# Which Linear Transformations Have Isomorphic Hyperinvariant Subspace Lattices?

Pei Yuan Wu*
*Department of Applied Mathematics*
*National Chiao Tung University*
*Hsinchu, Taiwan, Republic of China*

Submitted by Chandler Davis

ABSTRACT

Let $A$ be a linear transformation on a finite-dimensional complex vector space with the associated algebra $\operatorname{Alg} A$, commutant $\{A\}'$, and hyperinvariant subspace lattice $\operatorname{Hyperlat} A$. We determine $\operatorname{Alg} A$, $\{A\}'$ (up to algebra isomorphism), and $\operatorname{Hyperlat} A$ (up to lattice isomorphism) in terms of the parameters in the Jordan form of $A$.

## 1. INTRODUCTION

Let $A$ be a linear transformation on a finite-dimensional complex vector space $V$. There are two lattices naturally associated with $A$: the *invariant subspace lattice* $\operatorname{Lat} A$, consisting of those subspaces of $V$ that are invariant for $A$, and the *hyperinvariant subspace lattice* $\operatorname{Hyperlat} A$, consisting of subspaces that are invariant for any linear transformation commuting with $A$. An interesting question in this respect is: to what extent do these lattices determine $A$? For the invariant subspace lattice, the answer has been known: if $A$ and $B$ are linear transformations on finite-dimensional spaces, then $\operatorname{Lat} A$ is isomorphic to $\operatorname{Lat} B$ if and only if $A$ and $B$ have the same Jordan structure, and in this case the (lattice) isomorphism can be chosen to be implemented by an invertible transformation (see [9, Corollary 2.3.1], [5, Theorem 2.1], or [4, Theorem 16.1.2]). In this paper, we undertake a

*Research partially supported by the National Science Council of the Republic of China.




655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

corresponding study for the hyperinvariant subspace lattice. We completely determine when two linear transformations have isomorphic hyperinvariant subspace lattices in terms of the parameters in their Jordan forms. The situation is more delicate than the invariant-subspace one. This is to be expected, since for a linear transformation the number of hyperinvariant subspaces is much smaller than that of invariant subspaces: the former is always finite, and the latter uncountable in general (cf. [3] and [11]). We are able to show that, except in certain special cases, the Jordan structure of $A$ is more or less determined by Hyperlat $A$. We present the precise statement in Section 3 below.

Given any linear transformation $A$, we can associate two algebras with $A$: Alg $A$, the algebra of all polynomials in $A$, and $\{A\}'$, the *commutant* of $A$, consisting of all linear transformations commuting with $A$. To warm up for later developments, we answer in Section 2 the questions when two linear transformations have (algebraically) isomorphic or equal such algebras, and relate these to the isomorphism of their invariant subspace lattices.

In the following, $\sigma(A)$ denotes the set of eigenvalues of $A$. Two linear transformations $A$ and $B$ are *similar* ($A \approx B$) if there exists an invertible transformation $X$ such that $XA = BX$. For any $n \geqslant 1$ and complex number $a$, let

$$J_n(a) = \begin{bmatrix} a & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & a \end{bmatrix}$$

denote the $n \times n$ *Jordan cell* with eigenvalue $a$. Any linear transformation $A$ is similar to a unique *Jordan form*

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} \oplus J_{n_{ij}}(a_i),$$

where $a_i$ are the (distinct) eigenvalues of $A$ and, for each $i$, $n_{i1} \geqslant \cdots \geqslant n_{ir_i}$ are the *partial multiplicities* of $A$ at $a_i$. In this case, $n_{i1}$ is the *ascent* of $A$ at $a_i$ and $\prod_{i=1}^{n}(\lambda - a_i)^{n_{i1}}$ is the *minimal polynomial* of $A$. The *reduced partial multiplicities* of $A$ at $a_i$ are obtained from $n_{i1} \geqslant \cdots \geqslant n_{ir_i}$ by retaining only the distinct ones. $A$ is *cyclic* if there is a vector $x$ in $V$ such that the linear span of $x, Ax, A^2x, \ldots$ is $V$. Every linear transformation is similar to a unique *rational form*, which is, in particular, a direct sum of cyclic transformations. Our reference for linear algebra is [4].

## 2. INVARIANT SUBSPACE LATTICE

Let $A$ and $B$ be linear transformations on (possibly different) finite-dimensional vector spaces $V$ and $W$. We start by considering the equality of their associated algebras.

THEOREM 2.1. *The following statements are equivalent for A and B*:

(1) $\operatorname{Alg} A = \operatorname{Alg} B$;

(2) $\{A\}' = \{B\}'$;

(3) $A = p(B)$ *and* $B = q(A)$ *for some polynomials* $p$ *and* $q$;

(4) $\operatorname{Lat} A = \operatorname{Lat} B$, *and* $A = p(B)$ *for some polynomial* $p$;

(5) $A = p(B)$ *for some polynomial* $p$ *which defines a one-to-one mapping from* $\sigma(B)$ *onto* $\sigma(A)$ *and is such that* $p'(b) \neq 0$ *for any* $b$ *in* $\sigma(B)$ *with ascent greater than one*.

Recall that if $\mathscr{A}$ is any set of linear transformations, $\mathscr{A}'$ denotes its *commutant*, that is, the algebra of linear transformations which commute with every transformation in $\mathscr{A}$. It is well known that $\{A\}''$, the commutant of $\{A\}'$, is always equal to $\operatorname{Alg} A$ for any $A$.

*Proof*. (1) $\Rightarrow$ (2): $\{A\}' = (\operatorname{Alg} A)' = (\operatorname{Alg} B)' = \{B\}'$.

(2) $\Rightarrow$ (3): Since $\operatorname{Alg} A = \{A\}'' = \{B\}'' = \operatorname{Alg} B$, (3) follows immediately.

(3) $\Rightarrow$ (4): This is trivial.

(4) $\Rightarrow$ (1): That $\operatorname{Alg} A \subseteq \operatorname{Alg} B$ is trivial. To prove the converse, let $T \in \operatorname{Alg} B$. Then $TA = AT$, and $T$ leaves invariant every invariant subspace of $B$, whence every invariant subspace of $A$. We infer from [1, Theorem 10] that $T \in \operatorname{Alg} A$.

