1 - T_3)(T_2 - T_3)]$, $\alpha_4 = [T_2^4 - 2T_1T_2^3 + (aT_1 - b)T_2^2 + c(2T_2 - T_1)]/[(T_1 - T_2)^2T_2^2]$, $\alpha_5 = (-T_2^3 + aT_2^2 - T_2 b + c)/[(T_1 - T_2)T_2^3]$, $\alpha_6 = [T_1^4 - 2T_1^3T_3 + T_1^2(aT_3 - b) + c(2T_1 - T_3)]/[(T_1 - T_3)^2T_1^2]$, $\alpha_7 = (T_1^3 - aT_1^2 + T_1 b - c)/[(T_1 - T_3)T_1^3]$, $\alpha_8 = 1 - c/T_1^3$, $\alpha_9 = (T_1^3 - T_1 b + 2c)/T_1^4$, and $\alpha_{10} = (T_1^3 - aT_1^2 + T_1 b - c)/2T_1^5$. The denominators of these coefficients are all positive. Therefore, the sign of any coefficient is also the sign of its numerator.

According to (1) and (3), we have one more type:

- *Type E:* For a negative real pole and a pair of complex poles, $T_1 > 0$, $\omega > 0$, and $0 < \zeta < 1$
$$\varepsilon(t) = \beta_1 e^{-t/T_1} - \beta_2 e^{-\zeta\omega t}\sin\left(\omega\sqrt{1-\zeta^2}\, t + \psi\right) \tag{8}$$

where $\psi = \mathrm{atan2}(y, x)$ with $y = \sqrt{1-\zeta^2}[2T_1\zeta\omega - 1 - aT_1\omega^2 + b\omega^2 - c\omega^3(2\zeta - T_1\omega)]$ and $x = \zeta - T_1\omega(2\zeta^2 - 1) + a\omega(T_1\zeta\omega - 1) - b\omega^2(T_1\omega - \zeta) + c\omega^3[T_1\zeta\omega - (2\zeta^2 - 1)]$, and

$$\beta_1 = \frac{\omega^2\left(T_1^3 - aT_1^2 + T_1 b - c\right)}{T_1\left(1 - 2T_1\zeta\omega + T_1^2\omega^2\right)} \tag{9}$$

$$\beta_2 = \left(\frac{A^2+B^2}{(1-\zeta^2)\left(1 - 2T_1\zeta\omega + T_1^2\omega^2\right)}\right)^{1/2} \geq 0 \tag{10}$$

in which $A = 1 - a\zeta\omega + b\omega^2(2\zeta^2 - 1) - c\zeta\omega^3(4\zeta^2 - 3)$ and $B = \omega\sqrt{1-\zeta^2}[-a+2b\zeta\omega - c\omega^2(4\zeta^2-1)]$. Note that $\mathrm{atan2}(y,x) \equiv \tan^{-1}(y/x)$, and the signs of x and y determine the region of the codomain; i.e., let $\theta = \mathrm{atan}(y,x)$, then $0 \le \theta \le \pi/2$ for $x \ge 0$ and $y \ge 0$, $\pi/2 < \theta \le \pi$ for $x < 0$ and $y \ge 0$, $-\pi \le \theta < -\pi/2$ for $x < 0$ and $y < 0$, and $-\pi/2 \le \theta < \pi$ for $x \ge 0$ and $y < 0$. It should also be remarked that the denominators in (9) and (10) are greater than zero and β_2 is nonnegative.

A necessary condition for the nonovershooting step response is that the initial response must not be greater than the dc gain, i.e., $y(0^+) \le 1$, since $K = 1$ in (1). The initial value theorem of the Laplace transform allows us to conclude the following result.

Lemma 1: For the step response of the transfer function (1), $y(0^+) \le 1$ if and only if: 1) $c \le T_1T_2T_3$ for the case of real poles and 2) $c \le T_1/\omega^2$ for the case of a pair of complex poles.

First, consider the case of real poles. When Lemma 1 is satisfied, then $\varepsilon(0^+) \ge 0$. In the Appendix, Lemmas A1–A4 show sufficient and necessary conditions in terms of the coefficients α_i for (4)–(7) possessing $\varepsilon(t) \ge 0$ for all $t \ge 0$. Substituting coefficients α_i into the conditions of Lemmas A1 to A4, we obtain the following theorem.

Theorem 2: A necessary and sufficient condition for the nonovershooting step response of (1) with only real negative poles is that $c \le T_1T_2T_3$ and one of the following individual conditions corresponding to each type of real poles holds.

