An Identification Method for Estimating the Inertia Parameters of a Manipulator

Shir-Kuan Lin
Institute of Control Engineering
National Chiao Tung University
Hsinchu 30050, Taiwan, ROC

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This article presents an off-line identification method to estimate the minimal knowledge of the inertia parameters for determining the dynamic model of a manipulator. A new approach is proposed to find a set of the minimal knowledge of the inertia parameters. This set is recursively estimated by moving one joint at a time. The off-line identification procedure also provides a sufficient condition for a persistently exciting trajectory. A simulation example of Stanford arm illustrates the validity and simplicity of the identification procedure.

この発表では、マニピュレータのダイナミック・モデルを構築するのに必要な、慣性パラメータに関する最小限の認識情報を予測するオフライン認識法について説明する。この新しい方法は、慣性パラメータの最小限の認識情報をセットとして得るために考案された。1つのジョイントが動くとともに、慣性パラメータの最小限の情報セットは再帰的手法により予測される。また、オフライン認識法は、軌道を持続的に生成するための充分条件を特定する。Stanfordアームでのシミュレーション例を使って、この認識法の妥当性と簡易性について説明している。

1. INTRODUCTION

The dynamic system of a manipulator is a nonlinear, coupled multivariable system. Conventionally, each joint of a manipulator is controlled by an independent PD algorithm.[1] Because of the nonlinearity and coupling, the proportional derivative (PD) algorithm is only justified for some nominal trajectories. On the other hand, the computed torque control uses the inverse dynamics to compensate for the nonlinearity so that the tracking error in the whole workspace can be reduced to zero.[2] Recently, many advanced control schemes[3–8] for manipulators are all based on the computed torque method. However, good

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performance of these control schemes can be ensured only if the inverse dynamics of manipulators are known.

Some algorithms based on the efficient recursive Newton–Euler formulation[9] satisfied the sampling rate criterion for computing the manipulator inverse dynamics on a multiprocessor system[10] or on a single processor.[11] We are then asked if we can get the exact values of the inertia parameters (mass, center of mass, and inertia tensor) required for the recursive Newton–Euler formulation. Armstrong et al.[12] used a mechanical method to measure the inertia parameters of PUMA 560. Their approach is tedious and requires to disassemble the manipulator. Therefore, a lot of identification methods are proposed for the inertia parameters of manipulators. Atkeson et al.[13] showed that the actuator forces of a manipulator are linear functions of the inertia parameters. All works dealing with the inertia parameter identification tried to explicitly[14–18] or implicitly[13,19–21] formulate the linear equations. Some regrouping rules are also presented[16,17,22] to make the number of inertia parameters appearing in the linear equations minimum since it is found that not all inertia parameters are required to determine the actuator forces. On the other hand, Craig et al.[23] developed an identification algorithm for a parameter-adaptive control scheme.

Among these identification methods, Khosla and Kanade[18] exploited the property that the actuator force of joint i is dependent merely on the inertia parameters of link i to link n. They then proposed an off-line identification method by letting only one joint (from joint n to 1) move at a time such that the combinations of the inertia parameters required in the symbolic dynamic equations are estimated recursively from link n to link 1. However, it is cumbersome to form the symbolic dynamic equations and difficult to regroup the terms in the symbolic equations, especially for a robot with six joints.

In this article, we present a new off-line identification method to estimate the inertia parameters for determining the dynamic model of manipulators. A new approach to finding the minimal knowledge of the inertia parameters of a manipulator is proposed, although the result is substantially equivalent to the earlier ones.[22,24–28] An identification procedure is then to estimate the minimal knowledge of the inertia parameters. Although the off-line identification method is also to move one joint at a time, only the first rotational joint is required to move in the largest part of the identification procedure. An analytic method is presented to investigate the linear independence of the columns in the linear equations of the dynamic model while only one rotational joint or one translational joint of a manipulator moves. This analysis provides us with the persistently exciting trajectories for identifying the minimal knowledge of the inertia parameters. Another advantage of the present method is that it does not require the symbolic dynamic equations.

The next section introduces the inertia constants of composite bodies, which are found to be able to constitute a set of the minimal knowledge of the inertia parameters for determining the dynamic model of manipulators. In Section 3, we let one rotational joint or one translational joint move alone and relate the actuator forces to the minimal knowledge of the inertia parameters. The analysis of these relations allows us to establish an off-line identification procedure.

The Stanford arm is taken as an example to illustrate the identification procedure in Section 4. The computer simulation verifies the theory.

We first introduce the following definition:

Definition. *A set of columns* $\mathbf{a}_i(\boldsymbol{\theta}) : R^m \to R^n$ *is said to be linearly dependent over* R^m *if there exist constants* α_i, $i = 1, \ldots, n$, *not all zero such that*

$$\sum_{i=1}^{n} \alpha_i \mathbf{a}_i(\boldsymbol{\theta}) = \mathbf{0} \qquad \forall \boldsymbol{\theta} \in R^m \tag{1}$$

If α_i *are all zero, the set is said to be linearly independent over* R^m.

2. BACKGROUND

We consider a manipulator with n low-pair joints (i.e., connections with a single degree of freedom), which are labeled as joint 1 to n outward from the base. Assign a body-fixed frame on each joint (i.e., frame E_i is fixed on joint i) in accord with the normal driving-axis coordinate system.[11,29] The distance from the origin of E_i to that of E_j is designated as ${}^j_i\mathbf{s}$, and that to the center of mass of link i as $\mathbf{c}_i$.

In the normal driving-axis coordinate system (see Fig. 1), the z-axis of a body-fixed frame is the driving axis of the corresponding link, i.e., the unit vector along joint i is

$$\mathbf{u}_i^{(i)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

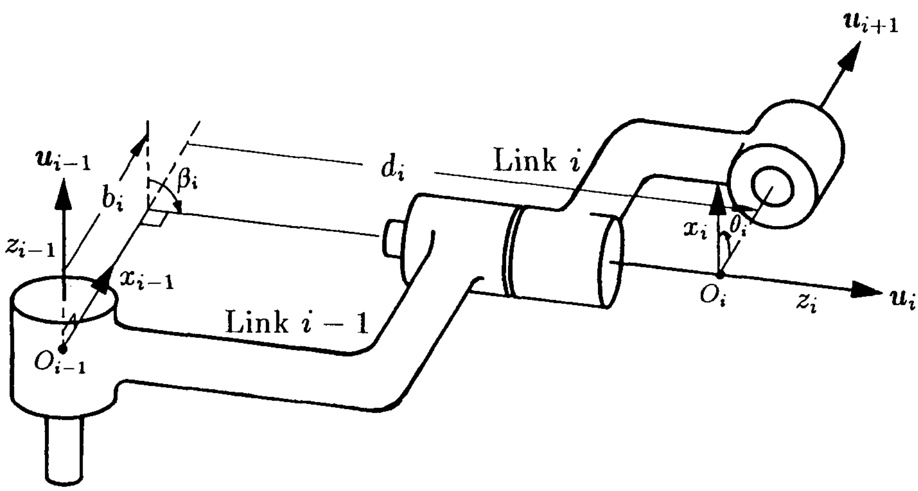


Figure 1. Normal driving-axis coordinate system.

where superscript $\langle i \rangle$ denotes the representation of a vector with respect to frame E_i. The distance from the origin of frame E_{i-1} to frame E_i is shown to be

$$ {}_{i-1}^{\ i}\mathbf{s}^{\langle i-1\rangle} = \begin{bmatrix} b_i \\ -d_i S\beta_i \\ d_i C\beta_i \end{bmatrix} \quad \text{or} \quad {}_{i-1}^{\ i}\mathbf{s}^{\langle i\rangle} = \begin{bmatrix} b_i C\theta_i \\ -b_i S\theta_i \\ d_i \end{bmatrix} \tag{3} $$

where $S\theta_i \equiv \sin\theta_i$, $C\theta_i \equiv \cos\theta_i$, and b_i, d_i, β_i, and θ_i are the geometrical parameters of the coordinate system and shown in Figure 1. Note, $d_i = d_i' + q_i$, $\theta_i = \theta_i'$ if joint i is a translational joint; otherwise, $d_i = d_i'$, $\theta_i = \theta_i' + q_i$, where q_i is the displacement of joint i. That means d_i' and θ_i' are the null-position values of d_i and θ_i, respectively. The coordinate transformation matrix from E_{i-1} to E_i is then

$$ {}_{i-1}^{\ i}\mathbf{R} = \begin{bmatrix} C\theta_i & -S\theta_i & 0 \\ C\beta_i S\theta_i & C\beta_i C\theta_i & -S\beta_i \\ S\beta_i S\theta_i & S\beta_i C\theta_i & C\beta_i \end{bmatrix} \tag{4} $$

The *composite body* i is defined as the union of link i to link n. Let the mass of the composite body i and the first moment of the composite body about the origin of E_i be denoted as $\hat{m}_i$ and $\hat{\mathbf{c}}_i$, respectively, which are

$$ \hat{m}_i = \sum_{j=i}^{m} m_j \tag{5} $$

$$ \hat{\mathbf{c}}_i^{\langle i\rangle} = \sum_{j=i}^{n} m_j({}_i^j\mathbf{s}^{\langle i\rangle} + \mathbf{c}_j^{\langle i\rangle}) \tag{6} $$

where m_j is the mass of link j. According to Huygeno–Steiner formula,[30] the inertia tensor of the composite body i about the origin of frame E_i is

$$ \hat{\mathbf{J}}_i^{\langle i\rangle} = \sum_{j=i}^{n} {}_i^j\mathbf{R}\, \mathbf{I}_j^{\langle j\rangle}\, {}_i^j\mathbf{R}^T - m_j[({}_i^j\mathbf{s}^{\langle i\rangle} + \mathbf{c}_j^{\langle i\rangle})\times][({}_i^j\mathbf{s}^{\langle i\rangle} + \mathbf{c}_j^{\langle i\rangle})\times] \tag{7} $$

where $\mathbf{I}_j^{\langle j\rangle}$ is the representation of the inertia tensor of link j, about the center of mass, with respect to frame E_j, and $[\mathbf{a}\times]$ denotes a skew-symmetric matrix representing the vector multiplication, i.e., $[\mathbf{a}\times]\mathbf{b} = \mathbf{a} \times \mathbf{b}$. In the context, the overhead symbol ˆ is used to denote the inertia parameters (mass, first moment, and inertia tensor) of composite bodies.

