An $O(kn)$ Algorithm for a Circular Consecutive-k-out-of-n:F System

Jer-Shyan Wu
National Chiao Tung University, Hsinchu
Rong-Jaye Chen, Member IEEE
National Chiao Tung University, Hsinchu

Key Words — **Circular consecutive-k-out-of-n:F system, System reliability, Algorithm.**

Reader Aids —
Purpose: Report a new algorithm
Special math needed for explanations: Probability theory
Special math needed to use results: None
Result useful to: Reliability analysts and theoreticians

Abstract — **An $O(k\cdot n)$ algorithm is described for evaluating the reliability of a circular consecutive-k-out-of-n:F system.**

1. INTRODUCTION

A consecutive-k-out-of-n:F system is n ordered components such that the system fails if and only if at least k consecutive components fail. Such a system is relevant to telecommunication. There are two topologies for this system: a straight line and a circle. The reliability of this system was first studied by Chiang & Niu [2], and later by Derman, Lieberman, Ross [3], Hwang [4], Shanthikumar [5], and Antonopoulou & Papastavridis [1].

Hwang [4] proposed two recursive algorithms to evaluate the reliabilities of linear and circular consecutive-k-out-of-n:F systems. These two algorithms require $O(n)$ and $O(n\cdot k^2)$ computing time respectively. Antonopoulou & Papastavridis [1] announced an $O(n\cdot k)$ recursive algorithm for computing the reliability of a circular such system. This paper demonstrates an algorithm (Sys) to evaluate the reliability of a circular system which needs only $O(n\cdot k)$ computing time.

2. MODEL

Assumptions

1. Each component, subsystem and system either functions or fails.
2. All n component states are mutually s-independent.
3. Components $1, 2, \ldots, n$ are arranged on a circle in that order.
4. The system or subsystem fails if and only if at least k consecutive components all fail. □

Notation

n	number of components in a system
k	minimum number of consecutive failed components which causes system failure
i	index for a component; $i = 1, 2, \ldots, n$
p_i, q_i	probability that component i functions, fails; $p_i + q_i \equiv 1$
sys	circular consecutive-k-out-of-n:F system
sys-0	linear consecutive-k-out-of-n:F system
sys-i	linear consecutive-k-out-of-$(n+i)$:F systems, for $i = 1, 2, \ldots, k-1$; $q_j = 1$ for $j = 1, 2, \ldots, i, n+1, \ldots, n+i$
F_{sys}	Pr{sys is failed}
$F_{\text{sys-}i}$	Pr{sys-i is failed}, for $i = 0, 1, \ldots, k-1$
$F'_{\text{sys-}i}$	Pr{sys-i is failed}, for $i = 1, \ldots, k-1$, wherein the first $(n+i-1)$ components are considered
S_i, T_i	event that consecutive k components starting with component i all fail in a linear, circular system

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

3. COMPUTATION OF RELIABILITY

We express Pr{sys is failed} by using the Pr{sys-i is failed} formulas. Thus, (3-1) is Sys-algorithm.

$$F_{\text{sys}} = F_{\text{sys-0}} + q_1\cdot(F_{\text{sys-1}} - F'_{\text{sys-1}}) + q_1\cdot q_2\cdot(F_{\text{sys-2}} - F'_{\text{sys-2}}) + \cdots + q_1\cdot q_2\cdot\ldots\cdot q_{k-1}\cdot(F_{\text{sys-}(k-1)} - F'_{\text{sys-}(k-1)}). \quad (3\text{-}1)$$

Hwang [4] proved that the time complexity of computing the reliability of a linear consecutive-k-out-of-n:F system is $O(n)$. Because $n > k$, the computing time of each $F_{\text{sys-i}}$ or $F'_{\text{sys-}i}$ is $O(n+k) = O(n)$, for $i = 0, 1, \ldots, k-1$. Furthermore, the time complexity for calculating $q_1, q_1\cdot q_2, \ldots, q_1\cdot q_2\cdot\ldots\cdot q_{k-1}$ is $O(k)$. So the time complexity for (3-1) in the Sys-algorithm is $O(n) + O(k\cdot n) + O(k) = O(k\cdot n)$. Intuitively, the time complexity for the formula in [1] is $O(n^2\cdot k)$.

4. PROOF

4.1 *Lemma*

Before proving Sys-algorithm we need the lemma.