(3) $\Rightarrow$ (5): Note that the polynomial $q \circ p$ defines a mapping from $\sigma(B)$ onto $(q \circ p)(\sigma(B)) = q(p(\sigma(B))) = q(\sigma(p(B))) = q(\sigma(A)) = \sigma(q(A)) = \sigma(B)$. Since $\sigma(B)$ is a finite set, the mapping must be one-to-one. Hence $p$ defines a one-to-one mapping from $\sigma(B)$ onto $\sigma(A)$.

Assume that $p'(b) = 0$ for some $b$ in $\sigma(B)$ with ascent $m$ greater than one. Let $J = \sum_{i=1}^{n} \oplus J_i$ be the Jordan form of $B$, where the $J_i$'s are Jordan cells arranged so that

$$J_1 = \begin{bmatrix} b & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & b \end{bmatrix}$$

is one associated with $b$ with size $m$. Then $p(B)$ is similar to $p(J)=\sum_{i=1}^{n}\oplus p(J_i)$ with

$$p(J_1)=\begin{bmatrix} p(b) & 0 & \frac{1}{2!}p''(b) & \cdot & \cdot & \cdot & \frac{1}{(m-1)!}p^{(m-1)}(b) \\ & \cdot & \cdot & \cdot & & & \vdots \\ & & \cdot & \cdot & \cdot & & \vdots \\ & & & \cdot & \cdot & \cdot & \vdots \\ & & & & \cdot & \cdot & \frac{1}{2!}p''(b) \\ & & & & & \cdot & 0 \\ 0 & & & & & & p(b) \end{bmatrix}.$$

It is easily seen that $[p(J_1)-p(b)I]^l=0$ for some $l<m$, whence $p(J_1)$ is not cyclic. This, together with the fact that $p$ is one-to-one from $\sigma(B)$ onto $\sigma(A)$, implies that, in the Jordan form of $p(B)$, the maximal size of the Jordan cells associated with $p(b)$ is strictly less than $m$. Hence the same holds for $q(p(B))=B$ [with $p(b)$ replaced by $q(p(b))$], that is, the ascent of $B$ at $(q\circ p)(b)$ is strictly less than $m$. Since $(p\circ q\circ p)'(b)=0$, we may repeat the above arguments to obtain the same assertion for $(q\circ p\circ q\circ p)(b)$ or, more generally, for

$$(q\circ p)^{(n)}(b)\equiv(\underbrace{(q\circ p)\circ\cdots\circ(q\circ p)}_{n})(b)\qquad\text{for any}\quad n\geqslant 1.$$

It is well known that, on a finite set, a one-to-one mapping will map any element eventually back to itself. Thus $b=(q\circ p)^{(n)}(b)$ for some $n$, and we obtain a contradiction to our assertion on the ascent of $b$.

(5) ⇒ (3): Let $b_1,\ldots,b_n$ be the (distinct) eigenvalues of $B$, and let $a_i=p(b_i)$, $i=1,\ldots,n$. Since $A=p(B)$ and $p$ is one-to-one from $\sigma(B)$ onto $\sigma(A)$, the $a_i$'s are all the (distinct) eigenvalues of $A$. For each $i$, let $n_i$ be the ascent of $B$ at $b_i$. We need to construct a polynomial $q$ such that $q(A)=B$ or, equivalently, $q(p(B))=B$. This will be the case if, for all $i$, $(q\circ p)(b_i)=b_i$, $(q\circ p)'(b_i)=1$, and $(q\circ p)^{(j)}(b_i)=0$ for all $j$, $2\leqslant j\leqslant n_i-1$ (cf. [7, p. 305, Proposition 1]). A simple computation shows that these latter conditions are equivalent to the system of equations $q(a_i)=b_i$, $q'(a_i)=1/p'(b_i)$, and

$$q^{(j)}(a_i)=\frac{1}{p'(b_i)}L_j\big(q^{(j-1)}(a_i),\ldots,q'(a_i)\big),$$

$2 \leqslant j \leqslant n_i - 1$, where $L_j$ denotes some linear combination of its arguments. [Note that, by our assumption, $p'(b_i) \neq 0$ for any $i$ with $n_i > 1$.] This system has a polynomial solution $q$, namely, the so-called Hermite interpolating polynomial (cf. [7, p. 306, Proposition 1]). Hence we would have $q(A) = B$ as required. ■

Before passing on, a few remarks are in order. In the preceding theorem, the equivalence of (4) and (5) has been obtained before [4, Theorem 2.11.3]. However, our proof is completely different: it establishes their equivalence via (3). Also note that the proof for the equivalence of (3) and (5) remains valid with the equality signs "$=$" in $A = p(B)$ and $B = q(A)$ replaced by the similarity signs "$\approx$." This observation will be needed in proving Theorem 2.2 below. Mention should be made that, unfortunately, none of the results in [12] is exactly correct: Lemma 2.1 there should exclude $A = 0$; the conditions in Corollary 2.2 and Theorem 3.1 are neither sufficient nor necessary (they can be modified as here or as in [4, Theorem 2.11.3]).

We next give conditions under which $A$ and $B$ have isomorphic invariant subspace lattices. Recall that two lattices $L_1$ and $L_2$ are *isomorphic* ($L_1 \cong L_2$) if there is a one-to-one mapping from $L_1$ onto $L_2$ which preserves the lattice operations "join" and "meet." $A$ and $B$ are said to have the *same Jordan structure* if they have the same number of (distinct) eigenvalues, which can be ordered, say, $\sigma(A) = \{a_1, \dots, a_n\}$ and $\sigma(B) = \{b_1, \dots, b_n\}$, so that the partial multiplicities of $A$ at $a_i$ coincide with those of $B$ at $b_i$ for all $i$, $i = 1, \dots, n$ (cf. [4, p. 482]).