We introduce the following notation

$$ K_i^* \equiv (1 - K_i) \equiv \begin{cases} 1 & \text{for rotational joint } i \\ 0 & \text{for translational joint } i \end{cases} \tag{8} $$

Renaud's formulation relates the entries of the inertia matrix of a manipulator to the inertia parameters of composite bodies as[31–34]

$$
\begin{aligned}
h_{mi} = {} & K_m^* K_i^* \left(\mathbf{u}_m^{\langle i\rangle} \cdot \begin{bmatrix} (\hat{\mathbf{J}}_i^{\langle i\rangle})_{13} \\ (\hat{\mathbf{J}}_i^{\langle i\rangle})_{23} \\ (\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} \end{bmatrix} + \begin{bmatrix} (\mathbf{a}_{i,m}^{\langle i\rangle})_y \\ -(\mathbf{a}_{i,m}^{\langle i\rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i\rangle} \right) \\
& + K_m K_i^* \begin{bmatrix} (\mathbf{u}_m^{\langle i\rangle})_y \\ -(\mathbf{u}_m^{\langle i\rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i\rangle} \\
& + K_m^* K_i \left(\hat{m}_i (\mathbf{a}_{i,m}^{\langle i\rangle})_z - \begin{bmatrix} (\mathbf{u}_m^{\langle i\rangle})_y \\ -(\mathbf{u}_m^{\langle i\rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i\rangle} \right) \\
& + K_m K_i \, \hat{m}_i (\mathbf{u}_m^{\langle i\rangle})_z \qquad m \le i
\end{aligned} \tag{9}
$$

where h_{mi} is the (m, i)th entry of the inertia matrix, $(\cdot)_{ij}$ denotes the (i, j)th entry of a matrix, $(\cdot)_x$ does the x-component of a vector, and

$$
\mathbf{a}_{i,m}^{\langle i\rangle} \equiv \mathbf{u}_m^{\langle i\rangle} \times {}_m^i\mathbf{s}^{\langle i\rangle} \tag{10}
$$

is the acceleration of the origin of frame E_i due to a unit joint acceleration of joint m if joint m is a rotational joint.

The gravitational force of the composite body i is $\hat{m}_i\mathbf{g}$ acting at the center of mass of the composite body, where $\mathbf{g}$ is the gravitational acceleration. The gravity term of the actuator force (denoted by τ_i^g) applied on joint i is to resist the gravitational forces exerted on joint i by link i along the direction of joint i, i.e.

$$
\begin{aligned}
\tau_i^g &= -\mathbf{u}_i^{\langle i\rangle} \cdot (K_i^* \hat{\mathbf{c}}_i^{\langle i\rangle} \times \mathbf{g}^{\langle i\rangle} + K_i \hat{m}_i \mathbf{g}^{\langle i\rangle}) \\
&= K_i^* \begin{bmatrix} -(\mathbf{g}^{\langle i\rangle})_y \\ (\mathbf{g}^{\langle i\rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i\rangle} - K_i \hat{m}_i (\mathbf{g}^{\langle i\rangle})_z
\end{aligned} \tag{11}
$$

Lemma 1. *The first moments and the inertia tensors of composite bodies can be divided into the constant* $(\mathbf{k}_i, \mathbf{U}_i)$ *and varying* $(\ell_i, \mathbf{V}_i)$ *parts as*

$$
\hat{\mathbf{c}}_i^{\langle i\rangle} = \mathbf{k}_i + \ell_i \tag{12}
$$

$$
\hat{\mathbf{J}}_i^{\langle i\rangle} = \mathbf{U}_i + \mathbf{V}_i \tag{13}
$$

where

$$\mathbf{k}_n = m_c \mathbf{c}_n^{\langle n\rangle} \tag{14}$$

$$\ell_n = \mathbf{0} \tag{15}$$

$$\mathbf{U}_n = \mathbf{I}_n^{\langle n\rangle} - m_n[\mathbf{c}_n^{\langle n\rangle} \times][\mathbf{c}_n^{\langle n\rangle} \times] \tag{16}$$

$$\mathbf{V}_n = \mathbf{0} \tag{17}$$

and if $i < n$

$$\mathbf{k}_i = m_i \mathbf{c}_i^{\langle i\rangle} + \hat{m}_{i+1}\, {}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} + {}^{i+1}_{\ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z \tag{18}$$

$$\ell_i = {}^{i+1}_{\ i}\mathbf{R}\left(\ell_{i+1} + \begin{bmatrix} (\mathbf{k}_{i+1})_x \\ (\mathbf{k}_{i+1})_y \\ 0 \end{bmatrix}\right) \tag{19}$$

$$\begin{aligned}\mathbf{U}_i = {} & \mathbf{I}_i^{\langle i\rangle} - m_i[\mathbf{c}_i^{\langle i\rangle} \times][\mathbf{c}_i^{\langle i\rangle} \times] - \hat{m}_{i+1}[{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} \times][{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} \times] \\ & + {}^{i+1}_{\ i}\mathbf{R}\begin{bmatrix} (\mathbf{U}_{i+1})_{22} & 0 & 0 \\ 0 & (\mathbf{U}_{i+1})_{22} & 0 \\ 0 & 0 & (\mathbf{U}_{i+1})_{33} \end{bmatrix} {}^{i+1}_{\ i}\mathbf{R}^T \\ & - [{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} \times]\{[{}^{i+1}_{\ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z] \times\} - \{[{}^{i+1}_{\ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z] \times\}[{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} \times]\end{aligned} \tag{20}$$

$$\begin{aligned}\mathbf{V}_i = {} & {}^{i+1}_{\ i}\mathbf{R}\left(\mathbf{V}_{i+1} + \begin{bmatrix} (\mathbf{U}_{i+1})_{11} - (\mathbf{U}_{i+1})_{22} & (\mathbf{U}_{i+1})_{12} & (\mathbf{U}_{i+1})_{13} \\ (\mathbf{U}_{i+1})_{12} & 0 & (\mathbf{U}_{i+1})_{23} \\ (\mathbf{U}_{i+1})_{13} & (\mathbf{U}_{i+1})_{23} & 0 \end{bmatrix}\right) {}^{i+1}_{\ i}\mathbf{R}^T \\ & - [{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} \times][\ell_i \times] - [\ell_i \times][{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} \times]\end{aligned} \tag{21}$$

for rotational joint $i + 1$ (*i.e.*, $K^*_{i+1} = 1$), *while*

$$\mathbf{k}_i = m_i \mathbf{c}_i^{\langle i\rangle} + \hat{m}_{i+1}\begin{bmatrix} ({}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle})_x \\ 0 \\ 0 \end{bmatrix} + {}^{i+1}_{\ i}\mathbf{R}\,\mathbf{k}_{i+1} \tag{22}$$

$$\ell_i = {}^{i+1}_{\ i}\mathbf{R}\left(\ell_{i+1} + \begin{bmatrix} 0 \\ 0 \\ \hat{m}_{i+1} d_{i+1} \end{bmatrix}\right) \tag{23}$$

$$\mathbf{U}_i = \mathbf{I}_i^{\langle i\rangle} - m_i[\mathbf{c}_i^{\langle i\rangle}\times][\mathbf{c}_i^{\langle i\rangle}\times] + {}^{i+1}_{i}\mathbf{R}\,\mathbf{U}_{i+1}\,{}^{i+1}_{i}\mathbf{R}^T - \hat{m}_{i+1}[\mathbf{b}_{i+1}^{\langle i\rangle}\times][\mathbf{b}_{i+1}^{\langle i\rangle}\times]$$
$$- [\mathbf{b}_{i+1}^{\langle i\rangle}\times][({}^{i+1}_{i}\mathbf{R}\,\mathbf{k}_{i+1})\times] - [({}^{i+1}_{i}\mathbf{R}\,\mathbf{k}_{i+1})\times][\mathbf{b}_{i+1}^{\langle i\rangle}\times] \quad (24)$$

$$\mathbf{V}_i = {}^{i+1}_{i}\mathbf{R}(\mathbf{V}_{i+1} - \hat{m}_{i+1}[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times][\mathbf{d}_{i+1}^{\langle i+1\rangle}\times] - \hat{m}_{i+1}[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times][\mathbf{b}_{i+1}^{\langle i+1\rangle}\times]$$
$$- \hat{m}_{i+1}[\mathbf{b}_{i+1}^{\langle i+1\rangle}\times][\mathbf{d}_{i+1}^{\langle i+1\rangle}\times] - [\mathbf{d}_{i+1}^{\langle i+1\rangle}\times][\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}\times] - [\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}\times][\mathbf{d}_{i+1}^{\langle i+1\rangle}\times]$$
$$- [\mathbf{b}_{i+1}^{\langle i+1\rangle}\times][\boldsymbol{\ell}_{i+1}\times] - [\boldsymbol{\ell}_{i+1}\times][\mathbf{b}_{i+1}^{\langle i+1\rangle}\times])\,{}^{i+1}_{i}\mathbf{R}^T \quad (25)$$

for translational joint $i + 1$ (i.e., $K_{i+1} = 1$).