Lemma: In sys-i, for $i = 1, \ldots, k-1$:

$$\Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \ldots \cap \overline{S_{i-k}} \cap S_{i-k+1}\} = F_{\text{sys-}i} - F'_{\text{sys-}i}. \quad (4\text{-}1)$$

Proof: By the sum-of-disjoint method, the failed probability of a linear sys-i, for $i = 1, \ldots, k-1$, is:

$$F_{\text{sys-}i} = \Pr\{S_1 \cup S_2 \cup \ldots \cup S_{i-k+1}\}$$

$$= \Pr\{S_1\} + \Pr\{\bar{S}_1 \cap S_2\} + \ldots$$

$$+ \Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \ldots \cap \overline{S_{i-k-1}} \cap S_{i-k}\}$$

$$+ \Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \ldots \cap \overline{S_{i-k}} \cap S_{i-k+1}\}. \quad (4\text{-}2)$$

In the subsystem of the linear sys-i wherein the first $n+i-1$ components are considered. Similarly, the failed probability of this subsystem is

$$F'_{\text{sys-}i} = \Pr\{S_1 \cup S_2 \cup \ldots \cup S_{i-k}\}$$

$$= \Pr\{S_1\} + \Pr\{\bar{S}_1 \cap S_2\} + \ldots$$

$$+ \Pr\{\bar{S}_1 \cap \bar{S}_2 \cap \ldots \cap \overline{S_{i-k-1}} \cap S_{i-k}\}. \quad (4\text{-}3)$$

Eq (4-1) is obtained by subtracting (4-3) from (4-2). *Q.E.D.*

4.2 *Sys-Algorithm*

In the circular system,

$$F_{\text{sys}} = F_{\text{sys-0}} + q_1[F_{\text{sys-1}} - F'_{\text{sys-1}}]$$

$$+ q_1 q_2[F_{\text{sys-2}} - F'_{\text{sys-2}}]$$

$$+ \ldots + q_1 q_2 \ldots q_{k-1}[F_{\text{sys-}(k-1)} - F'_{\text{sys-}(k-1)}]. \quad (4\text{-}4)$$

Proof: By the sum-of-disjoint method, the failed probability of the circular system is:

$$F_{\text{sys}} = \Pr\{T_1 \cup T_2 \cup \ldots \cup T_n\}$$

$$= \Pr\{T_1\} + \Pr\{\bar{T}_1 \cap T_2\} + \ldots$$

$$+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k}} \cap T_{n-k+1}\}$$

$$+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+1}} \cap T_{n-k+2}\}$$

$$+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+2}} \cap T_{n-k+3}\}$$

$$+ \ldots + \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-1}} \cap T_n\}. \quad (4\text{-}5)$$

By (4-2), we can express the failed probability of sys-0 as:

$$F_{\text{sys-0}} = \Pr\{T_1\} + \Pr\{\bar{T}_1 \cap T_2\} + \ldots$$

$$+ \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k}} \cap T_{n-k+1}\}. \quad (4\text{-}6)$$

Now we consider,

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \bar{T}_{n-k+1} \cap T_{n-k+2}\}$$

in (4-5).

Notation

E event: $\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \bar{T}_{n-k+1} \cap T_{n-k+2}$

F event: component 1 fails. □

By applying —

$$\Pr\{E\} = \Pr\{EF\} + \Pr\{E\bar{F}\} = \Pr\{E/F\} \Pr\{F\}$$

$$+ \Pr\{E/\bar{F}\} \Pr\{\bar{F}\}. \quad (4\text{-}7)$$

We have —

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+1}} \cap T_{n-k+2}\}$$

$$= \Pr\{1 \text{ fails}\} \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+1}} \cap T_{n-k+2} | 1 \text{ fails}\}$$

$$+ \Pr\{1 \text{ functions}\} \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+1}}$$

$$\cap T_{n-k+2} | 1 \text{ functions}\}$$

$$= \Pr\{1 \text{ fails}\} \Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+1}}$$

$$\cap T_{n-k+2} | 1 \text{ fails}\} + 0$$

$$= q_1(F_{\text{sys-1}} - F'_{\text{sys-1}}). \quad (4\text{-}8)$$

In (4-8),

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \bar{T}_{n-k+1} \cap T_{n-k+2} | \text{component-1 functions}\}$$

$$= 0,$$

because T_{n-k+2} is the event that all the components from $(n-k+2)$ to n and component-1 all fail, which is against the condition that component-1 functions. And

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \bar{T}_{n-k+1} \cap T_{n-k+2} | \text{component-1 fails}\}$$

$$= F_{\text{sys-1}} - F'_{\text{sys-1}},$$

due to the lemma (4-1).

Notation

E event: $\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \bar{T}_{n-k+2} \cap T_{n-k+3}$

F event: components 1 & 2 both fail □

By (4-7) we have:

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+2}} \cap T_{n-k+3}\}$$

$$= q_1 q_2 (F_{\text{sys-2}} - F'_{\text{sys-2}}). \quad (4\text{-}9)$$

By (4-8) & (4-9) & the lemma, we get the general formula for other terms in (4-4):

$$\Pr\{\bar{T}_1 \cap \bar{T}_2 \cap \ldots \cap \overline{T_{n-k+i}} \cap T_{n-k+1+i}\}$$

$$= q_1 q_2 \ldots q_i (F_{\text{sys-}i} - F'_{\text{sys-}i}), \text{ for } i=1,2,\ldots,k-1. \quad (4\text{-}10)$$

Apply (4-5) & (4-10). *Q.E.D*

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AUTHORS

Jer-Shyan Wu; Department of Computer Science and Information Engineering; National Chiao-Tung University; Hsinchu, TAIWAN ROC.