- *Type A:*
  1) $T_1^2(a-T_1) - T_1b + c \le 0$ and $T_2^2(a-T_2) - T_2b + c \ge 0$.
  2) $T_2^2(a-T_2) - T_2b + c < 0$ and $[T_2T_3/(T_2+T_3)](b - T_2T_3) \le c$.
  3) $T_1^2(a-T_1) - T_1b + c \le 0$, $T_2^2(a-T_2) - T_2b + c < 0$, $[T_2T_3/(T_2+T_3)](b-T_2T_3) > c$, and

$$\frac{T_1(T_2-T_3)}{T_3(T_1-T_2)} \ln \frac{T_2^2\left[T_1^2(a-T_1) - T_1b + c\right]}{T_1^2\left[T_2^2(a-T_2) - T_2b + c\right]} \ge \ln \frac{T_3^2\left[T_2^2(a-T_2) - T_2b + c\right]}{T_2^2\left[T_3^2(a-T_3) - T_3b + c\right]} < 0. \tag{11}$$

- *Type B:*
  1) $T_1^2(a-T_1) - T_1b + c \le 0$ and $[T_1T_2/(T_1+T_2)](b - T_1T_2) \le c$.
  2) $T_1^2(a-T_1) - T_1b + c < 0$, $[T_1T_2/(T_1+T_2)](b-T_1T_2) > c$, and

$$\frac{(T_1-T_2)\left(T_2^3 - T_2b + 2c\right)}{T_1\left[T_2^2(a-T_2) - T_2b + c\right]} \ge \ln \frac{T_1^2\left[T_2^2(a-T_2) - T_2b + c\right]}{T_2^2\left[T_1^2(a-T_1) - T_1b + c\right]} > 0. \tag{12}$$

- *Type C:*
  1) $T_1^2(a-T_1) - bT_1 + c = 0$ and $T_1^2(a - 2T_1) \le c$.
  2) $T_1^2(a-T_1) - T_1b + c < 0$ and $[T_1T_3/(T_1+T_3)](b - T_1T_3) \le c$.
  3) $T_1^2(a-T_1) - T_1b + c < 0$, $[T_1T_3/(T_1+T_3)](b-T_1T_3) > c$, and

$$\frac{-(T_1-T_3)\left(T_1^3 - T_1b + 2c\right)}{T_3\left[T_1^2(a-T_1) - T_1b + c\right]} \ge \ln \frac{T_3^2\left[T_1^2(a-T_1) - T_1b + c\right]}{T_1^2\left[T_3^2(a-T_3) - T_3b + c\right]} < 0. \tag{13}$$

- *Type D:*
  1) $T_1^2(a-T_1) - bT_1 + c \le 0$ and $T_1(b - T_1^2) \le 2c$.
  2) $c < T_1^3$, $T_1^2(a-T_1) - bT_1 + c \le 0$, and $(T_1^3 - c)[2T_1^2(a - 5T_1/2) - c] + [T_1(b - 2T_1^2) - c]^2 \le 0$.

Theorem 2 provides a way of using simple calculations to directly determine whether (1) with real poles has a nonovershooting step response.

Return to examine Type E. Because the second term on the right-hand side of (8) is sinusoidal, if $\beta_1 \le 0$ or $e^{-t/T_1} < e^{-\zeta\omega t}$ for some $t > 0$, then $\varepsilon(t)$ must be less than zero for some $t > 0$. Note that $e^{-t/T_1} < e^{-\zeta\omega t}$ for some $t > 0$ if and only if $1/T_1 > \zeta\omega$. Therefore, $1/T_1 \le \zeta\omega$ and $\beta_1 > 0$ are necessary conditions for the nonovershooting step response of Type E. The condition of $1/T_1 \le \zeta\omega$ indicates that the main pole should be real if a nonovershooting step response is requested. Combining Lemma 1 and Lemma A5 in the Appendix yields one sufficient and two necessary conditions as follows.

Theorem 3: Suppose that the poles of the transfer function (1) belong to Type E and the main pole is negative real (i.e., $0 > -1/T_1 \ge -\zeta\omega$).

