

An $O(kn)$ Algorithm for a Circular Consecutive- k -out-of- n :F System

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Key Words — Circular consecutive- k -out-of- n :F system, System reliability, Algorithm.

Reader Aids —

Purpose: Report a new algorithm

Special math needed for explanations: Probability theory

Special math needed to use results: None

Result useful to: Reliability analysts and theoreticians

Abstract — An $O(k \cdot n)$ algorithm is described for evaluating the reliability of a circular consecutive- k -out-of- n :F system.

1. INTRODUCTION

A consecutive- k -out-of- n :F system is n ordered components such that the system fails if and only if at least k consecutive components fail. Such a system is relevant to telecommunication. There are two topologies for this system: a straight line and a circle. The reliability of this system was first studied by Chiang & Niu [2], and later by Derman, Lieberman, Ross [3], Hwang [4], Shanthikumar [5], and Antonopoulou & Papastavridis [1].

Hwang [4] proposed two recursive algorithms to evaluate the reliabilities of linear and circular consecutive- k -out-of- n :F systems. These two algorithms require $O(n)$ and $O(n \cdot k^2)$ computing time respectively. Antonopoulou & Papastavridis [1] announced an $O(n \cdot k)$ recursive algorithm for computing the reliability of a circular such system. This paper demonstrates an algorithm (Sys) to evaluate the reliability of a circular system which needs only $O(n \cdot k)$ computing time.

2. MODEL

Assumptions

1. Each component, subsystem and system either functions or fails.
2. All n component states are mutually s -independent.
3. Components 1, 2, ..., n are arranged on a circle in that order.
4. The system or subsystem fails if and only if at least k consecutive components all fail. \square

Notation

n	number of components in a system
k	minimum number of consecutive failed components which causes system failure
i	index for a component; $i = 1, 2, \dots, n$
p_i, q_i	probability that component i functions, fails; $p_i + q_i \equiv 1$
sys	circular consecutive- k -out-of- n :F system
sys-0	linear consecutive- k -out-of- n :F system
sys- i	linear consecutive- k -out-of- $(n+i)$:F systems, for $i = 1, 2, \dots, k-1$; $q_j = 1$ for $j = 1, 2, \dots, i, n+1, \dots, n+i$
F_{sys}	$\Pr\{\text{sys is failed}\}$
$F_{\text{sys-}i}$	$\Pr\{\text{sys-}i \text{ is failed}\}$, for $i = 0, 1, \dots, k-1$
$F'_{\text{sys-}i}$	$\Pr\{\text{sys-}i \text{ is failed}\}$, for $i = 1, \dots, k-1$, wherein the first $(n+i-1)$ components are considered
S_i, T_i	event that consecutive k components starting with component i all fail in a linear, circular system

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

3. COMPUTATION OF RELIABILITY

We express $\Pr\{\text{sys is failed}\}$ by using the $\Pr\{\text{sys-}i \text{ is failed}\}$ formulas. Thus, (3-1) is Sys-algorithm.

$$\begin{aligned}
 F_{\text{sys}} &= F_{\text{sys-}0} + q_1 \cdot (F_{\text{sys-}1} - F'_{\text{sys-}1}) \\
 &+ q_1 \cdot q_2 \cdot (F_{\text{sys-}2} - F'_{\text{sys-}2}) + \dots \\
 &+ q_1 \cdot q_2 \cdot \dots \cdot q_{k-1} \cdot (F_{\text{sys-}(k-1)} - F'_{\text{sys-}(k-1)}). \quad (3-1)
 \end{aligned}$$

Hwang [4] proved that the time complexity of computing the reliability of a linear consecutive- k -out-of- n :F system is $O(n)$. Because $n > k$, the computing time of each $F_{\text{sys-}i}$ or $F'_{\text{sys-}i}$ is $O(n+k) = O(n)$, for $i = 0, 1, \dots, k-1$. Furthermore, the time complexity for calculating $q_1, q_1 \cdot q_2, \dots, q_1 \cdot q_2 \cdot \dots \cdot q_{k-1}$ is $O(k)$. So the time complexity for (3-1) in the Sys-algorithm is $O(n) + O(k \cdot n) + O(k) = O(k \cdot n)$. Intuitively, the time complexity for the formula in [1] is $O(n^2 \cdot k)$.

4. PROOF

4.1 Lemma

Before proving Sys-algorithm we need the lemma.

\square **Lemma:** In sys- i , for $i = 1, \dots, k-1$:

$$\Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \dots \cap \bar{S}_{i-k} \cap S_{i-k+1}\} = F_{\text{sys-}i} - F'_{\text{sys-}i}. \quad (4-1)$$

Proof: By the sum-of-disjoint method, the failed probability of a linear sys- i , for $i = 1, \dots, k-1$, is:

$$\begin{aligned} F_{\text{sys-}i} &= \Pr\{S_1 \cup S_2 \cup \dots \cup S_{i-k+1}\} \\ &= \Pr\{S_1\} + \Pr\{\bar{S}_1 \cap S_2\} + \dots \\ &+ \Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \dots \cap \bar{S}_{i-k-1} \cap S_{i-k}\} \\ &+ \Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \dots \cap \bar{S}_{i-k} \cap S_{i-k+1}\}. \end{aligned} \quad (4-2)$$

In the subsystem of the linear sys- i wherein the first $n+i-1$ components are considered. Similarly, the failed probability of this subsystem is

$$\begin{aligned} F'_{\text{sys-}i} &= \Pr\{S_1 \cup S_2 \cup \dots \cup S_{i-k}\} \\ &= \Pr\{S_1\} + \Pr\{\bar{S}_1 \cap S_2\} + \dots \\ &+ \Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \dots \cap \bar{S}_{i-k-1} \cap S_{i-k}\}. \end{aligned} \quad (4-3)$$