THEOREM 2.2. *The following statements are equivalent for $A$ and $B$:*

(1) Lat $A$ *and* Lat $B$ *are isomorphic*;

(2) *$A$ and $B$ have the same Jordan structure*;

(3) *there exists an invertible transformation $X$ such that* $\operatorname{Lat} B = \{XK : K \in \operatorname{Lat} A\}$;

(4) $A \approx p(B)$ *and* $B \approx q(A)$ *for some polynomials $p$ and $q$*;

(5) $A \approx p(B)$ *for some polynomial $p$ which defines a one-to-one mapping from $\sigma(B)$ onto $\sigma(A)$ and for which $p'(b) \neq 0$ for any $b$ in $\sigma(B)$ with ascent greater than one.*

*Proof.* As noted before, the equivalence of (1), (2), and (3) is known (cf. [4, Theorem 16.1.2]). The proof for the equivalence of (4) and (5) follows the same line of argument as that for (3) and (5) of Theorem 2.1. That (5) implies (1) is a consequence of [4, Theorem 2.11.3]. Hence to complete the proof, we need only show that (2) implies (4).

Let $\sigma(A)=\{a_1,\ldots,a_n\}$ and $\sigma(B)=\{b_1,\ldots,b_n\}$ consist of (distinct) eigenvalues of $A$ and $B$, respectively, We may assume that $A$ and $B$ have the same Jordan structure at $a_i$ and $b_i$ for all $i$. Let $p$ be a polynomial satisfying $p(b_i)=a_i$ and $p'(b_i)\neq 0$ for all $i$. For any Jordan cell

$$J=\begin{bmatrix} b_i & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & b_i \end{bmatrix},$$

we have

$$p(J)=\begin{bmatrix} a_i & p'(b_i) & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & p'(b_i) \\ 0 & & & & a_i \end{bmatrix},$$

which is similar to

$$\begin{bmatrix} a_i & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & a_i \end{bmatrix},$$

since $p'(b_i)\neq 0$. Using the assumption that $A$ and $B$ have the same Jordan structure, we infer that $A\approx p(B)$. In a similar fashion, $B\approx q(A)$ for some polynomial $q$. ■

Ong proved in [10] that $\operatorname{Lat} A=\operatorname{Lat} B$ implies that $A\approx p(B)$ and $B\approx q(A)$ for some polynomials $p$ and $q$. The equivalence of (1) and (4) in Theorem 2.2 strengthens this.

We conclude this section by considering conditions under which $A$ and $B$ have isomorphic associated algebras. We start with the following definitions: $\operatorname{Alg} A$ and $\operatorname{Alg} B$ are *isomorphic* if there is an algebra isomorphism $\alpha$ from $\operatorname{Alg} A$ onto $\operatorname{Alg} B$ which maps $I$ to $I$; $(\operatorname{Alg} A, A)$ and $(\operatorname{Alg} B, B)$ are *isomorphic* if, furthermore, $\alpha$ maps $A$ to $B$ (this terminology was suggested by N. C. Phillips). Similar definitions can be made for $\{A\}'$ and $\{B\}'$. The next two lemmas deal with these latter, more restricted isomorphisms.

LEMMA 2.3. *$(\mathrm{Alg}\, A, A)$ and $(\mathrm{Alg}\, B, B)$ are isomorphic if and only if the minimal polynomials of $A$ and $B$ coincide.*

*Proof.* It is easily seen that $(\mathrm{Alg}\, A, A)$ and $(\mathrm{Alg}\, B, B)$ being isomorphic is equivalent to the condition that, for any polynomial $p$, $p(A)=0$ if and only if $p(B)=0$. This latter condition is, in turn, equivalent to the equality of the minimal polynomials of $A$ and $B$. ∎

LEMMA 2.4. *$(\{A\}', A)$ and $(\{B\}', B)$ are isomorphic if and only if $A$ and $B$ are similar.*

*Proof.* One direction is trivial: if $A$ and $B$ are similar via the invertible $X$, then $\alpha(T)=XTX^{-1}$ for $T\in\{A\}'$ defines an algebra isomorphism from $\{A\}'$ onto $\{B\}'$ satisfying $\alpha(A)=B$ and $\alpha(I)=I$.

To prove the converse, let $\alpha:\{A\}'\to\{B\}'$ be an algebra isomorphism such that $\alpha(A)=B$ and $\alpha(I)=I$. We may assume, by the above observation, that $A$ is in rational form: $A=\sum_{i=1}^{n}\oplus A_i$ on $V=\sum_{i=1}^{n}\oplus V_i$, where each $A_i$ is a cyclic transformation. Let

$$P_i = 0\oplus\cdots\oplus 0\oplus \underset{i\text{th}}{I}\oplus 0\oplus\cdots\oplus 0,\qquad i=1,\dots,n.$$

Since the $P_i$'s are in $\{A\}'$ and satisfy $P_iP_j=\delta_{ij}P_i$ and $\sum_i P_i = I$ ($\delta_{ij}$ being the Kronecker delta), the $\alpha(P_i)$'s will belong to $\{B\}'$ and satisfy analogous conditions. Hence, letting $Q_i=\alpha(P_i)$ and noting that $B=B_1\dot{+}\cdots\dot{+}B_n$, where $B_i=B|Q_iW$, we may further assume that $B=\sum_{i=1}^{n}\oplus B_i$ on $W=\sum_{i=1}^{n}\oplus Q_iW$.

Next we show that, for each $i$, $\alpha$ induces an algebra isomorphism from $\{A_i\}'$ onto $\{B_i\}'$. Indeed, for any $C\in\{A_i\}'$, letting

$$\hat{C} = 0\oplus\cdots\oplus 0\oplus \underset{i\text{th}}{C}\oplus 0\oplus\cdots\oplus 0,$$

we have $\hat{C}\in\{A\}'$ and, for any $j,k=1,\dots,n$,

$$P_j\hat{C}P_k=\begin{cases} C & \text{if } j=k=i,\\ 0 & \text{otherwise.}\end{cases}$$

Thus $\hat{D} \equiv \alpha(\hat{C})$ is in $\{B\}'$, and $Q_j \hat{D} Q_k = 0$ for any $j, k$, $(j,k) \neq (i,i)$. Therefore, $\hat{D}$ is of the form

$$0 \oplus \cdots \oplus 0 \oplus \underset{i\text{th}}{D} \oplus 0 \oplus \cdots \oplus 0$$

on $W$. It is easily seen that the induced mapping $\alpha_i(C) = D$ for $C \in \{A_i\}'$ is an algebra isomorphism from $\{A_i\}'$ onto $\{B_i\}'$. Note that a linear transformation $T$ is cyclic if and only if $\{T\}' = \operatorname{Alg} T$. Hence the cyclicity of $A_i$ implies that of $B_i$, and thus $(\operatorname{Alg} A_i, A_i)$ is isomorphic to $(\operatorname{Alg} B_i, B_i)$. We infer from Lemma 2.3 that the minimal polynomials of $A_i$ and $B_i$ coincide. For cyclic transformations, this is equivalent to their similarity. We conclude that $A$ and $B$ are similar. ■

Finally, we are ready for our promised conditions.