1) The step response of the system has no overshoot if $c \le T_1/\omega^2$, $T_1^2(a-T_1) - bT_1 + c \le 0$, and $\beta_1 \ge \beta_2$.
2) Suppose that $c \le T_1/\omega^2$ and $T_1^2(a-T_1) - bT_1 + c \le 0$ hold. If $\beta_1 < \beta_2$ and the nonovershooting step response of the system is still requested, then $\beta_2 > 0$, $\beta_1/\beta_2 > \eta$, and

$$1 > \frac{\beta_1}{\beta_2} \ge \sqrt{1-\zeta^2}\, e^{-t_x(\zeta\omega - 1/T_1)} \tag{14}$$

where $\eta = \exp[-2\pi(\zeta\omega - 1/T_1)/(\omega\sqrt{1-\zeta^2})]$ and $t_x = (\omega\sqrt{1-\zeta^2})^{-1}[\mathrm{atan2}(\sqrt{1-\zeta^2}, \zeta) - \psi]$.

Remark: In fact, it can be shown that a necessary and sufficient condition for the nonovershooting step response of Type E is that either $\beta_1 \ge \beta_2$ or $\varepsilon(t_s) \ge 0$ for the smallest one of t_s satisfying $\dot{\varepsilon}(t_s) = 0$ and $\sin(\omega\sqrt{1-\zeta^2}\, t_s + \psi) > 0$. However, no explicit solution to $\dot{\varepsilon}(t_s) = 0$ can be obtained. The condition of $\varepsilon(t_s) \ge 0$ is then useless.

III. Monotone Nondecreasing Step Response

The above theory can be extended to the monotone nondecreasing step response. System (1) has a monotone nondecreasing step response if and only if the first derivative of the output response $y(t)$ for $R(s) = 1/s$ is greater than or equal to zero for all $t \ge 0$. Obviously, $\dot{y}(0^+) \ge 0$ is a necessary condition. By the initial value theorem, $\dot{y}(0^+) = \lim_{s\to\infty} s[sY(s) - y(0^+)]$, we obtain the following lemma.

Lemma 4: For the step response of the transfer function (1), $\dot{y}(0^+) \ge 0$ if and only if: 1) $c \le bT_1T_2T_3/(T_1T_2 + T_2T_3 + T_3T_1)$ for the cases of real poles, and 2) $c \le bT_1/(1 + 2T_1\zeta\omega)$ for the case of a pair of complex poles.

The first derivatives of $y(t)$ for different types of poles are obtained by $\dot{y}(t) = d(1 - \varepsilon(t))/dt$ as follows.

- *Type A:*

$$\dot{y}(t) = \frac{\alpha_1}{T_1}e^{-t/T_1} + \frac{\alpha_2}{T_2}e^{-t/T_2} + \frac{\alpha_3}{T_3}e^{-t/T_3}. \tag{15}$$

- *Type B:*

$$\dot{y}(t) = \frac{\alpha_1}{T_1}e^{-t/T_1} + \left[\left(\frac{\alpha_4}{T_2} - \alpha_5\right) + \frac{\alpha_5}{T_2}t\right]e^{-t/T_2}. \tag{16}$$

- *Type C:*

$$\dot{y}(t) = \left[\left(\frac{\alpha_6}{T_1} - \alpha_7\right) + \frac{\alpha_7}{T_1}t\right]e^{-t/T_1} + \frac{\alpha_3}{T_3}e^{-t/T_3}. \tag{17}$$

- *Type D:*

$$\dot{y}(t) = \left(\frac{\alpha_8 - T_1\alpha_9}{T_1} + \frac{\alpha_9 - 2T_1\alpha_{10}}{T_1}t + \frac{\alpha_{10}}{T_1}t^2\right)e^{-t/T_1}. \tag{18}$$

- *Type E:*

$$\dot{y}(t) = \frac{\beta_1}{T_1}e^{-t/T_1} - \beta_2\omega e^{-\zeta\omega t}\sin\left(\omega\sqrt{1-\zeta^2}\,t + \psi + \phi\right) \tag{19}$$

where $\phi = \operatorname{atan2}(\sqrt{1-\zeta^2}, -\zeta)$.

Note that $(\alpha_8 - T_1\alpha_9) \geq 0$ in (18) when the condition in Lemma 4 is satisfied. Comparing (15)–(19) with (4)–(8), we find that $\dot{y}(t)$ and $\varepsilon(t)$ have the same forms but with different coefficients. When the condition of Lemma 4 is satisfied, the initial condition $\dot{y}(0) \geq 0$, too. Replacing $f(t)$ and the coefficients in Lemmas A1–A5 with $\dot{y}(t)$ and the corresponding coefficients in (15)–(19), we straightforwardly conclude the analogous results for the monotone nondecreasing step response.