Jer-Shyan Wu was born in Taipei, Taiwan in 1967. He received his BS (1989) in Computer Science from National Taiwan University, and his MS (1991) in Computer Science from National Chiao-Tung University. His research interests include reliability theory, queuing theory, and algorithms.

Dr. Rong-Jaye Chen; Department of Computer Science and Information Engineering; National Chiao-Tung University; Hsinchu, TAIWAN ROC.

Rong-Jaye Chen (M'90) was born in Taiwan in 1952. He received his BS (1977) in Mathematics from National Tsing-Hua University, and his PhD (1987) in Computer Science from University of Wisconsin - Madison. Dr. Chen is Associate Professor in the Department of Computer Science and Information Engineering in National Chiao-Tung University. He is a member of IEEE. His research interests include reliability theory, algorithms, mathematical programming, and computer networking.

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Mean Time to Failure for a Consecutive-k-out-of-n:F System

Masafumi Sasaki, Senior Member IEEE
National Defense Academy
Yukari Nakai
National Defense Academy
Tetsushi Yuge
National Defense Academy

Formulas for μ, μ_c, μ_w in [1], contain some typographical errors. They must be the following formulas.

Notation

n	number of components in the system
k	minimum number of consecutive failed components that cause system failure
$p(t)$	reliability of component at time t
$\Gamma(x)$	Gamma function
c,w	as a subscript, implies the circular, Weibull case

Formulas

$$\mu = \sum_{m=0}^{(k+2)n} (-1)^m \int_0^\infty p^m(t)\, dt \sum_{i=m-n}^{m}$$

$$\cdot \left[\binom{n-ik}{i} \binom{ik}{m-i} - \binom{m-ik-k}{i} \binom{(i+1)k}{m-1} \right]$$

$$\mu_w = a\Gamma\left(\frac{b+1}{b}\right) \sum_{m=0}^{(k+2)n} (-1)^m m^{-1/b} \sum_{i=m-n}^{m}$$

$$\cdot \left[\binom{n-ik}{i} \binom{ik}{m-i} - \binom{n-ik-k}{i} \binom{ik+k}{m-i} \right]$$

$$\mu_{cw} = a\Gamma\left(\frac{b+1}{b}\right) \left[\sum_{m=0}^{(k+2)n} (-1)^m \sum_{i=m-n}^{m} \right.$$

$$\cdot \left\{ m^{-1/b} \binom{n-ik}{i} \binom{ik}{m-i} - k(m+1)^{-1/b} \right.$$

$$\left. \cdot \binom{n-ik-k-1}{i} \binom{ik+k}{m-i} \right\} - \left. \sum_{i=0}^{n} \binom{n}{i} (-1)^i i^{-1/b} \right].$$

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AUTHORS

Dr. Mas. Sasaki; Dept. of Electrical Engineering; National Defense Academy; Yokosuka 239 JAPAN.

Masafumi Sasaki (M'73, SM'84) was born in Japan in 1929. He received his DSc in 1959 in Mathematics from Hokkaido University. He is a professor of Operations Research and System Engineering at National Defense Academy. He has published numerous papers in reliability engineering. He has written several books in this field which were published in Japan. He is a member of the Organization Committee of the R & M Symposium in Japan since 1977, and is the chair'n of its referee committee for manuscripts since 1990. He was the chair'n of the Tokyo Chapter of the Reliability Society, IEEE for two years beginning in 1987. In 1988, he acted as the chair'n of the Reliability Research Commission of the Institute of Electronics, Information and Communication of Japan. From 1981 to 1988 he served as a member of the Program Committee of the Int'l Conf. on R&M held in France.

T. Yuge; Dept. of Electrical Engineering; National Defense Academy, Yokosuka 239 JAPAN.

Tetsushi Yuge was born in Japan on 1965 October 14. He received the BS & ME from Hokkaido University, Sapporo in 1989 and 1991. He is working at National Defense Academy since 1991 and has engaged in research in the reliability theory and system engineering. He is a research associate of the Department of Electrical Engineering. He is a member of the Operations Research Society of Japan and the Reliability Engineering Association of Japan.

Y. Nakai; c/o Prof. Mas. Sasaki.

Yukari Nakai was born in Japan on 1962 March 7. She received the BA from Tezukayama University, Nara in 1985. Now she is studying operations research in a postgraduate course at National Defense Academy of Japan. She is a Lieutenant Junior in the Japan Maritime Self Defense Force.

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