THEOREM 2.5.

(1) $\operatorname{Alg} A$ *and* $\operatorname{Alg} B$ *are isomorphic if and only if* $A$ *and* $B$ *have the same number of (distinct) eigenvalues, which can be ordered, say,* $\sigma(A) = \{a_1, \ldots, a_n\}$ *and* $\sigma(B) = \{b_1, \ldots, b_n\}$, *so that for each* $i$ *the ascent of* $A$ *at* $a_i$ *and that of* $B$ *at* $b_i$ *are equal.*

(2) $\{A\}'$ *and* $\{B\}'$ *are isomorphic if and only if* $\operatorname{Lat} A$ *and* $\operatorname{Lat} B$ *are isomorphic.*

*Proof.* (1): If $\operatorname{Alg} A$ and $\operatorname{Alg} B$ are isomorphic, then there exists a linear transformation $C$ in $\operatorname{Alg} B$ such that $(\operatorname{Alg} A, A)$ and $(\operatorname{Alg} C, C)$ are isomorphic and $\operatorname{Alg} C = \operatorname{Alg} B$. The former implies that the minimal polynomials of $A$ and $C$ coincide by Lemma 2.3, and the latter that $C$ and $B$ have the same Jordan structure by Theorems 2.1 and 2.2. Our condition then follows immediately.

Conversely, assume that the condition holds. Let $p$ be a polynomial which defines a one-to-one mapping from $\sigma(B)$ onto $\sigma(A)$ and satisfies $p'(b) \neq 0$ for any $b$ in $\sigma(B)$ with ascent greater than one, and let $C = p(B)$. Theorem 2.2 implies that $B$ and $C$ have the same Jordan structure. Therefore, by our assumption and the fact that $\sigma(C) = \sigma(A)$, the minimal polynomials of $A$ and $C$ coincide. Hence $(\operatorname{Alg} A, A)$ and $(\operatorname{Alg} C, C)$ are isomorphic by Lemma 2.3. This, together with $\operatorname{Alg} B = \operatorname{Alg} C$, implies that $\operatorname{Alg} A$ and $\operatorname{Alg} B$ are isomorphic.

(2): If $\{A\}'$ and $\{B\}'$ are isomorphic then there exists a linear transformation $C$ in $\{B\}'$ such that $(\{A\}', A)$ and $(\{C\}', C)$ are isomorphic and $\{C\}' = \{B\}'$.

The former implies that $A$ and $C$ are similar by Lemma 2.4, and the latter that Lat $C$ = Lat $B$ by Theorem 2.1. Hence Lat $A$ is isomorphic to Lat $B$.

Conversely, if Lat $A$ and Lat $B$ are isomorphic, let $p$ be a polynomial as in Theorem 2.2(5) such that $A \approx p(B)$. Letting $C = p(B)$, we infer from Lemma 2.4 that $(\{A\}', A)$ is isomorphic to $(\{C\}', C)$. On the other hand, $\{C\}' = \{B\}'$ by Theorem 2.1. Hence $\{A\}'$ is isomorphic to $\{B\}'$. ∎

## 3. HYPERINVARIANT SUBSPACE LATTICE

In this section, we come to the major theme of this paper: characterizing linear transformations with isomorphic hyperinvariant subspace lattices. The next theorem is our main result.

THEOREM 3.1. *Assume that $A$ [respectively, $B$] has (distinct) eigenvalues $a_1, \dots, a_n$ [$b_1, \dots, b_m$] with reduced partial multiplicities $n_{i1} > \cdots > n_{ir_i}$, $i = 1, \dots, n$ [$m_{j1} > \cdots > m_{js_j}$, $j = 1, \dots, m$]. Then* Hyperlat $A$ *is isomorphic to* Hyperlat $B$ *if and only if $n = m$ and, after a reordering, the eigenvalues $a_i$ and $b_i$ are matched so that, for each $i$, their reduced partial multiplicities $n_{i1} > \cdots > n_{ir_i}$ and $m_{i1} > \cdots > m_{is_i}$ satisfy one of the following conditions:*

(1) *they coincide with $5 > 2$ and $4 > 2 > 1$;*
(2) *they coincide with $l > l - 1$ and $2l - 1$ for some $l \geqslant 2$;*
(3) *$r_i = s_i$ and $n_{ik} = m_{ik}$ for all $k$.*

The theorem says that, up to the ordering of the eigenvalues and except for some special cases, Hyperlat $A$ determines the reduced partial multiplicities of the eigenvalues of A. (The appearance of such exceptional cases can best be explained by Richard Guy's strong law of small numbers [6]: there aren't enough small numbers to meet the many demands made on them.) Thus the dependence of $A$ on Hyperlat $A$ is looser than that of $A$ on Lat $A$, as should be the case. In particular, we have the following result of Longstaff [8]:

COROLLARY 3.2. *If* Lat $A$ *is isomorphic to* Lat $B$, *then* Hyperlat $A$ *is isomorphic to* Hyperlat $B$.

For normal transformations, the conditions for isomorphic lattices are much easier to state.