Note that ${}^{i+1}_{i}\mathbf{R}_b$ is the third column of the coordinate transformation matrix ${}^{i+1}_{i}\mathbf{R}$

$${}^{i+1}_{i}\mathbf{R}_b \equiv \begin{bmatrix} 0 \\ -S\beta_{i+1} \\ C\beta_{i+1} \end{bmatrix} \quad (26)$$

and

$$\mathbf{b}_{i+1}^{\langle i\rangle} \equiv \begin{bmatrix} b_{i+1} \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{b}_{i+1}^{\langle i+1\rangle} = \begin{bmatrix} b_{i+1}C\theta_{i+1} \\ -b_{i+1}S\theta_{i+1} \\ 0 \end{bmatrix} \quad (27)$$

$$\mathbf{d}_{i+1}^{\langle i+1\rangle} \equiv \begin{bmatrix} 0 \\ 0 \\ d_{i+1} \end{bmatrix} \quad (28)$$

This lemma can be straightforwardly proved by the principle of mathematical induction, for which we refer to Lin.[35] For convenience, we name $\hat{m}_i$, $\mathbf{k}_i$ and $\mathbf{U}_i$ *inertia constants of the composite body i.*

Theorem 2. *For a manipulator with n low-pair joints in which joint r is the first rotational joint counting from the base and joint s is the nearest rotational joint not parallel to joint r, a* sufficient *knowledge of the inertia parameters for determining the actuator forces $\boldsymbol{\tau}$ is the information of the set $\mathcal{S}$ consisting of*

1. $K_j^*(\mathbf{U}_j)_{33}$, $\delta_j K_j^*(\mathbf{k}_j)_x$, $\delta_j K_j^*(\mathbf{k}_j)_y$ for $r \le j < s$
2. $K_j^*[(\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22}]$, $K_j^*(\mathbf{U}_j)_{33}$, $K_j^*(\mathbf{U}_j)_{12}$, $K_j^*(\mathbf{U}_j)_{13}$, $K_j^*(\mathbf{U}_j)_{23}$, $K_j^*(\mathbf{k}_j)_x$, $K_j^*(\mathbf{k}_j)_y$ for $s \le j \le n$
3. $K_i\hat{m}_i$ for $i = 1, \ldots, n$

4. $K_i(\mathbf{k}_i)_x$, $K_i(\mathbf{k}_i)_y$, $K_i(\mathbf{k}_i)_z$ for $s < i \leq n$

and

5. $\sigma_i K_i[-(\mathbf{u}_r^{(i)})_y(\mathbf{k}_i)_x + (\mathbf{u}_r^{(i)})_x(\mathbf{k}_i)_y]$, $\sigma_i K_i\{-(\mathbf{u}_r^{(i)})_z[(\mathbf{u}_r^{(i)})_x(\mathbf{k}_i)_x + (\mathbf{u}_r^{(i)})_y(\mathbf{k}_i)_y] + [1 - (\mathbf{u}_r^{(i)})_z^2](\mathbf{k}_i)_z\}$ for $r < i < s$

where $\delta_j = 0$ for the case that $\mathbf{u}_r /\!/ \mathbf{u}_k /\!/ \mathbf{g}$, $\forall k < j < s$, and ${}_m^j\mathbf{s}$ is zero or parallel to $\mathbf{u}_r$ for every rotational joint m, $r \leq m < j$; otherwise, $\delta_j = 1$; and $\sigma_i = 0$ for the case of $\mathbf{u}_i /\!/ \mathbf{u}_r$, $r < i < s$; otherwise $\sigma_i = 1$.

Proof: We recognize that the dynamic equations of a manipulator with n joints are

$$\mathbf{H}(\mathbf{q}, \mathbf{x})\ddot{\mathbf{q}} + \boldsymbol{\tau}^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x}) + \boldsymbol{\tau}^g(\mathbf{q}, \mathbf{x}) = \boldsymbol{\tau} \tag{29}$$

where $\mathbf{q} \in R^n$ consists of the joint displacements, $\mathbf{x} \in R^{10n}$ consists of the $10n$ inertia parameters, $\boldsymbol{\tau} \in R^n$ consists of the actuator forces, $\mathbf{H}(\mathbf{q}, \mathbf{x}) : R^{n+p} \rightarrow R^{n\times n}$ is the symmetric inertia matrix, $\boldsymbol{\tau}^g(\mathbf{q}, \mathbf{x}) : R^{11n} \rightarrow R^n$ consists of the gravitational forces, $\boldsymbol{\tau} \in R^n$ consists of the actuator forces, and $\boldsymbol{\tau}^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x}) : R^{12n} \rightarrow R^{n\times n}$ consists of the Coriolis, centrifugal forces, which can also be related to the inertia matrix with Christoffel symbols (c_{ijk})[33]:

$$\tau_i^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x}) = \sum_{j=1}^{n} \sum_{k=j}^{n} c_{ijk}\dot{q}_j\dot{q}_k \tag{30}$$

$$c_{ijk} = \left(\frac{\partial h_{ij}}{\partial q_k} + \frac{\partial h_{ik}}{\partial q_j} - \frac{\partial h_{jk}}{\partial q_i}\right) \quad \text{for } j \neq k \tag{31}$$

$$c_{ijj} = \left(\frac{\partial h_{ij}}{\partial q_j} - \frac{1}{2}\frac{\partial h_{jj}}{\partial q_i}\right) \tag{32}$$

where τ_i^C is the ith element of $\boldsymbol{\tau}^C$, q_i is that of $\mathbf{q}$, and h_{ij} is the (i, j)th entry of $\mathbf{H}$. According to (30)–(32), the knowledge of the inertia parameters for determining $\mathbf{H}$ is sufficient to determine τ^C. Thereafter, we just need to investigate the inertia parameters in (9) and (11).

According to (12), (19), and (23), ℓ_i and then $\hat{\mathbf{c}}_i^{(i)}$ can be calculated with the x- and y-components of $K_j^*\mathbf{k}_j$ and $K_j\hat{m}_j$, $j = i + 1, \ldots, n$, i.e.

$$(\hat{\mathbf{c}}_i^{(i)})_x = (\mathbf{k}_i)_x + \sum_{j=i+1}^{n} a_{ij1}(\mathbf{q})K_j^*(\mathbf{k}_j)_x + b_{ij1}(\mathbf{q})K_j^*(\mathbf{k}_j)_y + c_{ij1}(\mathbf{q})K_j\hat{m}_j \tag{33}$$

$$(\hat{\mathbf{c}}_i^{(i)})_y = (\mathbf{k}_i)_y + \sum_{j=i+1}^{n} a_{ij2}(\mathbf{q})K_j^*(\mathbf{k}_j)_x + b_{ij2}(\mathbf{q})K_j^*(\mathbf{k}_j)_y + c_{ij2}(\mathbf{q})K_j\hat{m}_j \tag{34}$$

$$(\hat{\mathbf{c}}_i^{\langle i\rangle})_z = (\mathbf{k}_i)_z + \sum_{j=i+1}^{n} a_{ij3}(\mathbf{q})K_j^*(\mathbf{k}_j)_x + b_{ij3}(\mathbf{q})K_j^*(\mathbf{k}_j)_y + c_{ij3}(\mathbf{q})K_j\hat{m}_j \quad (35)$$

where a_{ijk}, b_{ijk}, and c_{ijk} are some appropriate functions.

We can rewrite (25) for translational joint $i + 1$ in the form of

$$\mathbf{V}_i = {}^{i+1}_{\ i}\mathbf{R}\left(\mathbf{V}_{i+1} + \mathbf{T}_{i+1} + \begin{bmatrix} 2d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_z & 0 & -d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x \\ & 2d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_z & -d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y \\ \text{Symmetry} & & 0 \end{bmatrix}\right) {}^{i+1}_{\ i}\mathbf{R}^T$$

$$\text{for } K_{i+1} = 1 \quad (36)$$

where

$$(\mathbf{T}_{i+1})_{11} = \hat{m}_{i+1}d_{i+1}^2 - 2b_{i+1}S\theta_{i+1}(\ell_{i+1})_y$$

$$(\mathbf{T}_{i+1})_{22} = \hat{m}_{i+1}d_{i+1}^2 + 2b_{i+1}C\theta_{i+1}(\ell_{i+1})_x$$

$$(\mathbf{T}_{i+1})_{33} = 2b_{i+1}[C\theta_{i+1}(\ell_{i+1})_x - S\theta_{i+1}(\ell_{i+1})_y]$$

$$(\mathbf{T}_{i+1})_{21} = (\mathbf{T}_{i+1})_{12} = b_{i+1}[S\theta_{i+1}(\ell_{i+1})_x - C\theta_{i+1}(\ell_{i+1})_y]$$

$$(\mathbf{T}_{i+1})_{31} = (\mathbf{T}_{i+1})_{13} = -b_{i+1}C\theta_{i+1}(\ell_{i+1})_z - \hat{m}_{i+1}d_{i+1}b_{i+1}C\theta_{i+1}$$

$$(\mathbf{T}_{i+1})_{32} = (\mathbf{T}_{i+1})_{23} = b_{i+1}S\theta_{i+1}(\ell_{i+1})_z + \hat{m}_{i+1}d_{i+1}b_{i+1}S\theta_{i+1}$$

Equations (21) and (36) reveal that $\mathbf{V}_i$ is independent of $K_j\mathbf{U}_j$, $j > i$, and is a function of $K_j\hat{m}_j$, $K_j\hat{\mathbf{c}}_j^{\langle j\rangle}$, the x- and y-components of $K_j^*\mathbf{k}_j$ and the (3,3)th, (1,2)th, (1,3)th, (2,3)th entries and the difference of the (1,1)th and (2,2)th entries of $K_j^*\mathbf{U}_j$, $j > i$.

We examine (9) and (11) and find that h_{mi} and τ_i^g are not explicitly related to the z-component of $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and that only one column, the third column, of $K_i^*\hat{\mathbf{J}}_i^{\langle i\rangle}$ is required in calculating the inertia matrix. The observations for (9), (11), (21), and (36) allow us to conclude that the knowledge of

1. $(\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22}$, $(\mathbf{U}_j)_{33}$, $(\mathbf{U}_j)_{12}$, $(\mathbf{U}_j)_{13}$, $(\mathbf{U}_j)_{23}$, $(\mathbf{k}_j)_x$, $(\mathbf{k}_j)_y$ of all rotational joints
2. $\hat{m}_i$, $(\mathbf{k}_i)_x$, $(\mathbf{k}_i)_y$, $(\mathbf{k}_i)_z$ of all translational joints

are sufficient to determine the inertia matrix and gravity load. However, we can still eliminate some elements for link i, $i < s$.

For the case of $i < r$, there are only translational joints. Equations (9) and (11) are reduced to

$$h_{mi} = \hat{m}_i(\mathbf{u}_m^{\langle i\rangle})_z, \qquad m \le i < r \quad (37)$$

$$\tau_i^g = -\hat{m}_i(\mathbf{g}^{\langle i\rangle})_z \qquad i < r \tag{38}$$

which implies that $\hat{\mathbf{J}}_i^{\langle i\rangle}$ and $\hat{\mathbf{c}}_i^{\langle i\rangle}$, $i < r$, are unnecessary in determining the inertia matrix and the gravity load, nor are $\mathbf{U}_i$ and $\mathbf{k}_i$ necessary.

