Systolic Block Householder Transformation for RLS Algorithm with Two-Level Pipelined Implementation

KuoJuey Ray Liu, *Member, IEEE,* Shih-Fu Hsieh, *Member, IEEE,* and Kung Yao, *Senior Member, IEEE*

***Abstract*—The *QR* decomposition, recursive least squares (QRD RLS) algorithm is one of the most promising RLS algorithms, due to its robust numerical stability and suitability for VLSI implementation based on a systolic array architecture. Up to now, among many techniques to implement the *QR* decomposition, only the Givens rotation and modified Gram–Schmidt methods have been successfully applied to the development of the QRD RLS systolic array. It is well known that Householder transformation (HT) outperforms the Givens rotation method under finite precision computations. Presently, there is no known technique to implement the HT on a systolic array architecture. In this paper, we propose a systolic block Householder transformation (SBHT) approach, to implement the HT on a systolic array as well as its application to the RLS algorithm. Since the data is fetched in a block manner, vector operations are in general required for the vectorized array. However, a modified HT algorithm permits a two-level pipelined implementation of the SBHT systolic array at both the vector and word levels. The throughput rate can be as fast as that of the Givens rotation method. Our approach makes the HT amenable for VLSI implementation as well as applicable to real-time high throughput applications of modern signal processing. The constrained RLS problem using the SBHT RLS systolic array is also considered in this paper.**

I. Introduction

LEAST squares (LS) technique constitutes an integral part of modern signal processing and communications methodology as used in adaptive filtering, beamforming, array signal processing, channel equalization, etc. [6]. Efficient implementation of the LS algorithm, particularly the recursive LS algorithm (RLS), is needed to meet the high throughput and speed requirements of modern signal processing. There are many possible approaches, such as the fast transversal method and the lattice method, which can perform RLS algorithm efficiently [1], [6]. Unfortunately, these methods can encounter numerical difficulties due to the accumulation of roundoff errors under a finite-precision implementation as summarized in [2]. This may lead to a divergence of the computations of the RLS algorithm [2]. A new type of systolic algorithm based on the *QR* decomposition (QRD), known as the QRD RLS, was first proposed by McWhirter in [18]. This algorithm is one of the most promising algorithms in that it is numerically stable [1], [12] as well as suitable for parallel processing implementation on a systolic array [6], [18].

Up to now, most of the QRD RLS implementations were based on the Givens rotation method and modified Gram–Schmidt method, which are both rank-1 update approaches [2], [4], [7], [13], [16], [18], [9]. It is well known that the Householder transformation (HT), which is a rank-k update approach, is one of the most computationally efficient methods to compute QRD. The error analysis carried out by Wilkinson [26], [8] showed that the HT outperforms the Givens method under finite precision computations. Presently, there is no known technique to implement the HT on a systolic array parallel processing architecture, since there is a belief that nonlocal connections in the implementation are necessary due to the vector processing nature of the Householder transformation. One of the purposes of this paper is to show that we can implement the HT on a systolic array with only local connections. Thus, it is amenable to VLSI implementation and is applicable to real-time high throughput applications of modern signal processing.

In this paper, we first propose a systolic Householder algorithm called a systolic block Householder transformation (SBHT) to compute the QRD with an implementation on a vectorized systolic array. Then a RLS algorithm based on the SBHT called SBHT RLS algorithm is proposed to perform RLS operations on the array. We shall show that the SBHT array and the SBHT RLS array are generalizations of Gentleman–Kung's QRD array [4] and McWhirter's QRD RLS systolic array [18] (see Fig. 1), respectively. The difficulty in the applications of the above arrays is mainly due to the vectorized operations of the processing cells. This results in a high cell complexity as well as a high I/O bandwidth. By using a modified HT algorithm proposed by Tsao [25], a two-level pipelined implementation of the SBHT RLS algorithm can be achieved. That is, the algorithm is pipelined at the vector level as well as at the word level. The complexity of the processing cell and the I/O bandwidth are thus reduced.

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K. J. R. Liu is with the Department of Electrical Engineering, Systems Research Center, University of Maryland, College Park, MD 20742.

S.-F. Hsieh is with the Department of Communication Engineering, National Chiao Tung University, Hsinchu, Taiwan 30039, Republic of China.