COROLLARY 3.3. *Let $A$ and $B$ be normal transformations. Then*

(1) Lat $A$ *is isomorphic to* Lat $B$ *if and only if the eigenvalues of $A$ and $B$ are the same (including multiplicities), and*

(2) Hyperlat $A$ *is isomorphic to* Hyperlat $B$ *if and only if $A$ and $B$ have the same number of (distinct) eigenvalues.*

Before embarking on the proof of Theorem 3.1, we draw the reader's attention to an error in [4, p. 338, Exercise 10.3]. If

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\text{ and }B=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\qquad\text{on}\quad\mathbb{C}^3,$$

then, since $A=J_3(0)$ and $B\approx J_2(0)\oplus J_1(0)$, they are not similar to each other. But it is easily seen that Hyperlat $A=$ Hyperlat $B$ consists of the four subspaces $\mathbb{C}^3$, $\{(\lambda_1,\lambda_2,0):\lambda_1,\lambda_2\in\mathbb{C}\}$, $\{(\lambda_1,0,0):\lambda_1\in\mathbb{C}\}$, and $\{(0,0,0)\}$. Actually, this example corresponds to case (2) (with $l=2$) in Theorem 3.1.

To prove Theorem 3.1, we need a series of lemmas. We start with a result from [3] which yields a more concrete representation for the hyperinvariant subspace lattice. For positive integers $n_1\geqslant\cdots\geqslant n_r$, $L(n_1,\ldots,n_r)$ denotes the set of $r$-tuples $(u_1,\ldots,u_r)$ of integers satisfying $u_1\geqslant\cdots\geqslant u_r\geqslant 0$ and $n_1-u_1\geqslant\cdots\geqslant n_r-u_r\geqslant 0$. Under the operations

$$(u_1,\ldots,u_r)\vee(v_1,\ldots,v_r)=(\max(u_1,v_1),\ldots,\max(u_r,v_r))$$

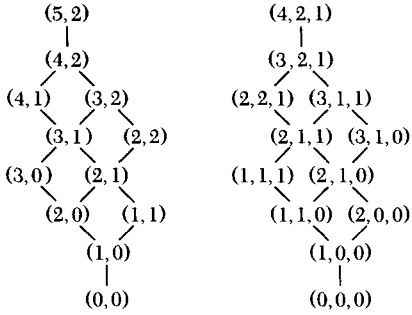
and

$$(u_1,\ldots,u_r)\wedge(v_1,\ldots,v_r)=(\min(u_1,v_1),\ldots,\min(u_r,v_r)),$$

$L(n_1,\ldots,n_r)$ is a lattice. In particular, the partial order in $L(n_1,\ldots,n_r)$ is given by $(u_1,\ldots,u_r)\geqslant(v_1,\ldots,v_r)$ if $u_i\geqslant v_i$ for all $i$. As proved in [3], such lattices are exactly (isomorphic to) the hyperinvariant subspace lattices of nilpotent linear transformations.

LEMMA 3.4. *If $A$ is a nilpotent transformation with partial multiplicities $n_1\geqslant\cdots\geqslant n_p$ and reduced partial multiplicities $n'_1\geqslant\cdots\geqslant n'_r$, then* Hyperlat $A$, *$L(n_1,\ldots,n_p)$, and $L(n'_1,\ldots,n'_r)$ are isomorphic lattices.*

This is essentially [3, Theorem 3]; that $L(n'_1,\ldots,n'_r)$ is isomorphic to the other two lattices can be easily derived. It enables us to concentrate on lattices of the form $L(n_1,\ldots,n_r)$ with $n_1>\cdots>n_r$.




$L(5,2)$ $L(4,2,1)$

FIG. 1.

The next lemma takes care of the exceptional cases in Theorem 3.1. We use the symbol $K_1$-$K_2$-$\cdots$-$K_n$ to denote a chain in a lattice, where the $K_i$'s satisfy $K_i > K_{i+1}$ for all $i$, $1 \leqslant i \leqslant n-1$.

LEMMA 3.5. *$L(5,2) \cong L(4,2,1)$ and $L(l, l-1) \cong L(2l-1)$ for any $l \geqslant 2$.*

*Proof.* The lattices $L(5,2)$ and $L(4,2,1)$, as shown in Figure 1, are easily seen to be isomorphic. For any $l \geqslant 2$, $L(l, l-1)$ is a chain of $2l$ elements: $(l, l-1)$-$(l-1, l-1)$-$(l-1, l-2)$-$\cdots$-$(1,1)$-$(1,0)$-$(0,0)$, and hence is isomorphic to $L(2l-1)$. ∎

As observed in Figure 1, it seems that each element in the lattice $L(n_1, \dots, n_r)$ $(n_1 > \cdots > n_r)$ has at most two "sons" and two "parents." This is indeed the case, as we now show. Recall that, for two elements $K_1$ and $K_2$ in a lattice, $K_1$ is a *son* of $K_2$ ($K_2$ is a *parent* of $K_1$) if $K_1 < K_2$ and there is no other element between $K_1$ and $K_2$.

LEMMA 3.6. *Let $n_1 > \cdots > n_r \geqslant 1$ and $u = (u_1, \dots, u_r) \in L(n_1, \dots, n_r)$.*

(1) *$u$ has at most two sons.*

(2) *$u$ has no son if and only if $u = (0, \dots, 0)$.*

(3) *$u$ has exactly one son if and only if $u \neq (0, \dots, 0)$ and one of the following holds*:

(i) $u_2 = \cdots = u_r = 0$;
(ii) *there exists $i$, $1 \leqslant i \leqslant r$, such that $u_1 = \cdots = u_i$ and $n_i - u_i = \cdots = n_r - u_r$.*

*In this case, the son is* $(u_1 - 1, 0, \ldots, 0)$ *or* $(u_1, \ldots, u_{i-1}, u_i - 1, u_{i+1}, \ldots, u_r)$, *depending on whether* (i) *or* (ii) *holds.*

Note that the corresponding results for the parent can be obtained through the self-duality in $L(n_1, \ldots, n_r)$ under the antiisomorphism $(u_1, \ldots, u_r) \to (n_1 - u_1, \ldots, n_r - u_r)$.