The rotational joints remaining in front of s are parallel to one another. Thus, $\mathbf{u}_m^{\langle i\rangle} = [0, 0, 1]^T$ for rotational joints m and i, $r \le m < i < s$. If joint m is a rotational joint and joint i is a translational joint, $r \le m < i < s$, $\mathbf{u}_m^{\langle i\rangle} = \mathbf{u}_r^{\langle i\rangle}$ is constant since the body-fixed frame on a translational joint is invariant to the motion of the nearest rotational joint in front of the translational joint. Equation (9) is then reduced to

$$\begin{aligned} h_{mi} = {} & K_m^* K_i^* \left((\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} + \begin{bmatrix} -({}_m^i\mathbf{s}^{\langle i\rangle})_x \\ -({}_m^i\mathbf{s}^{\langle i\rangle})_y \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i\rangle} \right) \\ & + K_m K_i^* \begin{bmatrix} (\mathbf{u}_m^{\langle i\rangle})_y \\ -(\mathbf{u}_m^{\langle i\rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i\rangle} \\ & + K_m^* K_i \{\hat{m}_i(\mathbf{a}_{i,m}^{\langle i\rangle})_z - (\mathbf{u}_m^{\langle i\rangle})_y(\mathbf{k}_i + \boldsymbol{\ell}_i)_x + (\mathbf{u}_m^{\langle i\rangle})_x(\mathbf{k}_i + \boldsymbol{\ell}_i)_y\} \\ & + K_m K_i\, \hat{m}_i(\mathbf{u}_m^{\langle i\rangle})_z \qquad r \le m \le i < s \end{aligned} \tag{39}$$

Equations (11) and (39) allow us to delete $\mathbf{k}_j$ of any rotational joint j, $r \le j < s$ out of the sufficient knowledge of the inertia parameters if $\mathbf{u}_r /\!/ \mathbf{u}_k /\!/ \mathbf{g}$, $\forall k < j < s$, and ${}_m^j\mathbf{s}$ is zero or parallel to $\mathbf{u}_r$ for every rotational joint m, $r \le m < j$. Instead of $\mathbf{k}_i$ for translational joint i, $r < i < s$, a combination of

$$-(\mathbf{u}_m^{\langle i\rangle})_y(\mathbf{k}_i)_x + (\mathbf{u}_m^{\langle i\rangle})_x(\mathbf{k}_i)_y \tag{40}$$

is sufficient since $\mathbf{u}_m^{\langle i\rangle}$ is constant. Besides, we only need the (3,3)th entry of $\hat{\mathbf{J}}_i^{\langle i\rangle}$ for the rotational joints, which contains some $\mathbf{U}_j$ and $\mathbf{k}_j$, $j \ge i$.

Suppose that joints i and m, $i < m \le s$, are rotational joints and joints k, $i < k < m$, are translational joints. Combining (21) and (36), we get

$$\begin{aligned} (\mathbf{V}_i)_{33} = (\mathbf{u}_i^{\langle m\rangle})^T & \left(\mathbf{V}_m + \begin{bmatrix} (\mathbf{U}_m)_{11} - (\mathbf{U}_m)_{22} & (\mathbf{U}_m)_{12} & (\mathbf{U}_m)_{13} \\ (\mathbf{U}_m)_{12} & 0 & (\mathbf{U}_m)_{23} \\ (\mathbf{U}_m)_{13} & (\mathbf{U}_m)_{23} & 0 \end{bmatrix} \right. \\ & - [{}_{m-1}^{m}\mathbf{s}^{\langle m\rangle} \times] \left[\left(\boldsymbol{\ell}_m + \begin{bmatrix} (\mathbf{k}_m)_x \\ (\mathbf{k}_m)_y \\ 0 \end{bmatrix} \right) \times \right] \end{aligned}$$

$$\times \left[\left(\ell_m + \begin{bmatrix}(\mathbf{k}_m)_x\\(\mathbf{k}_m)_y\\0\end{bmatrix}\right) \times \right][{}_{m-1}^{m}\mathbf{s}^{\langle m\rangle} \times]\Bigg)\mathbf{u}_i^{\langle m\rangle}$$

$$+ \sum_{k=i+1}^{m-1} (\mathbf{u}_i^{\langle k\rangle})^T\left(\mathbf{T}_k + \begin{bmatrix}2d_k(\ell_k)_z & 0 & -d_k(\ell_k)_x\\ & 2d_k(\ell_k)_z & -d_k(\ell_k)_y\\ \text{Symmetry} & & 0\end{bmatrix}\right)\mathbf{u}_i^{\langle k\rangle}$$

$$+ \sum_{k=i+1}^{m-1} 2d_k\{[1 - (\mathbf{u}_i^{\langle k\rangle})_z^2](\mathbf{k}_k)_z - (\mathbf{u}_i^{\langle k\rangle})_z[(\mathbf{u}_i^{\langle k\rangle})_x(\mathbf{k}_k)_x + (\mathbf{u}_i^{\langle k\rangle})_y(\mathbf{k}_k)_y]\} \quad (41)$$

since

$$\mathbf{u}_i^{\langle m\rangle} = {}_m^i\mathbf{R}\,\mathbf{u}_i^{\langle i\rangle} = {}_i^m\mathbf{R}^T\begin{bmatrix}0\\0\\1\end{bmatrix} \quad (42)$$

implies that the third row of ${}_i^m\mathbf{R}$ is $(\mathbf{u}_i^{\langle m\rangle})^T$. $(\mathbf{V}_i)_{33}$ is independent of $\mathbf{U}_m$ because of $\mathbf{u}_i^{\langle m\rangle} = [0, 0, 1]^T$ and it additionally contains the constants

$$-(\mathbf{u}_r^{\langle k\rangle})_z[(\mathbf{u}_r^{\langle k\rangle})_x(\mathbf{k}_k)_x + (\mathbf{u}_r^{\langle k\rangle})_y(\mathbf{k}_k)_y] + [1 - (\mathbf{u}_r^{\langle k\rangle})_z^2](\mathbf{k}_k)_z \quad (43)$$

Note that $\mathbf{u}_r /\!/ \mathbf{u}_i$. Therefore, only one entry, the (3,3)th entry, of $K_j^*\mathbf{U}_j$, $r \le j < s$, and the combination (43) of the components of $\mathbf{k}_k$ for translational joint k, $r \le k < s$, should be included in the sufficient knowledge of the inertia parameters. However, the two combinations (40) and (43) of the components of $\mathbf{k}_i$ for any translational joint i remaining between joints r and s are unnecessary if the translational joint is parallel to joint r since the x- and y-components of $\mathbf{u}_r^{\langle i\rangle}$ are zero in this case. This completes the proof. ■

Remark: In the literature, Gautier et al.[22,24,25] and Mayeda et al.[26–28] investigated the minimal knowledge of the inertia parameters for the manipulator dynamic model. The inertia constants listed in Theorem 2 are substantially equivalent to the results of Gautier et al. and Mayeda et al., although some minor terms are different since the set of the minimal knowledge of the inertia parameters is not unique. Another difference is that our approach does not say the inertia constants in Theorem 2 are the minimal knowledge of the inertia parameters for the dynamic model. However, an off-line identification method for estimating these inertia constants will be presented in the next section. Since these inertia constants are all identifiable, they form a set of minimal knowledge of the inertia parameters for determining the dynamic model. The off-line identification method requires an algorithm to compute ℓ_i and $\mathbf{V}_i$, which is proposed as follows.

Algorithm 1. To compute ℓ_j and $\mathbf{V}_j$, $j \geq r$.

Step 1: *Let* $\ell_n = \mathbf{0}$ *and* $\mathbf{V}_n = \mathbf{0}$. *If* $j = n$, *go to step* 5; *otherwise, go to step* 2 *for* $n - 1 \geq s$ *or step* 4 *for* $n - 1 < s$.

Step 2: *If joint n is a rotational joint*

$$\ell_{n-1} = {}_{n-1}^{\ n}\mathbf{R} \begin{bmatrix} (\mathbf{k}_n)_x \\ (\mathbf{k}_n)_y \\ 0 \end{bmatrix}$$

$$\mathbf{V}_{n-1} = {}_{n-1}^{\ n}\mathbf{R} \begin{bmatrix} (\mathbf{U}_n)_{11} - (\mathbf{U}_n)_{22} & (\mathbf{U}_n)_{12} & (\mathbf{U}_n)_{13} \\ (\mathbf{U}_n)_{12} & 0 & (\mathbf{U}_n)_{23} \\ (\mathbf{U}_n)_{13} & (\mathbf{U}_n)_{23} & 0 \end{bmatrix} {}_{n-1}^{\ n}\mathbf{R}^T$$
$$-[{}_{n-1}^{\ n}\mathbf{s}^{\langle n-1\rangle} \times][\ell_{n-1} \times] - [\ell_{n-1} \times][{}_{n-1}^{\ n}\mathbf{s}^{\langle n-1\rangle} \times]$$

Otherwise

$$\ell_{n-1} = \hat{m}_n \begin{bmatrix} 0 \\ -S\beta_n d_n \\ C\beta_n d_n \end{bmatrix}$$

$$\mathbf{V}_{n-1} = {}_{n-1}^{\ n}\mathbf{R} \begin{bmatrix} \hat{m}_n d_n^2 + 2d_n(\mathbf{k}_n)_z & 0 & -d_n(\mathbf{k}_n)_x - \hat{m}_n d_n b_n C\theta_n \\ & \hat{m}_n d_n^2 + 2d_n(\mathbf{k}_n)_z & -d_n(\mathbf{k}_n)_y - \hat{m}_n d_n b_n S\theta_n \\ \text{Symmetry} & & 0 \end{bmatrix} {}_{n-1}^{\ n}\mathbf{R}^T$$

If $j = n - 1$, *go to step* 5; *otherwise, let* $i =: n - 2$ *and go to step* 3 *for* $i \geq s$ *or step* 4 *for* $i < s$.

Step 3: *Compute* ℓ_i *and* $\mathbf{V}_i$ *using* (19) *and* (21) *if joint* $i + 1$ *is a rotational joint. If joint* $i + 1$ *is a translational joint, compute* ℓ_i, $\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}$ *and* $\mathbf{V}_i$ *using* (23), (12), *and* (36). *If* $j = i$, *go to step* 5; *otherwise, let* $i =: i - 1$ *and do step* 3 *again for* $i \geq s$ *or go to step* 4 *for* $i < s$.