Theorem 5: A necessary and sufficient condition for the monotone nondecreasing step response of (1) with only real negative poles is that $c \leq bT_1T_2T_3/(T_1T_2 + T_2T_3 + T_3T_1)$ and one of the following individual conditions corresponding to each type of real poles holds.

- *Type A:*
  1) $T_1^2(a - T_1) - T_1b + c \leq 0$ and $T_2^2(a - T_2) - T_2b + c \geq 0$.
  2) $T_2^2(a - T_2) - T_2b + c < 0$ and $T_2T_3a - (T_2 + T_3)[b - (1/T_2 + 1/T_3)c] \geq c$.
  3) $T_1^2(a - T_1) - T_1b + c \leq 0$, $T_2^2(a - T_2) - T_2b + c < 0$, $T_2T_3a - (T_2 + T_3)[b - (1/T_2 + 1/T_3)c] < c$, and

$$\frac{T_1(T_2 - T_3)}{T_3(T_1 - T_2)}\ln\frac{T_2^3\left[T_1^2(a - T_1) - T_1b + c\right]}{T_1^3\left[T_2^2(a - T_2) - T_2b + c\right]} \geq \ln\frac{T_3^3\left[T_2^2(a - T_2) - T_2b + c\right]}{T_2^3\left[T_3^2(a - T_3) - T_3b + c\right]} < 0. \tag{20}$$

- *Type B:*
  1) $T_1^2(a - T_1) - T_1b + c \leq 0$ and $T_1T_2a - (T_1 + T_2)[b - (1/T_1 + 1/T_2)c] \geq c$.
  2) $T_1^2(a - T_1) - T_1b + c < 0$, $T_1T_2a - (T_1 + T_2)[b - (1/T_1 + 1/T_2)c] < c$, and

$$\frac{(T_1 - T_2)\left(T_2^2a - 2T_2b + 3c\right)}{T_1\left[T_2^2(a - T_2) - T_2b + c\right]} \geq \ln\frac{T_1^3\left[T_2^2(a - T_2) - T_2b + c\right]}{T_2^3\left[T_1^2(a - T_1) - T_1b + c\right]} > 0. \tag{21}$$

- *Type C:*
  1) $T_1^2(a - T_1) - bT_1 + c = 0$ and $T_1^2(a - 2T_1) \leq c$.
  2) $T_1^2(a - T_1) - T_1b + c < 0$ and $T_1T_3a - (T_1 + T_3)[b - (1/T_1 + 1/T_3)c] \geq c$.
  3) $T_1^2(a - T_1) - T_1b + c < 0$, $T_1T_3a - (T_1 + T_3)[b - (1/T_1 + 1/T_3)c] < c$, and

$$\frac{-(T_1 - T_3)\left(T_1^2a - 2T_1b + 3c\right)}{T_3\left[T_1^2(a - T_1) - T_1b + c\right]} \geq \ln\frac{T_3^3\left[T_1^2(a - T_1) - T_1b + c\right]}{T_1^3\left[T_3^2(a - T_3) - T_3b + c\right]} < 0. \tag{22}$$

- *Type D:*
  1) $T_1^2(a - T_1) - bT_1 + c \leq 0$ and $T_1(2b - T_1a) \leq 3c$.
  2) $c < bT_1/3$, $T_1^2(a - T_1) - bT_1 + c \leq 0$, and $T_1^4a^2 - 2T_1^3ab + 2T_1^2b^2 - 2T_1^4b - 4T_1bc + 3c^2 + 6T_1^3c \leq 0$.

Theorem 6: Suppose that the poles of the transfer function (1) belong to Type E, and the main pole is negative real (i.e., $0 > -1/T_1 \geq -\zeta\omega$).

1) The step response of the system is monotone nondecreasing if $c \leq bT_1/(1 + 2T_1\zeta\omega)$, $T_1^2(a - T_1) - bT_1 + c \leq 0$, and $\beta_1 \geq T_1\omega\beta_2$.
2) Suppose that $c \leq bT_1/(1+2T_1\zeta\omega)$ and $T_1^2(a-T_1)-bT_1+c \leq 0$. If $\beta_1 < T_1\omega\beta_2$ and the monotone nondecreasing step response of the system is still requested, then $\beta_2 > 0$ and $\beta_1/\beta_2 > T_1\omega\eta$ and

$$1 > \frac{\beta_1}{\beta_2T_1\omega} \geq \sqrt{1-\zeta^2}\,e^{-t_{x^*}(\zeta\omega - 1/T_1)} \tag{23}$$

where $t_{x^*} = (\omega\sqrt{1-\zeta^2})^{-1}[\operatorname{atan2}(\sqrt{1-\zeta^2}, \zeta) - \psi - \phi]$ and $\eta = \exp[-2\pi(\zeta\omega - 1/T_1)/(\omega\sqrt{1-\zeta^2})]$ identical to that in Theorem 3.