*Proof.* Obviously, $(u_1, 0, \ldots, 0)$ $(u_1 \neq 0)$ has only one son $(u_1 - 1, 0, \ldots, 0)$. Otherwise, for $u \neq (0, \ldots, 0)$, let $i$ (respectively, $j$), $1 \leqslant i \leqslant r$ $(1 \leqslant j \leqslant r)$, be the smallest (largest) index for which $u_i > u_{i+1}$ $(n_j - u_j < n_{j-1} - u_{j-1})$. In general, we have $i \geqslant j$. If $i > j$, then $u$ has two sons $(u_1, \ldots, u_{i-1}, u_i - 1, u_{i+1}, \ldots, u_r)$ and $(u_1, \ldots, u_{j-1}, u_j - 1, u_{j+1}, \ldots, u_r)$; if $i = j$, then it has only one son $(u_1, \ldots, u_{i-1}, u_i - 1, u_{i+1}, \ldots, u_r)$. ■

LEMMA 3.7. *Assume that* $L(n_1, \ldots, n_r)$ *and* $L(m_1, \ldots, m_s)$ *are isomorphic, where* $n_1 > \cdots > n_r \geqslant 1$ *and* $m_1 > \cdots > m_s \geqslant 1$. *If* $(u_1, \ldots, u_r) \in L(n_1, \ldots, n_r)$ *and* $(v_1, \ldots, v_s) \in L(m_1, \ldots, m_s)$ *correspond under an isomorphism, then* $u_1 + \cdots + u_r = v_1 + \cdots + v_s$. *In particular,* $L(n_1, \ldots, n_r) \cong L(m_1, \ldots, m_s)$ *implies that* $n_1 + \cdots + n_r = m_1 + \cdots + m_s$.

*Proof.* For any $K = (u_1, \ldots, u_r)$ in $L(n_1, \ldots, n_r)$, let $\dim K = u_1 + \cdots + u_r$. It follows from Lemma 3.6 (and its dual) that every $K$ belongs to a chain $C: K_t\text{-}K_{t-1}\text{-} \cdots \text{-}K_1\text{-}K_0$ of length $t + 1 = n_1 + \cdots + n_r + 1$ in $L(n_1, \ldots, n_r)$ with $\dim K_i = i$ for all $i$, $0 \leqslant i \leqslant t$. Note that such chains are exactly maximal chains in $L(n_1, \ldots, n_r)$, whence they are preserved under isomorphism. We infer that $\dim K$, an indicator of the relative position of $K$ in $C$, is invariant under isomorphism. This completes the proof. ■

To proceed further, we need to consider some special constructs in the lattice $L(n_1, \ldots, n_r)$. A chain $K_1\text{-}K_2\text{-} \cdots \text{-}K_n$ in $L(n_1, \ldots, n_r)$ $(n_1 > \cdots > n_r \geqslant 1)$ is *special* if it is maximal with the property that each $K_i$, $1 \leqslant i \leqslant n - 1$, has only one son, namely, $K_{i+1}$. The next lemma shows the existence of special chains which end at the zero element.

LEMMA 3.8. *In any* $L(n_1, \ldots, n_r)$ $(n_1 > \cdots > n_r \geqslant 1)$, *there exist at most two special chains ending at* $(0, \ldots, 0)$. *There exists exactly one such chain if and only if* $r = 1$ *or* $r = 2$ *and* $n_1 - n_2 = 1$. *In this case, the chain is of length* $n_1 + 1$ *or* $2n_1 - 1$; *otherwise, the two chains are of lengths* $r + 1$ *and* $n_1 - n_2 + 1$.

*Proof.* It is easily seen that $(1, \ldots, 1)\text{-}(1, \ldots, 1, 0)\text{-}(1, \ldots, 1, 0, 0)\text{-} \cdots \text{-}(0, \ldots, 0)$ and $(n_1 - n_2, 0, \ldots, 0)\text{-}(n_1 - n_2 - 1, 0, \ldots, 0)\text{-} \cdots \text{-}(1, 0, \ldots, 0)\text{-}(0, \ldots, 0)$

are two special chains ending at the zero element, and that they coincide if and only if $r = 1$ or $r = 2$ and $n_1 - n_2 = 1$. ∎

Note that in $L(n_1, \ldots, n_r)$, these two chains, denoted by $C_1$ and $C_2$ henceforth, encode the parameters $r$ and $n_1 - n_2$, respectively. Also, any isomorphism between such lattices must map special chains among themselves. To distinguish $C_1$ from $C_2$, we need another construct. We say that a chain $C: K_1\text{-}K_2\text{-}\cdots\text{-}K_n$ in $L(n_1, \ldots, n_r)$ *rides* on $C_i$, $i = 1, 2$, if

(i) there is some $j$, $0 \leqslant j \leqslant n-1$, such that each $K_l$ has only one son $K_{l+1}$ for $1 \leqslant l \leqslant j$ and has two sons, one being $K_{l+1}$ and the other being in $C_i$, for $j+1 \leqslant l \leqslant n-1$;
(ii) $K_n$ has only one son $(1, 0, \ldots, 0)$; and
(iii) $C$ is maximal with properties (i) and (ii).

As examples, in $L(5,2)$, the chains (4,1)-(3,1)-(2,1)-(1,1) and (2,2)-(2,1)-(2,0) ride on special chains $C_1$:(1,1)-(1,0)-(0,0) and $C_2$:(3,0)-(2,0)-(1,0)-(0,0), respectively (cf. Figure 1). It is easy to see that, in any $L(n_1, \ldots, n_r)$, $C_2$ has at most one chain, namely, $(n_1 - n_2 + 1, 1, \ldots, 1)\text{-}(n_1 - n_2, 1, \ldots, 1)\text{-}\cdots\text{-}(2, 1, \ldots, 1)\text{-}(1, \ldots, 1)$, riding on it; moreover, it is also of length $n_1 - n_2 + 1$.

In the next lemma, we use the notion of riding chain to show that, under certain conditions, an isomorphism from $L(n_1, \ldots, n_r)$ to $L(m_1, \ldots, m_s)$ must map $C_i$ to $C_i$ $(i = 1, 2)$.

LEMMA 3.9. *There is no isomorphism from $L(n_1, \ldots, n_r)$ onto $L(m_1, \ldots, m_s)$ taking $C_1$ (in $L(n_1, \ldots, n_r)$) to $C_2$ (in $L(m_1, \ldots, m_s)$) under either of the following conditions*:

(1) $r \geqslant 2$, $s = 3$, *and* $m_2 - m_3 \geqslant 2$;
(2) $r \geqslant 2$ *and* $s \geqslant 4$.