Step 4: *If joint* $i + 1$ *is a rotational joint, compute* ℓ_i *using* (19), *otherwise using* (23). *If joint i is a rotational joint, in addition compute* $(\mathbf{V}_i)_{33}$ *using* (41). *If* $j = i$, *go to step* 5; *otherwise, let* $i =: i - 1$ *and do step* 4 *again.*

Step 5: *Output* ℓ_j *and* $\mathbf{V}_j$.

In this algorithm, we only compute the (3,3)th entry of $K_j^*\mathbf{V}_j$ for $j < s$ since the other entries are unnecessary in determining the inertia matrix. The required inertia constants for links in front of joint r are only $\hat{m}_j$, $j < r$; therefore $\mathbf{V}_j$ and ℓ_j, $j < r$, are not required in determining the inertia matrix and the gravity load.

3. OFF-LINE IDENTIFICATION

We intend to identify all the inertia constants listed in Theorem 2 by rotating/translating one joint at a time. The identification method is backward recursive. The inertia constants of the composite body n are first estimated and then used as known values to estimate those of the composite body $n - 1$. Finally, all required inertia constants are recursively estimated. The identification procedure requires the linear equations. The linear independence of the columns of the matrix in the linear equations implies that the inertia constants in the linear equations are identifiable. In the following, we derive the linear equations for the motion of only one joint and investigate the columns of the matrix in the equations.

Suppose that joints i and j, $i < j$, are rotational joints, and joints k, $i < k < j$, are translational joints. We lock joints $j + 1$ to n so that the composite body j can be seen as a rigid body. Applying Newton–Euler equations, we obtain the inertia force $\mathbf{f}_{Tj}$ and torque $\mathbf{t}_{Tj}$ of the composite body j, acting at the center of mass of the composite body j, as follows

$$-\mathbf{f}_{Tj}^{\langle j\rangle} = \hat{m}_j\, {}_0^j\mathbf{R}^T \frac{d^2\, {}_0^j\mathbf{s}^{\langle 0\rangle}}{dt^2} + \dot{\boldsymbol{\omega}}_j^{\langle j\rangle} \times \hat{\mathbf{c}}_j^{\langle j\rangle} + \boldsymbol{\omega}_j^{\langle j\rangle} \times (\boldsymbol{\omega}_j^{\langle j\rangle} \times \hat{\mathbf{c}}_j^{\langle j\rangle}) \tag{44}$$

$$-\mathbf{t}_{Tj}^{\langle j\rangle} = \hat{\mathbf{I}}_j^{\langle j\rangle}\dot{\boldsymbol{\omega}}_j^{\langle j\rangle} + \boldsymbol{\omega}_j^{\langle j\rangle} \times (\hat{\mathbf{I}}_j^{\langle j\rangle}\boldsymbol{\omega}_j^{\langle j\rangle}) \tag{45}$$

where $\hat{\mathbf{I}}_j$ is the inertia tensor of the composite body j about the center of mass of the composite body j, i.e.

$$\begin{aligned}\hat{\mathbf{I}}_j^{\langle j\rangle} &= \sum_{m=j}^{n} {}_j^m\mathbf{R}\, \mathbf{I}_m^{\langle m\rangle}\, {}_j^m\mathbf{R}^T - m_m \left[\left({}_j^m\mathbf{s}^{\langle j\rangle} + \mathbf{c}_m^{\langle j\rangle} - \frac{\hat{\mathbf{c}}_j^{\langle j\rangle}}{\hat{m}_j}\right)\times\right]\left[\left({}_j^m\mathbf{s}^{\langle j\rangle} + \mathbf{c}_m^{\langle j\rangle} - \frac{\hat{\mathbf{c}}_j^{\langle j\rangle}}{\hat{m}_j}\right)\times\right] \\ &= \hat{\mathbf{J}}_j^{\langle j\rangle} + \frac{1}{\hat{m}_j}[\hat{\mathbf{c}}_j^{\langle j\rangle}\times][\hat{\mathbf{c}}_j^{\langle j\rangle}\times]\end{aligned} \tag{46}$$

and $\boldsymbol{\omega}_j$ and $\dot{\boldsymbol{\omega}}_j$ are the angular velocity and acceleration of link j, respectively. Let

$$\mathbf{a}_j^{\langle j\rangle} \equiv {}_0^j\mathbf{R}^T \frac{d^2\, {}_0^j\mathbf{s}^{\langle 0\rangle}}{dt^2} \tag{47}$$

The dynamic equilibrium states that the force $\mathbf{f}_{Ej}$ and torque $\mathbf{t}_{Ej}$ applied by joint j on link j is in equilibrium with the inertia force and torque and the gravity force of the composite body j, i.e.

$$\begin{aligned} \mathbf{f}_{Ej}^{\langle j \rangle} &= -\mathbf{f}_{Tj}^{\langle j \rangle} - \hat{m}_j \mathbf{g}^{\langle j \rangle} \\ &= \hat{m}_j(\mathbf{a}_j^{\langle j \rangle} - \mathbf{g}^{\langle j \rangle}) + \dot{\boldsymbol{\omega}}_j^{\langle j \rangle} \times \hat{\mathbf{c}}_j^{\langle j \rangle} + \boldsymbol{\omega}_j^{\langle j \rangle} \times (\boldsymbol{\omega}_j^{\langle j \rangle} \times \hat{\mathbf{c}}_j^{\langle j \rangle}) \end{aligned} \tag{48}$$

$$\begin{aligned} \mathbf{t}_{Ej}^{\langle j \rangle} &= -\mathbf{t}_{Tj}^{\langle j \rangle} - \frac{\hat{\mathbf{c}}_j^{\langle j \rangle}}{\hat{m}_j} \times (\mathbf{f}_{Tj}^{\langle j \rangle} + \hat{m}_j \mathbf{g}^{\langle j \rangle}) \\ &= \hat{\mathbf{J}}_j^{\langle j \rangle} \dot{\boldsymbol{\omega}}_j^{\langle j \rangle} + \boldsymbol{\omega}_j^{\langle j \rangle} \times (\hat{\mathbf{J}}_j^{\langle j \rangle} \boldsymbol{\omega}_j^{\langle j \rangle}) + \hat{\mathbf{c}}_j^{\langle j \rangle} \times (\mathbf{a}_j^{\langle j \rangle} - \mathbf{g}^{\langle j \rangle}) \end{aligned} \tag{49}$$

since

$$\begin{aligned} \mathbf{c} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{c})] &= -(\mathbf{c} \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{c}) + [\mathbf{c} \cdot (\boldsymbol{\omega} \times \mathbf{c})]\boldsymbol{\omega} \\ &= (\boldsymbol{\omega} \cdot \mathbf{c})(\mathbf{c} \times \boldsymbol{\omega}) - [\boldsymbol{\omega} \cdot (\mathbf{c} \times \boldsymbol{\omega})]\mathbf{c} \\ &= -\boldsymbol{\omega} \times [\mathbf{c} \times (\mathbf{c} \times \boldsymbol{\omega})] \end{aligned} \tag{50}$$

3.1. Rotating One Rotational Joint

We rotate joint i but lock all other joints. Under such a condition, the composite body k or j can be seen as a rigid body. The actuator force of joint j due to the motion of joint i is the component of $\mathbf{t}_{Ej}^{\langle j \rangle}$ along the direction of the joint, i.e.

$$\tau_j = \mathbf{u}_j^{\langle j \rangle} \cdot \mathbf{t}_{Ej}^{\langle j \rangle} = \begin{bmatrix} -(\boldsymbol{\omega}_j^{\langle j \rangle})_x(\boldsymbol{\omega}_j^{\langle j \rangle})_y \\ (\dot{\boldsymbol{\omega}}_j^{\langle j \rangle})_z \\ (\boldsymbol{\omega}_j^{\langle j \rangle})_x^2 - (\boldsymbol{\omega}_j^{\langle j \rangle})_y^2 \\ (\dot{\boldsymbol{\omega}}_j^{\langle j \rangle})_x - (\boldsymbol{\omega}_j^{\langle j \rangle})_y(\boldsymbol{\omega}_j^{\langle j \rangle})_z \\ (\dot{\boldsymbol{\omega}}_j^{\langle j \rangle})_y - (\boldsymbol{\omega}_j^{\langle j \rangle})_x(\boldsymbol{\omega}_j^{\langle j \rangle})_z \\ (\mathbf{a}_j^{\langle j \rangle} - \mathbf{g}^{\langle j \rangle})_y \\ -(\mathbf{a}_j^{\langle j \rangle} - \mathbf{g}^{\langle j \rangle})_x \end{bmatrix} \cdot \begin{bmatrix} (\hat{\mathbf{J}}_j^{\langle j \rangle})_{11} - (\hat{\mathbf{J}}_j^{\langle j \rangle})_{22} \\ (\hat{\mathbf{J}}_j^{\langle j \rangle})_{33} \\ (\hat{\mathbf{J}}_j^{\langle j \rangle})_{12} \\ (\hat{\mathbf{J}}_j^{\langle j \rangle})_{13} \\ (\hat{\mathbf{J}}_j^{\langle j \rangle})_{23} \\ (\hat{\mathbf{c}}_j^{\langle j \rangle})_x \\ (\hat{\mathbf{c}}_j^{\langle j \rangle})_y \end{bmatrix} \tag{51}$$

Similarly, we also get the equation for the actuator force of every translational joint k, $i < k < j$, in the form of

$$\tau_k = \mathbf{u}_k^{\langle k \rangle} \cdot \mathbf{f}_{Ek}^{\langle k \rangle} = \begin{bmatrix} -(\dot{\boldsymbol{\omega}}_k^{\langle k \rangle})_y + (\boldsymbol{\omega}_k^{\langle k \rangle})_x(\boldsymbol{\omega}_k^{\langle k \rangle})_z \\ (\dot{\boldsymbol{\omega}}_k^{\langle k \rangle})_x + (\boldsymbol{\omega}_k^{\langle k \rangle})_x(\boldsymbol{\omega}_k^{\langle k \rangle})_z \\ -[(\boldsymbol{\omega}_k^{\langle k \rangle})_x^2 + (\boldsymbol{\omega}_k^{\langle k \rangle})_y^2] \\ (\mathbf{a}_k^{\langle k \rangle} - \mathbf{g}^{\langle k \rangle})_z \end{bmatrix} \cdot \begin{bmatrix} (\hat{\mathbf{c}}_k^{\langle k \rangle})_x \\ (\hat{\mathbf{c}}_k^{\langle k \rangle})_y \\ (\hat{\mathbf{c}}_k^{\langle k \rangle})_z \\ \hat{m}_k \end{bmatrix} \tag{52}$$