Eq (4-1) is obtained by subtracting (4-3) from (4-2). *Q.E.D.*

4.2 Sys-Algorithm

In the circular system,

$$\begin{aligned} F_{\text{sys}} &= F_{\text{sys-0}} + q_1[F_{\text{sys-1}} - F'_{\text{sys-1}}] \\ &+ q_1 q_2 [F_{\text{sys-2}} - F'_{\text{sys-2}}] \\ &+ \dots + q_1 q_2 \dots q_{k-1} [F_{\text{sys-(}k-1)} - F'_{\text{sys-(}k-1)}]. \end{aligned} \quad (4-4)$$

Proof: By the sum-of-disjoint method, the failed probability of the circular system is:

$$\begin{aligned} F_{\text{sys}} &= \Pr\{T_1 \cup T_2 \cup \dots \cup T_n\} \\ &= \Pr\{T_1\} + \Pr\{\bar{T}_1 \cap T_2\} + \dots \\ &+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k} \cap T_{n-k+1}\} \\ &+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2}\} \\ &+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+2} \cap T_{n-k+3}\} \\ &+ \dots + \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-1} \cap T_n\}. \end{aligned} \quad (4-5)$$

By (4-2), we can express the failed probability of sys-0 as:

$$\begin{aligned} F_{\text{sys-0}} &= \Pr\{T_1\} + \Pr\{\bar{T}_1 \cap T_2\} + \dots \\ &+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k} \cap T_{n-k+1}\}. \end{aligned} \quad (4-6)$$

Now we consider,

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2}\}$$

in (4-5).

Notation

$$\begin{aligned} E &\quad \text{event: } \bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2} \\ F &\quad \text{event: component 1 fails.} \end{aligned} \quad \square$$

By applying —

$$\begin{aligned} \Pr\{E\} &= \Pr\{EF\} + \Pr\{E\bar{F}\} = \Pr\{E/F\} \Pr\{F\} \\ &+ \Pr\{E/\bar{F}\} \Pr\{\bar{F}\}. \end{aligned} \quad (4-7)$$

We have —

$$\begin{aligned} &\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2}\} \\ &= \Pr\{1 \text{ fails}\} \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2} | 1 \text{ fails}\} \\ &+ \Pr\{1 \text{ functions}\} \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \\ &\cap T_{n-k+2} | 1 \text{ functions}\} \\ &= \Pr\{1 \text{ fails}\} \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \\ &\cap T_{n-k+2} | 1 \text{ fails}\} + 0 \\ &= q_1 (F_{\text{sys-1}} - F'_{\text{sys-1}}). \end{aligned} \quad (4-8)$$

In (4-8),

$$\begin{aligned} &\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2} | \text{component-1 functions}\} \\ &= 0, \end{aligned}$$

because T_{n-k+2} is the event that all the components from $(n-k+2)$ to n and component-1 all fail, which is against the condition that component-1 functions. And

$$\begin{aligned} &\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+1} \cap T_{n-k+2} | \text{component-1 fails}\} \\ &= F_{\text{sys-1}} - F'_{\text{sys-1}}, \end{aligned}$$

due to the lemma (4-1).

Notation

$$\begin{aligned} E &\quad \text{event: } \bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+2} \cap T_{n-k+3} \\ F &\quad \text{event: components 1 \& 2 both fail} \end{aligned} \quad \square$$

By (4-7) we have:

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \bar{T}_{n-k+2} \cap T_{n-k+3}\}$$

$$= q_1 q_2 (F_{\text{sys-2}} - F'_{\text{sys-2}}). \quad (4-9)$$

By (4-8) & (4-9) & the lemma, we get the general formula for other terms in (4-4):

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \dots \cap \overline{T_{n-k+i}} \cap T_{n-k+1+i}\} \\ = q_1 q_2 \dots q_i (F_{\text{sys-}i} - F'_{\text{sys-}i}), \text{ for } i = 1, 2, \dots, k-1. \quad (4-10)$$

Apply (4-5) & (4-10).

Q.E.D

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Mean Time to Failure for a Consecutive- k -out-of- n :F System

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Formulas for μ , μ_c , μ_w in [1], contain some typographical errors. They must be the following formulas.

Notation

n	number of components in the system
k	minimum number of consecutive failed components that cause system failure
$p(t)$	reliability of component at time t
$\Gamma(x)$	Gamma function
c, w	as a subscript, implies the circular, Weibull case

Formulas

$$\mu = \sum_{m=0}^{(k+2)n} (-1)^m \int_0^{\infty} p^m(t) dt \sum_{i=m-n}^m \left[\binom{n-ik}{i} \binom{ik}{m-i} - \binom{m-ik-k}{i} \binom{(i+1)k}{m-1} \right]$$

$$\mu_w = a\Gamma\left(\frac{b+1}{b}\right) \sum_{m=0}^{(k+2)n} (-1)^m m^{-1/b} \sum_{i=m-n}^m$$

$$\left[\binom{n-ik}{i} \binom{ik}{m-i} - \binom{n-ik-k}{i} \binom{ik+k}{m-i} \right]$$

$$\mu_{cw} = a\Gamma\left(\frac{b+1}{b}\right) \left[\sum_{m=0}^{(k+2)n} (-1)^m \sum_{i=m-n}^m$$

$$\left. \cdot \left\{ m^{-1/b} \binom{n-ik}{i} \binom{ik}{m-i} - k(m+1)^{-1/b} \right. \right. \\ \left. \left. \cdot \left(\binom{n-ik-k-1}{i} \binom{ik+k}{m-i} \right) \right\} - \sum_{i=0}^n \binom{n}{i} (-1)^i i^{-1/b} \right].$$

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