*Proof*. Assume that such an isomorphism exists. By Lemma 3.8, we obtain $n_1 - n_2 = s$ and $m_1 - m_2 = r$. Let $C$ be the chain $(2, 1, \ldots, 1)\text{-}(2, 1, \ldots, 1.0)\text{-}(2, 1, \ldots, 1, 0, 0)\text{-}\cdots\text{-}(2, 0, \ldots, 0)$ in $L(m_1, \ldots, m_s)$. Since $m_1 - m_2 = r \geqslant 2$, $(3, 1, \ldots, 1)$ is an element of $L(m_1, \ldots, m_s)$. Thus $(2, 1, \ldots, 1)$ has two parents: $(2, 2, 1, \ldots, 1)$ and $(3, 1, \ldots, 1)$. Under condition (1), (2,2,0) is in $L(m_1, m_2, m_3)$, and therefore both (2,2,1) and (3,1,1) have two sons: (2,1,1) and (2,2,0) for the former, and (2,1,1) and (3,1,0) for the latter. On the other hand, if (2) holds, then $(2, 2, 1, \ldots, 1, 0)$ is in $L(m_1, \ldots, m_s)$. Thus, again, both $(2, 2, 1, \ldots, 1)$ and $(3, 1, \ldots, 1)$ have two sons: $(2, 1, \ldots, 1)$ and $(2, 2, 1, \ldots, 1, 0)$ for the former, and $(2, 1, \ldots, 1)$ and $(3, 1, \ldots, 1, 0)$ for the latter. In any case, it is easy to see that the chain $C$ rides on $C_1$ with length $s$. However, by our assumption, $C_1$ [in $L(m_1, \ldots, m_s)$] and $C_2$ [in $L(n_1, \ldots, n_r)$] correspond under the isomorphism, whence the chain corresponding to $C$

rides on $C_2$ and is of length $s$. This contradicts the fact that the chain riding on $C_2$ must be of length $n_1 - n_2 + 1 = s + 1$. Therefore, no such isomorphism exists. ■

LEMMA 3.10. *Let $r \geqslant 1$, $n_1 > \cdots > n_r \geqslant 1$, and $m_1 > \cdots > m_r \geqslant 1$. Then $L(n_1,\dots,n_r)$ and $L(m_1,\dots,m_r)$ are isomorphic if and only if $n_i = m_i$ for all $i$, $1 \leqslant i \leqslant r$.*

*Proof.* We prove the necessity by induction on r. If $L(n_1,\dots,n_r)$ has only one special chain (ending at the zero element), then this is trivial. Assume next that $r = 2$ and $L(n_1, n_2)$ has two special chains $C_1$ and $C_2$. Then $L(n_1,n_2) \cong L(m_1,m_2)$ implies that $n_1 - n_2 = m_1 - m_2$ or $n_1 - n_2 = 2$ and $m_1 - m_2 = 2$, depending on whether $C_1$ [in $L(n_1,n_2)$] is mapped to $C_1$ or $C_2$ [in $L(m_1,m_2)$]. On the other hand, we also have $n_1 + n_2 = m_1 + m_2$ by Lemma 3.7. Solving these equations yields $n_1 = m_1$ and $n_2 = m_2$.

Assuming that the assertion is true for $r - 1$ ($r \geqslant 3$) and $L(n_1,\dots,n_r) \cong L(m_1,\dots,m_r)$, we proceed to prove it for $r$. If the isomorphism between them maps $C_1$ to $C_2$, then, in view of Lemma 3.9, we must have $r = 3$ and $n_2 - n_3 = m_2 - m_3 = 1$. This equation, together with $n_1 - n_2 = m_1 - m_2 = 3$ (Lemma 3.8) and $n_1 + n_2 + n_3 = m_1 + m_2 + m_3$ (Lemma 3.7), yields $n_i = m_i$ for all $i$. Hence, for the rest of the proof, we may assume that the isomorphism maps $C_i$ to $C_i$, $i = 1, 2$. Consider the congruence relation in $L(n_1,\dots,n_r)$ defined by $(u_1,\dots,u_r) \sim (v_1,\dots,v_r)$ if $u_i = v_i$ for all $i$, $2 \leqslant i \leqslant r$, or, equivalently, $(u_1 - v_1,\dots,u_r - v_r)$ belongs to $C_2$. The resulting factor lattice $L(n_1,\dots,n_r)/C_2$ is easily seen to be isomorphic to $L(n_2,\dots,n_r)$. (For these lattice-theoretic notions, the reader may consult [2, p. 73].) Similarly, we have $L(m_1,\dots,m_r)/C_2 \cong L(m_2,\dots,m_r)$. Since the isomorphism between $L(n_1,\dots,n_r)$ and $L(m_1,\dots,m_r)$ maps $C_2$ to $C_2$, it induces an isomorphism on the factor lattices. Thus $L(n_2,\dots,n_r) \cong L(m_2,\dots,m_r)$. The induction hypothesis then implies that $n_i = m_i$ for all $i$, $2 \leqslant i \leqslant r$. It follows from $n_1 + \cdots + n_r = m_1 + \cdots + m_r$ (by Lemma 3.7) that $n_1 = m_1$. This completes the proof. ■

LEMMA 3.11. *If $L(n_1, n_2)$ is isomorphic to $L(m_1,\dots,m_s)$, where $n_1 > n_2 \geqslant 1$, $n_1 - n_2 \geqslant 2$, $s \geqslant 3$, and $m_1 > \cdots > m_s \geqslant 2$, then $m_{s-1} - m_s \geqslant 2$.*

*Proof.* If $m_{s-1} - m_s = 1$, then, by Lemma 3.6, it is easily seen that $(m_s,\dots,m_s)$-$(m_s,\dots,m_s,m_s - 1)$-$(m_s,\dots,m_s,m_s - 1,m_s - 1)$ is a special chain in $L(m_1,\dots,m_s)$ which is not part of $C_1$ or $C_2$. We check that $L(n_1,n_2)$ has no special chain of length bigger than two except parts of these two special chains. It will then follow that $L(n_1,n_2)$ and $L(m_1,\dots,m_s)$ cannot be isomorphic.