Since only joint i rotates, we apply kinematics to get

$$\boldsymbol{\omega}_j^{\langle j \rangle} = \mathbf{u}_i^{\langle j \rangle} \dot{q}_i \tag{53}$$

$$\dot{\boldsymbol{\omega}}_j^{\langle j\rangle} = \mathbf{u}_i^{\langle j\rangle}\ddot{q}_i \tag{54}$$

$$\mathbf{a}_j^{\langle j\rangle} = \mathbf{u}_i^{\langle j\rangle} \times {}_i^j\mathbf{s}^{\langle j\rangle}\ddot{q}_i + \mathbf{u}_i^{\langle j\rangle} \times (\mathbf{u}_i^{\langle j\rangle} \times {}_i^j\mathbf{s}^{\langle j\rangle})\dot{q}_i^2 \tag{55}$$

where q_i is the displacement of joint i. Substituting (53)–(55) into (51) and (52) yields

$$\tau_j = \begin{bmatrix} -(\mathbf{u}_i^{\langle j\rangle})_x(\mathbf{u}_i^{\langle j\rangle})_y\dot{q}_i^2 \\ (\mathbf{u}_i^{\langle j\rangle})_z\ddot{q}_i \\ [(\mathbf{u}_i^{\langle j\rangle})_x^2 - (\mathbf{u}_i^{\langle j\rangle})_y^2]\dot{q}_i^2 \\ (\mathbf{u}_i^{\langle j\rangle})_x\ddot{q}_i - (\mathbf{u}_i^{\langle j\rangle})_y(\mathbf{u}_i^{\langle j\rangle})_z\dot{q}_i^2 \\ (\mathbf{u}_i^{\langle j\rangle})_y\ddot{q}_i + (\mathbf{u}_i^{\langle j\rangle})_x(\mathbf{u}_i^{\langle j\rangle})_z\dot{q}_i^2 \\ (\mathbf{a}_j^{\langle j\rangle} - \mathbf{g}^{\langle j\rangle})_y \\ -(\mathbf{a}_j^{\langle j\rangle} - \mathbf{g}^{\langle j\rangle})_x \end{bmatrix} \cdot \begin{bmatrix} (\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22} + (\mathbf{V}_j)_{11} - (\mathbf{V}_j)_{22} \\ (\mathbf{U}_j)_{33} + (\mathbf{V}_j)_{33} \\ (\mathbf{U}_j)_{12} + (\mathbf{V}_j)_{12} \\ (\mathbf{U}_j)_{13} + (\mathbf{V}_j)_{13} \\ (\mathbf{U}_j)_{23} + (\mathbf{V}_j)_{23} \\ (\mathbf{k}_j)_x + (\ell_j)_x \\ (\mathbf{k}_j)_y + (\ell_j)_y \end{bmatrix} \tag{56}$$

$$\tau_k = \begin{bmatrix} -(\mathbf{u}_i^{\langle k\rangle})_y\ddot{q}_i + (\mathbf{u}_i^{\langle k\rangle})_x(\mathbf{u}_i^{\langle k\rangle})_z\dot{q}_i^2 \\ (\mathbf{u}_i^{\langle k\rangle})_x\ddot{q}_i + (\mathbf{u}_i^{\langle k\rangle})_y(\mathbf{u}_i^{\langle k\rangle})_z\dot{q}_i^2 \\ -[(\mathbf{u}_i^{\langle k\rangle})_x^2 + (\mathbf{u}_i^{\langle k\rangle})_y^2]\dot{q}_i^2 \end{bmatrix} \cdot \begin{bmatrix} (\ell_k)_x \\ (\ell_k)_y \\ (\ell_k)_z \end{bmatrix} + \begin{bmatrix} \ddot{q}_i \\ \dot{q}_i^2 \\ (\mathbf{a}_i^{\langle k\rangle} - \mathbf{g}^{\langle k\rangle})_z \end{bmatrix}$$
$$\cdot \begin{bmatrix} -(\mathbf{u}_i^{\langle k\rangle})_y(\mathbf{k}_k)_x + (\mathbf{u}_i^{\langle k\rangle})_x(\mathbf{k}_k)_y \\ (\mathbf{u}_i^{\langle k\rangle})_z[(\mathbf{u}_i^{\langle k\rangle})_x(\mathbf{k}_k)_x + (\mathbf{u}_i^{\langle k\rangle})_y(\mathbf{k}_k)_y] - [1 - (\mathbf{u}_i^{\langle k\rangle})_z^2](\mathbf{k}_k)_z \\ \hat{m}_k \end{bmatrix} \tag{57}$$

Note that $\mathbf{u}_i^{\langle k\rangle}$ is invariant since joint k is a translational joint. We are also interested in the actuator force of joint i, which is

$$\tau_i = \mathbf{u}_i^{\langle i\rangle} \cdot \mathbf{t}_{Ei}^{\langle i\rangle} = \begin{bmatrix} \ddot{q}_i \\ -(\mathbf{g}^{\langle i\rangle})_y \\ (\mathbf{g}^{\langle i\rangle})_x \end{bmatrix} \cdot \begin{bmatrix} (\mathbf{U}_i)_{33} + (\mathbf{V}_i)_{33} \\ (\mathbf{k}_i)_x + (\ell_i)_x \\ (\mathbf{k}_i)_y + (\ell_i)_y \end{bmatrix} \tag{58}$$

since $\boldsymbol{\omega}_i^{\langle i\rangle} = \mathbf{u}_i^{\langle i\rangle}\dot{q}_i$, $\dot{\boldsymbol{\omega}}_i^{\langle i\rangle} = \mathbf{u}_i^{\langle i\rangle}\ddot{q}_i$ and $\mathbf{a}_i^{\langle i\rangle} = \mathbf{0}$.

The next effort is devoted to examining the linear independence of the coefficients on the right sides of (56)–(58). We remark that the x- and y-components of $\mathbf{g}^{\langle j\rangle}$ are linear independent if rotational joints i and j are not parallel to each other since these two components are linearly independent for the case that joint j is not parallel to the gravity direction. If $\mathbf{u}_i$ is neither parallel nor perpendicular to $\mathbf{u}_j$, the three components of $\mathbf{u}_i^{\langle j\rangle}$ are nonzero. However, $(\mathbf{u}_i^{\langle j\rangle})_z = 0$ for $\mathbf{u}_i \perp \mathbf{u}_j$, whereas $(\mathbf{u}_i^{\langle j\rangle})_x = (\mathbf{u}_i^{\langle j\rangle})_y = 0$ for $\mathbf{u}_i /\!/ \mathbf{u}_j$. Thus, we conclude the following.

Property 1. *Rotating joint i and locking all other joints.*

1. Actuator force of rotational joint j, i.e., (56).
   (a) The coefficients in (56) are linearly independent over $\{\dot{q}_i, \ddot{q}_i, q_j\} \subset R^3$ if joint j is neither parallel nor perpendicular to joint i.
   (b) The coefficient of $(\mathbf{U}_j)_{33}$ is zero; the other coefficients are linearly independent if $\mathbf{u}_j \perp \mathbf{u}_i$.
   (c) Only the coefficient of $(\mathbf{U}_j)_{33}$ is nonzero if $\mathbf{u}_j /\!/ \mathbf{u}_i /\!/ \mathbf{g}$ and ${}^j_i\mathbf{s} /\!/ \mathbf{u}_i$.
   (d) Only the coefficients of $(\mathbf{U}_j)_{33}$, $(\mathbf{k}_j)_x$, and $(\mathbf{k}_j)_y$ are nonzero and linearly independent if $\mathbf{u}_j /\!/ \mathbf{u}_i$ but $\mathbf{u}_i \not/\!/ \mathbf{g}$ or ${}^j_i\mathbf{s} \not/\!/ \mathbf{u}_i$.
2. Actuator force of translational joint k, i.e., (57).
   (a) $\mathbf{k}_k$ has no effect on τ_k if $\mathbf{u}_k /\!/ \mathbf{u}_i$.
   (b) The coefficients, $\ddot{q}_i$ and $\dot{q}_i^2$, are linearly independent if $\mathbf{u}_k \not/\!/ \mathbf{u}_i$.
3. Actuator force of rotating joint i, i.e., (58).
   (a) The coefficients in (58) are linearly independent over $\{q_i, \ddot{q}_i\} \subset R^2$ if $\mathbf{u}_i \not/\!/ \mathbf{g}$. Otherwise, only the coefficient of $(\mathbf{U}_i)_{33}$ is nonzero.

3.2. Moving One Translational Joint

We now consider another case that all joints except a translational joint k, $i < k < j$, are locked. Since only a translational joint is in motion, there is no angular velocity and acceleration. The velocities and accelerations of the links remaining behind joint k are all $\mathbf{u}_k\dot{q}_k$ and $\mathbf{u}_k\ddot{q}_k$, respectively. Analogous to (51) and (52), we get the actuator forces of joint j and joint k in the forms of

$$\tau_j = \begin{bmatrix} (\mathbf{u}_k^{\langle j\rangle})_y\ddot{q}_k - (\mathbf{g}^{\langle j\rangle})_y \\ -(\mathbf{u}_k^{\langle j\rangle})_x\ddot{q}_k + (\mathbf{g}^{\langle j\rangle})_x \end{bmatrix} \cdot \begin{bmatrix} (\mathbf{k}_j)_x + (\ell_j)_x \\ (\mathbf{k}_j)_y + (\ell_j)_y \end{bmatrix} \tag{59}$$

$$\tau_k = \hat{m}_k[\ddot{q}_k - (\mathbf{g}^{\langle k\rangle})_z] \tag{60}$$

If joint k is not parallel to joint j, the x- and y-components of $\mathbf{u}_k^{\langle j\rangle}$ are linearly independent; otherwise, they are zero.