IV. Conclusion

This paper presents the necessary and sufficient conditions for the coefficients of the numerator of a third-order transfer function (1) so that the nonovershooting and the monotone nondecreasing step responses are ensured. If the poles are all negative real, a necessary and sufficient condition is found and presented in Theorems 2 and 5. If the system has a pair of complex poles, one sufficient condition and two necessary conditions are obtained in Theorems 3 and 6. These results allow us to use simple calculations to determine whether the system (1) has a nonovershooting and a monotone nondecreasing step response.

The approach presented in this paper can also be applied to a second-order SISO linear system of $Y(s)/R(s) = (bs^2 + as + 1)/(qs^2 + rs + 1)$. If this system has two negative real poles of $-1/T_1$ and $-1/T_2$ with $T_1 \geq T_2 > 0$, a necessary and sufficient condition for the nonovershooting step response is the condition of $T_1(a - T_1) - b \leq 0$ and $b \leq T_1T_2$, while that for the monotone nondecreasing step response is the condition of $T_1(a - T_1) - b \leq 0$ and $b \leq aT_1T_2/(T_1 + T_2)$. However, if the second-order system has a pair of complex poles, the step response has overshoot for any values of a and b, and then there is no monotone nondecreasing step response.

Furthermore, the present results can also be transferred to the ones in terms of pole-zero configurations. For instance, consider the above second-order system with two negative poles of $\rho_1 = -1/T_1$ and $\rho_2 = -1/T_2$. Let two zeros be z_1 and z_2 (when they are real) or $x \pm jy$. Obviously, $a = -1/z_1 - 1/z_2$ and $b = 1/(z_1z_2)$ for the case of real zeros, while in the case of complex zeros $a = -2x/(x^2 + y^2)$ and $b = 1/(x^2 + y^2)$. A necessary and sufficient condition for the nonovershooting step response is either: 1) $(z_1z_2)^{-1}(z_1 - \rho_1)(z_2 - \rho_1) \geq 0$ and $(z_1z_2)^{-1} \leq (\rho_1\rho_2)^{-1}$, or 2) $x^2 + y^2 \geq \rho_1\rho_2$; while either: 1) $(z_1z_2)^{-1}(z_1 - \rho_1)(z_2 - \rho_1) \geq 0$ and $(z_1z_2)(z_1 + z_2) \leq (z_1z_2)(\rho_1 + \rho_2)$, or 2) $2x \leq \rho_1 + \rho_2$ is a necessary and sufficient condition for the monotone nondecreasing step response. However, such a condition for a third-order system is much more complicated.

Appendix

Lemma A1: Consider $f(t) = \alpha_1e^{-t/T_1} + \alpha_2e^{-t/T_2} + \alpha_3e^{-t/T_3}$ with $T_1 > T_2 > T_3 > 0$ and $f(0) \geq 0$. Then $f(t) \geq 0$ for all $t \geq 0$ if and only if one of the following conditions holds: 1) $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$; 2) $\alpha_2 < 0$ and $\alpha_3 \leq -[(1/T_2 - 1/T_1)/(1/T_3 - 1/T_1)]\alpha_2$;

3) $\alpha_1 \geq 0$, $\alpha_2 < 0$, $\alpha_3 > -[(1/T_2 - 1/T_1)/(1/T_3 - 1/T_1)]\alpha_2$, and

$$\frac{1/T_3 - 1/T_2}{1/T_2 - 1/T_1} \ln \frac{(1/T_3 - 1/T_1)\alpha_1}{1/T_3 - 1/T_2)(-\alpha_2)} \geq \ln \frac{(1/T_2 - 1/T_1)(-\alpha_2)}{(1/T_3 - 1/T_1)\alpha_3}. \tag{A1}$$

Note that the term on the right-hand side of (A1) is less than zero.