Assume to the contrary that $K_1$-$K_2$-$K_3$ is a chain in $L(n_1, n_2)$, where both $K_1$ and $K_2$ have only one son. By Lemma 3.6, $K_1 \equiv (u_1, u_2)$ has only one son if and only if $u_2 = 0$, $u_1 = u_2$, or $n_1 - u_2 = n_2 - u_2$. We consider the three cases separately:

(1) If $u_2 = 0$, then, obviously, $K_1$-$K_2$-$K_3$ is part of $C_2$.

(2) If $u_1 = u_2$, then $K_2 = (u_1, u_2 - 1)$. Again, by Lemma 3.6, either $u_2 - 1 = 0$ or $n_1 - u_1 = n_2 - (u_2 - 1)$. The former implies that $K_1$-$K_2$-$K_3$ is the special chain $(1,1)$-$(1,0)$-$(0,0)$; the latter, that $n_1 - n_2 = 1$, contradicting our hypothesis.

(3) If $n_1 - u_1 = n_2 - u_2$, then $K_2 = (u_1 - 1, u_2)$. By Lemma 3.6, either $u_2 = 0$ or $u_1 - 1 = u_2$. The former implies that $K_1$-$K_2$-$K_3$ is part of $C_2$; the latter, that $n_1 - n_2 = 1$, a contradiction. ∎

LEMMA 3.12. *Let $r, s \geqslant 1$, $n_1 > \cdots > n_r \geqslant 1$, and $m_1 > \cdots m_s \geqslant 1$. Then $L(n_1, \ldots, n_r)$ is isomorphic to $L(m_1, \ldots, m_s)$ if and only if one of the following holds*:

(1) *the $n_i$'s and $m_j$'s coincide with $5 > 2$ and $4 > 2 > 1$;*

(2) *the $n_i$'s and $m_j$'s coincide with $l > l - 1$ and $2l - 1$ for some $l \geqslant 2$;*

(3) *$r = s$ and $n_i = m_i$ for all $i$.*

*Proof.* If $r = 1$, then $L(n_1) \cong L(m_1, \ldots, m_s)$ implies (2) or (3) by Lemma 3.8. Next assume that $r, s \geqslant 2$. Let $\alpha$ be an isomorphism from $L(n_1, \ldots, n_r)$ onto $L(m_1, \ldots, m_s)$. If it takes $C_i$ to $C_i$, $i = 1, 2$, then $r = s$, whence (3) follows from Lemma 3.10. Consequently, we may assume that $\alpha$ takes $C_1$ to $C_2$. This can happen only when $s = 2$ or $s = 3$ and $m_2 - m_3 = 1$, by Lemma 3.9.

(i) $s = 2$. If $r = 2$, then (3) follows as above. Hence we may further assume that $r \geqslant 3$. Using Lemma 3.9 again (by reversing the roles of $n_i$ and $m_j$ there), we need only consider the case $r = 3$ and $n_2 - n_3 = 1$. Since $C_1$ and $C_2$ are corresponded under $\alpha$, we obtain $n_1 - n_2 = s = 2$ and $m_1 - m_2 = r = 3$. On the other hand, Lemma 3.7 and Lemma 3.11 imply $n_1 + n_2 + n_3 = m_1 + m_2$ and $n_3 = 1$, respectively. Solving these equations yields the solution $n_1 = 4$, $n_2 = 2$, $n_3 = 1$, $m_1 = 5$ and $m_2 = 2$, that is, (1) holds.

(ii) $s = 3$ *and* $m_2 - m_3 = 1$. If $r = 2$, then an argument similar to that in (i) with the roles of $r$ and $s$ reversed yields (1). If $r = 3$, then (3) follows from Lemma 3.10, while $r \geqslant 4$ cannot happen by Lemma 3.9. This completes the proof. ∎

Finally, we are ready for the coup de grâce. The idea of the following proof is similar to that for the invariant subspace lattice [8, Theorem 3.2].

*Proof of Theorem 3.1.* If $A_1 \oplus \cdots \oplus A_n$ on $V_1 \oplus \cdots \oplus V_n$ is the primary decomposition of $A$, where $V_i = \ker(A - a_i I)^{n_{i1}}$, then Hyperlat $A$ is isomorphic to the direct product of Hyperlat $A_i$, $i = 1, \dots, n$ (cf. [3, Theorem 2]). Note that each Hyperlat $A_i$ is a nontrivial indecomposable sublattice of Hyperlat $A$ (cf. [3, p. 131, (iv)]) and the family of such sublattices is unique (cf. [2, pp. 92–93, 11.8]). A similar consideration applies to $B$. Thus Hyperlat $A \cong$ Hyperlat $B$ if and only if, after a reordering, Hyperlat $A_i \cong$ Hyperlat $B_i$ for all $i$. The proof is then completed by applying Lemma 3.12. ■

## REFERENCES


1 L. Brickman and P. A. Fillmore, The invariant subspace lattice of a linear transformation, *Canad. J. Math.* 19:810–822 (1967).
2 P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, N.J., 1973.
3 P. A. Fillmore, D. A. Herrero, and W. E. Longstaff, The hyperinvariant subspace lattice of a linear transformation, *Linear Algebra Appl.* 17:125–132 (1977).
4 I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.
5 I. Gohberg and L. Rodman, On distance between lattices of invariant subspaces of matrices, *Linear Algebra Appl.* 76:85–120 (1986).
6 R. K. Guy, The strong law of small numbers, *Amer. Math. Monthly* 95:697–712 (1988).
7 P. Lancaster and M. Tismenetsky, *The Theory of Matrices with Applications*, 2nd ed., Academic, New York, 1985.
8 W. E. Longstaff, A lattice-theoretic description of the lattice of hyperinvariant subspaces of a linear transformation, *Canad. J. Math.* 28:1062–1066 (1976).
9 M. J. McAsey and P. S. Muhly, On projective equivalence of invariant subspace lattices, *Linear Algebra Appl.* 43:167–179 (1982).
10 S.-C. Ong, Remarks on invariant subspaces for finite dimensional operators, *Linear Algebra Appl.* 42:99–101 (1982).
11 S.-C. Ong, What kind of operators have few invariant subspaces?, *Linear Algebra Appl.* 95:181–185 (1987).
12 S.-C. Ong, On equality of invariant subspace lattices of operators, *Linear Algebra Appl.* 144:23–27 (1991).


*Received 20 March 1991; final manuscript accepted 25 April 1991*