Property 2. *Moving translational joint k and locking all other joints.*

1. Actuator forces of rotational joint j, i.e., (59).
   (a) The coefficients in (59) are linearly independent over $\{\ddot{q}_k, q_j\} \subset R^2$ if either $\mathbf{u}_j \not/\!/ \mathbf{g}$ or $\mathbf{u}_j \not/\!/ \mathbf{u}_k$.
   (b) The coefficients in (59) are zero if $\mathbf{u}_j /\!/ \mathbf{g}$ and $\mathbf{u}_j /\!/ \mathbf{u}_k$.
2. Actuator forces of moving joint k, i.e., (60).
   (a) The coefficient in (60) is nonzero if $\ddot{q}_k$ is nonzero.

3.3. Off-Line Indentification Procedure

Lemma 3. *The inertia constants listed in Theorem 2 are all identifiable.*

Proof: The inertia constants listed in item 3 of Theorem 2 can be identified by moving one translational joint at a time for every translational joint and using (60).

For a rotational joint j, $j \geq s$, there exists at least one rotational joint, say joint r, in front of joint j such that joint j is not parallel to it. If the two joints are also not perpendicular to each other, item 1(a) in Property 1 implies that there exists a persistently exciting trajectory of q_i and q_j such that the columns of the matrix in the linear equations formed by (56) are linearly independent. Since ℓ_i and $\mathbf{V}_i$ can be calculated by Algorithm 1 if the inertia constants of links $j + 1$ to n are known, the inertia constants listed in item 2 of Theorem 2 can be estimated recursively by using the standard least-squares method.[36] If joint j is perpendicular to joint r, we can rotate joint j alone to identify $(\mathbf{U}_j)_{33}$ by using (58), while the other inertia constants are still estimated, according to item 1(b) of Property 1, in the same way as for the above case.

For a translational joint k, $k > s$, we rewrite (57) in the form of

$$\tau_k = \begin{bmatrix} -(\mathbf{u}_i^{\langle k\rangle})_y \ddot{q}_i + (\mathbf{u}_i^{\langle k\rangle})_x (\mathbf{u}_i^{\langle k\rangle})_z \dot{q}_i^2 \\ (\mathbf{u}_i^{\langle k\rangle})_x \ddot{q}_i + (\mathbf{u}_i^{\langle k\rangle})_y (\mathbf{u}_i^{\langle k\rangle})_z \dot{q}_i^2 \\ -[(\mathbf{u}_i^{\langle k\rangle})_x^2 + (\mathbf{u}_i^{\langle k\rangle})_y^2]\dot{q}_i^2 \\ (\mathbf{a}_i^{\langle k\rangle} - \mathbf{g}^{\langle k\rangle})_z \end{bmatrix} \cdot \begin{bmatrix} (\ell_k)_x + (\mathbf{k}_k)_x \\ (\ell_k)_y + (\mathbf{k}_k)_y \\ (\ell_k)_z + (\mathbf{k}_k)_z \\ \hat{m}_k \end{bmatrix} \tag{61}$$

We denote any two nonparallel rotational joints in front of joint k as joints i and j, $i < j$. There are at least two nonzero components of $\mathbf{u}_i^{\langle j\rangle}$ for some configurations according to (4) and (42). Since $\mathbf{u}_i^{\langle k\rangle} = {}_k^j\mathbf{R}\ \mathbf{u}_i^{\langle j\rangle}$ and ${}_k^j\mathbf{R}$ is an orthogonal matrix, there are also at least two nonzero components of $\mathbf{u}_i^{\langle k\rangle}$ for some configurations. It is then possible to rotate joint j to find two or more configurations in which at least two components of $\mathbf{u}_i^{\langle k\rangle}$ are nonzero. Under such configurations, the coefficients on the right side of (61) are linearly independent. Therefore, $\mathbf{k}_k$ for any translational joint k, $k > s$, can be estimated by rotating joint i under at least two such configurations and using (61). This shows that the inertia constants listed in item 4 of Theorem 2 are identifiable.

Equation (57) and item 2(b) of Property 1 imply that the inertia constants listed in item 5 of Theorem 2 are identifiable by rotating joint r.

At last, we consider the rotational joints remaining in front of joint s. In this case, all rotational joints are parallel to one another. However, $(\mathbf{U}_j)_{33}$ for rotational joint j, $j < s$, is still identifiable by rotating joint r according to items 1(c), 1(d), and 3(a) of Property 1. Item 1(d) of Property 1 also indicates that the x- and y-components of $\mathbf{k}_j$ are identifiable by rotating a rotational joint i, $i < j$, for the case that $\mathbf{u}_r \not\parallel \mathbf{g}$ or ${}_i^j\mathbf{s} \not\parallel \mathbf{u}_i$, whereas item 1(a) of Property 2 reveals that these components for the case of $\mathbf{u}_j \parallel \mathbf{u}_r \parallel \mathbf{g} \parallel {}_i^j\mathbf{s}$ for any rotational joint i remaining in front of joint j can still be estimated by moving a translational joint k, which remains in front of joint j and is not parallel to joint j, if it exists. This completes the proof. ∎

Theorem 4. *The inertia constants listed in Theorem 2 are a set of the minimal knowledge of the inertia parameters for determining the dynamic model of a manipulator.*

Proof: Since the inertia constants in Theorem 2 are all identifiable and are linearly independent, the claim is true. ∎

The proof of Lemma 3 provides us with an off-line identification procedure.

Algorithm 2. *Off-line identification procedure.*

Step 1: Move one translational joint at a time for every translational joint and use (60) to estimate $\hat{m}_k$ for every translational joint k.

Step 2: Do the following substeps recursively from joint n to joint s.

2.1. For rotational joint j, we compute $\mathbf{V}_j$ and ℓ_j using Algorithm 1. Let joint j be neither parallel nor perpendicular to joint r. Under such condition, find three or more configurations of q_j such that the components of $\mathbf{u}_r^{\langle j \rangle}$ are all nonzero and the first two are not equal. For each of such configurations, we rotate joint r alone and measure the values of τ_j, $\dot{q}_r$, and $\ddot{q}_r$. These values are substituted into (56) to form the linear equations. Applying the least-squares method, we then estimate $[(\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22}]$, $(U_j)_{33}$, $(\mathbf{U}_j)_{12}$, $(\mathbf{U}_j)_{13}$, $(\mathbf{U}_j)_{23}$, $(\mathbf{k}_j)_x$, and $(\mathbf{k}_j)_y$.

If joint j is always perpendicular to joint r, the above method cannot estimate $(\mathbf{U}_j)_{33}$. We additionally rotate joint j alone to estimate this value by using (58).

2.2. For translational joint j, we also compute $\mathbf{V}_j$ and ℓ_j using Algorithm 1. We rotate joint s to find two or more configurations such that at least two components of $\mathbf{u}_r^{\langle j \rangle}$ are nonzero. For each of such configurations, we rotate joint r alone and measure the values of τ_j, $\dot{q}_r$, and $\ddot{q}_r$. These values are substituted into (61) to form the linear equations. Applying the least-squares method, we then estimate the three components of $\mathbf{k}_j$.

Step 3: Do the following substeps recursively from joint $s - 1$ to joint r.

3.1. If joint j is a rotational joint, compute $(\mathbf{V}_j)_{33}$ and $(\ell)_j$ using Algorithm 1.

(a) If $\mathbf{u}_r \not\parallel \mathbf{g}$, we rotate joint r alone and use (56) to estimate $(\mathbf{U}_j)_{33}$, $(\mathbf{k}_j)_x$, and $(\mathbf{k}_j)_y$.

(b) If $\mathbf{u}_r \parallel \mathbf{g}$ and there exists a rotational joint i in front of joint j from which the distance to joint j (i.e., ${}_i^j\mathbf{s}$) is nonzero and not parallel to joint j, we rotate joint i alone and use (56) again to estimate $(\mathbf{U}_j)_{33}$, $(\mathbf{k}_j)_x$, and $(\mathbf{k}_j)_y$.

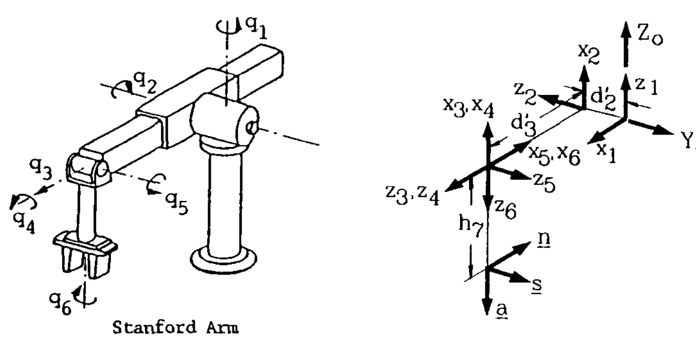
(c) If $\mathbf{u}_r \parallel \mathbf{g}$ and any rotational joint i in front of joint j satisfies ${}_i^j\mathbf{s} \parallel \mathbf{u}_j$, we still rotate joint r and use (56) to estimate $(\mathbf{U}_j)_{33}$. However, we check if there is a translational joint k in front of joint j that is not parallel to joint j. If there is, we move this translational joint for two or more configurations in which the x- and y-components of $\mathbf{u}_k^{\langle j\rangle}$ are linearly independent and then use (59) to estimate $(\mathbf{k}_j)_x$ and $(\mathbf{k}_j)_y$.

3.2. If joint j is a translational joint, compute ℓ_j using Algorithm 1. We rotate joint r alone and apply (57) and the least-squares method to estimate $-(\mathbf{u}_r^{\langle j\rangle})_z[(\mathbf{u}_r^{\langle j\rangle})_x(\mathbf{k}_j)_x + (\mathbf{u}_r^{\langle j\rangle})_y(\mathbf{k}_j)_y] + [1 - (\mathbf{u}_r^{\langle j\rangle})_z^2](\mathbf{k}_j)_z$ and $-(\mathbf{u}_r^{\langle j\rangle})_y(\mathbf{k}_j)_x + (\mathbf{u}_r^{\langle j\rangle})_x(\mathbf{k}_j)_y$.

The off-line identification procedure is not a unique method. However, it is simple because we rotate joint r alone in the most part of the identification procedure. Since the joint acceleration cannot be very accurately measured in comparison with the joint displacement and velocity, the allowance of one joint in motion at a time in the above procedure can reduce the identification error to some extent.