Lemma A2: Consider $f(t) = \alpha_1 e^{-t/T_1} + (\alpha_4 + \alpha_5 t)e^{-t/T_2}$ with $T_1 > T_2 > 0$ and $f(0) \geq 0$. Then $f(t) \geq 0$ for all $t \geq 0$ if and only if one of the following conditions holds: 1) $\alpha_1 \geq 0$ and $\alpha_5 \geq -(1/T_2 - 1/T_1)\alpha_1$; 2) $\alpha_1 > 0$, $\alpha_5 < -(1/T_2 - 1/T_1)\alpha_1$, and

$$1 - \left(\frac{1}{T_2} - \frac{1}{T_1}\right)\frac{\alpha_4}{\alpha_5} \geq \ln \frac{-\alpha_5}{(1/T_2 - 1/T_1)\alpha_1}. \tag{A2}$$

Note that the term on the right-hand side of (A2) is greater than zero.

Lemma A3: Consider $f(t) = (\alpha_6 + \alpha_7 t)e^{-t/T_1} + \alpha_3 e^{-t/T_3}$ with $T_1 > T_3 > 0$ and $f(0) \geq 0$. Then $f(t) \geq 0$ for all $t \geq 0$ if and only if one of the following conditions holds: 1) $\alpha_7 = 0$ and $\alpha_6 \geq 0$; 2) $\alpha_7 > 0$ and $\alpha_3 \leq \alpha_7/(1/T_3 - 1/T_1)$; 3) $0 < \alpha_7/(1/T_3 - 1/T_1) < \alpha_3$ and

$$1 + \left(\frac{1}{T_3} - \frac{1}{T_1}\right)\frac{\alpha_6}{\alpha_7} \geq \ln \frac{\alpha_7}{(1/T_3 - 1/T_1)\alpha_3}. \tag{A3}$$

Note that the term on the right-hand side of (A3) is less than zero.

Lemma A4: Consider $f(t) = (\alpha_8 + \alpha_9 t + \alpha_{10} t^2)e^{-t/T_1}$ with $T_1 > 0$ and $f(0) \geq 0$. Then $f(t) \geq 0$ for all $t \geq 0$ if and only if one of the following conditions holds: 1) $\alpha_9 \geq 0$ and $\alpha_{10} \geq 0$; 2) $\alpha_8 > 0$ and $4\alpha_8\alpha_{10} \geq \alpha_9^2$.

Lemma A5: Consider $f(t) = \beta_1 e^{-t/T_1} - \beta_2 e^{-\zeta\omega t} \sin(\omega\sqrt{1-\zeta^2}\, t + \psi)$ with $0 < \zeta < 1$, and $0 < 1/T_1 \leq \zeta\omega$.

1) If $\beta_1 \geq \beta_2 \geq 0$, then $f(t) \geq 0$ for all $t \geq 0$.
2) If $\beta_1 < \beta_2$ and $f(t) \geq 0$ for all $t \geq 0$ is requested, then $\beta_2 > 0$

$$1 > \frac{\beta_1}{\beta_2} > e^{-2\pi(\zeta\omega - 1/T_1)/(\omega\sqrt{1-\zeta^2})} \tag{A4}$$

and

$$1 > \frac{\beta_1}{\beta_2} \geq \sqrt{1-\zeta^2}\, e^{-t_x(\zeta\omega - 1/T_1)} \tag{A5}$$

where $t_x = (\omega\sqrt{1-\zeta^2})^{-1}[\text{atan2}(\sqrt{1-\zeta^2}, \zeta) - \psi]$.

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Numerical Improvements for Solving Riccati Equations

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Abstract— **In this paper, we discuss some ideas for improving the efficiency and accuracy of numerical methods for solving algebraic Riccati equations (ARE's) based on invariant or deflating subspace methods. The focus is on ARE's for which symmetric solutions exist, and our methods apply to both standard linear-quadratic-Gaussian (or H_2) ARE's and to so-called H_∞-type ARE's arising from either continuous-time or discrete-time models. The first technique is a new symmetric representation of a symmetric Riccati solution computed from an orthonormal basis of a certain invariant or deflating subspace. The symmetric representation does not require sign definiteness of the Riccati solution. The second technique relates to improving algorithm efficiency. Using a pencil-based approach, the solution of a Riccati equation can always be reformulated so that the deflating subspace whose basis is being sought corresponds to eigenvalues outside the unit circle. Thus, the natural tendency of the QZ algorithm to deflate these eigenvalues last, and hence, to appear in the upper left blocks of the appropriate pencils, then reduces the amount of reordering that must be done to a (generalized) Schur form.**

Index Terms—**Numerical methods, Riccati equations.**