4. ILLUSTRATIVE EXAMPLE

The Stanford arm has five rotational joints and one translational joint. The normal driving-axis coordinate system[16] and the kinematic and dynamic parameters of the Stanford arm are shown in Figure 2. We assume that the inertia parameters are unknown, and want to use the off-line identification procedure to estimate the minimal knowledge of the inertia parameters.



Link (Type)	θ	β	b (m)	d (m)	Mass (kg)	Center of Mass (m) $(\mathbf{c}_i^{<i>})_x$	$(\mathbf{c}_i^{<i>})_y$	$(\mathbf{c}_i^{<i>})_z$	Inertia Tensor (kg m^2) $(\mathbf{I}_i^{<i>})_{11}$	$(\mathbf{I}_i^{<i>})_{22}$	$(\mathbf{I}_i^{<i>})_{33}$
1 (R)	q_1	0.	0.	0.	9.29	0.	0.1105	0.0175	0.276	0.255	0.71
2 (R)	$q_2+90°$	90°	0.	0.1529	5.01	0.	0.	−0.1054	0.108	0.018	0.1
3 (T)	0.	90°	0.	$q_3+0.6447$	4.25	0.	0.	−0.6447	2.51	2.51	0.006
4 (R)	q_4	0.	0.	0.	1.08	0.0092	0.0054	0.	0.002	0.001	0.001
5 (R)	$q_5-90°$	90°	0.	0.	0.63	0.	−0.0566	0.	0.003	0.003	0.0004
6 (R)	q_6	90°	0.	0.	0.51	0.	0.	0.1554	0.013	0.013	0.0003

$h_7 = 0.2554$

Figure 2. Normal driving-axis coordinate system and technical data of the Stanford arm.

The first rotational joint is joint 1, and joint 2 is the nearest rotational joint not parallel to joint 1, i.e., $r = 1$ and $s = 2$. Note, joint 1 is parallel to the gravitational direction. According to Theorem 2, the minimal knowledge of inertia parameters is the set of

1. $(\mathbf{U}_1)_{33}$,
2. $[(\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22}], (\mathbf{U}_j)_{33}, (\mathbf{U}_j)_{12}, (\mathbf{U}_j)_{13}, (\mathbf{U}_j)_{23}, (\mathbf{k}_j)_x, (\mathbf{k}_j)_y, \quad j = 2, 4, 5, 6$,
3. $\hat{m}_3$,
4. $(\mathbf{k}_3)_x, (\mathbf{k}_3)_y, (\mathbf{k}_3)_z$.

Although $\hat{m}_3$ can be estimated individually according to step 1 in Algorithm 2, it can also be identified together with $\mathbf{k}_3$ by using step 2.2 in Algorithm 2. For convenience, we skip step 1 of Algorithm 2 and use step 2 to estimate the inertia constants of the composite bodies remaining behind joint 1.

It is easy to find some persistently exciting configurations described in step 2 of Algorithm 2. For example, the following configuration satisfies the requirement of Algorithm 2

$$\mathbf{q} = \begin{bmatrix} 0.7 \\ 0.7 \\ 0. \\ 0.7 \\ 0.7 \\ 0.7 \end{bmatrix} \tag{62}$$

where $\mathbf{q} = [q_1, \ldots, q_6]^T$. The values of $\mathbf{u}_1^{\langle j \rangle}$, $j = 2, 3, 4, 5, 6$, are listed in the first row of Table I. The other two configurations to form persistently exciting configurations for estimating the inertia constants of composite bodies 2, 4, 5, and 6 are selected the same as (62) except that the corresponding joint displace-

Table I. Representations of $\mathbf{u}_1$ with respect to the body-fixed frames in three individual exciting configurations.

	$\mathbf{u}_1^{\langle 2 \rangle}$	$\mathbf{u}_1^{\langle 3 \rangle}$	$\mathbf{u}_1^{\langle 4 \rangle}$	$\mathbf{u}_1^{\langle 5 \rangle}$	$\mathbf{u}_1^{\langle 6 \rangle}$
1	$\begin{bmatrix} 0.7648 \\ -0.6442 \\ 0. \end{bmatrix}$	$\begin{bmatrix} 0.7648 \\ 0. \\ 0.6442 \end{bmatrix}$	$\begin{bmatrix} 0.5850 \\ -0.4927 \\ 0.6442 \end{bmatrix}$	$\begin{bmatrix} -0.1159 \\ 0.8624 \\ 0.4927 \end{bmatrix}$	$\begin{bmatrix} 0.2288 \\ 0.4515 \\ -0.8624 \end{bmatrix}$
2	$\begin{bmatrix} 0.7648 \\ 0.6442 \\ 0. \end{bmatrix}$	$\begin{bmatrix} 0.7648 \\ 0. \\ -0.6442 \end{bmatrix}$	$\begin{bmatrix} 0.5850 \\ 0.4927 \\ 0.6442 \end{bmatrix}$	$\begin{bmatrix} -0.8696 \\ 0.0324 \\ 0.4927 \end{bmatrix}$	$\begin{bmatrix} -0.4060 \\ 0.3022 \\ -0.8624 \end{bmatrix}$
3	$\begin{bmatrix} 0.8253 \\ 0.5646 \\ 0. \end{bmatrix}$	—	$\begin{bmatrix} 0.6313 \\ 0.4319 \\ 0.6442 \end{bmatrix}$	$\begin{bmatrix} -0.8620 \\ 0.1190 \\ 0.4927 \end{bmatrix}$	$\begin{bmatrix} -0.3738 \\ 0.3412 \\ -0.8264 \end{bmatrix}$

ment is changed to −0.6 and −0.7. That means we use the following three configurations to estimate the inertia constants of composite body 6

$$\mathbf{q} = \begin{bmatrix} 0.7 \\ 0.7 \\ 0. \\ 0.7 \\ 0.7 \\ 0.7 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 0.7 \\ 0.7 \\ 0. \\ 0.7 \\ 0.7 \\ -0.6 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 0.7 \\ 0.7 \\ 0. \\ 0.7 \\ 0.7 \\ -0.7 \end{bmatrix} \tag{63}$$

Another configuration for estimating the inertia constants of composite body 3 is that the displacement of joint 2 is changed to −0.7 while the other joint displacements are kept the same as (62). The values of $\mathbf{u}_1^{(j)}$ for the second and third configurations of the individual identification process are also listed in Table I.

Under each of the above configurations, we rotate joint 1 from the displacement of 0.7 to 1.223 rad (i.e., a rotation of 30°). We assume there is a controller installed in the power driver of the actuator on joint 1 such that the response of the joint angle to a step input is a second-order critically damping with damping time constant $1/\sigma = 1/10$s, i.e.

$$q_1(t) = \frac{\pi}{6}[1 - (1 + \sigma t)e^{-\sigma t}] + 0.7 \tag{64}$$

$$\dot{q}_1(t) = \frac{\pi}{6}\sigma^2 t e^{-\sigma t} \tag{65}$$

$$\ddot{q}_1(t) = \frac{\pi}{6}(1 - \sigma t)\sigma^2 e^{-\sigma t} \tag{66}$$

We arbitrarily select three sets of the values of τ_j, $\dot{q}_1$, and $\ddot{q}_1$, say at $t = 0.2, 0.3$, and 0.4, and substitute them into (56) and (57) to form the linear equations. Using a package of the least-squares method, we then estimate the inertia constants of composite bodies 2, 3, 4, 5, and 6. Since joint 2 is perpendicular to joint 1, $(\mathbf{U}_2)_{33}$ is not yet identified. We rotate joint 2 alone, take one set of the values of τ_2 and $\ddot{q}_2$, and use (58) to estimate $(\mathbf{U}_2)_{33}$. $(\mathbf{U}_1)_{33}$ is also estimated in the same manner. The identified values of the minimal knowledge of the inertia parameters of the Stanford arm are shown in Table II.

It should be remarked that the example is performed in a computer simulation. The actuator forces are calculated by a software of the recursive Newton–Euler formulation when the joint displacements, velocities, and accelerations are given. We also write a program, which uses the identified values of the minimal knowledge of the inertia parameters and sets the inertia constants of

Table II. Identified values of inertia constants of the composite bodies.

Composite Body i	1	2	3	4	5	6
$\hat{m}_i$	—	—	6.4700	—	—	—
$\mathbf{k}_i$	—	$\begin{bmatrix} -0.00001 \\ 2.7400 \\ — \end{bmatrix}$	$\begin{bmatrix} 0. \\ 0. \\ -2.7400 \end{bmatrix}$	$\begin{bmatrix} 0.0099 \\ 0.0058 \\ — \end{bmatrix}$	$\begin{bmatrix} 0. \\ -0.1149 \\ — \end{bmatrix}$	$\begin{bmatrix} 0. \\ 0. \\ — \end{bmatrix}$
$(\mathbf{U}_i)_{11}-(\mathbf{U}_i)_{22}$	—	4.3850	—	−0.0235	0.0270	0.
$(\mathbf{U}_i)_{33}$	1.0144	4.4053	–	0.0044	0.0277	0.0003
$(\mathbf{U}_i)_{12}$	—	0.00003	—	−0.00005	0.	0.
$(\mathbf{U}_i)_{13}$	—	0.	—	0.	0.	0.
$(\mathbf{U}_i)_{23}$	—	−0.00003	—	0.	0.	0.

composite bodies other than those in Theorem 2 to zero, to compute the inertia matrix and the gravity load by using (9) and (11)–(13). We find the results are the same as those using another method presented in Lin.[32] This verifies our theorems and the off-line identification procedure.

5. CONCLUSION

We have presented a new approach to finding a set of the minimal knowledge of the inertia parameters for determining the manipulator dynamics. An identification procedure is only to estimate the inertia constants in the minimal knowledge of the inertia parameters. The central topic of this article is then to develop such an off-line identification method. The identification procedure demands only to move one joint (the first rotational joint in the most part of the procedure) at a time for recursively estimating the minimal knowledge of the inertia parameters. A simulation example of the Stanford arm verifies the off-line identification procedure. The main advantage of our method is that the identification procedure does not require the symbolic dynamic equations.

Finding a persistently exciting trajectory is always a difficult problem in the identification. Armstrong[37] addressed a method of generating the optimal identification trajectory, whereas trial and error methods are widely used in the literature.[13,15,17] On the contrary, our off-line identification procedure provides a sufficient condition for a persistently exciting trajectory and allows the identification method to be easily implemented.

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