K. Yao is with the Department of Electrical Engineering, University of California, Los Angeles, CA 90024-1594.

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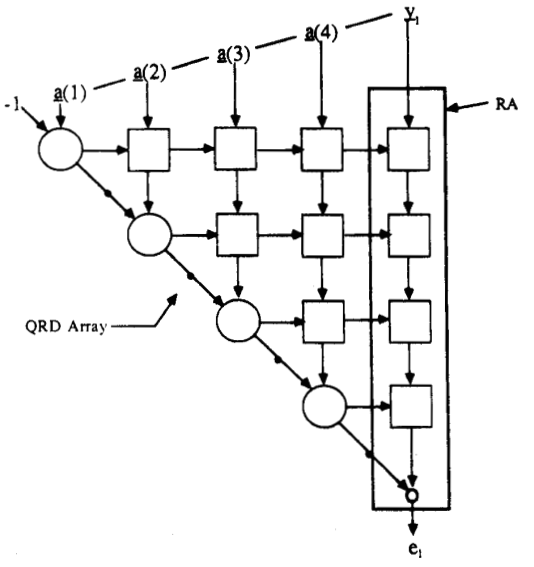
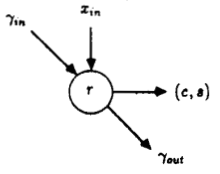


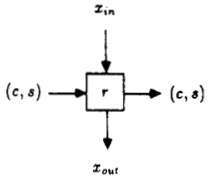
(a)

(1) Boundary Cell



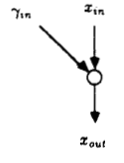
If $x_{in} = 0$ **then**
$c \leftarrow 1$; $s \leftarrow 0$; $\gamma_{out} \leftarrow \gamma_{in}$;
$r = \lambda r$,
otherwise
$r' = \sqrt{\lambda^2 r^2 + x_{in}^2}$;
$c \leftarrow \lambda r / r'$; $s \leftarrow x_{in}/r'$
$r \leftarrow r'$; $\gamma_{out} = c\gamma_{in}$
end

(2) Internal Cell



$x_{out} \leftarrow cx_{in} - s\lambda r$
$r \leftarrow sx_{in} + c\lambda r$

(3) Final Cell



$x_{out} \leftarrow \gamma_{in} x_{in}$

(b)

Fig. 1. (a) QRD RLS systolic array using Givens rotation method. (b) Processing cells of the Givens rotation method.

In general, the cell complexity of the SBHT array is higher and the system latency is longer than that of the conventional Givens rotation implementations. With the two-level pipelined implementation, the throughput of the SBHT RLS systolic array is as fast as that of McWhirter's Givens rotation array, and it offers better numerical stability than the Givens method. In addition, an extension of the SBHT RLS array to MVDR beamformation, which is a constrainted RLS problem, is also considered.

In Section II, a brief review of the QRD RLS algorithm is given. In Section III, the SBHT is presented, while the SBHT RLS algorithm is considered in Section IV. The two-level pipelined implementation of the SBHT RLS systolic array is discussed in Section V. In Section VI, the constrained RLS problem is applied to the MVDR beamformation, using an extension of the SBHT RLS array. Finally, the systolic array for the hyperbolic Householder transformation is considered in Section VII and a conclusion is given in Section VIII.

II. QRD RLS Algorithm

A full rank $m \times p$, $m > p$, rectangular matrix $\boldsymbol{X}$ can be uniquely factorized into two matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ such that $\boldsymbol{X} = \boldsymbol{QR}$, where $\boldsymbol{Q}$ is an $m \times p$ matrix with orthonormal columns and $\boldsymbol{R}$ is a $p \times p$ upper triangular matrix. Several different approaches of the QRD systolic arrays have been proposed by Gentleman and Kung [4], Heller and Ipsen [7], Luk [16], Ling *et al.* [13], and Kalson and Yao [9]. The first three approaches are based on the Givens rotations methods, while the last two are based on the modified Gram–Schmidt orthogonalization. Given an $m \times 1$ vector $\boldsymbol{y}$, the LS problem is to minimize the norm of the residual vector $\boldsymbol{\varepsilon}$

$$\|\boldsymbol{\varepsilon}(m)\| = \|\boldsymbol{X}(m)\boldsymbol{w}(m) - \boldsymbol{y}(m)\|$$

by choosing an optimal weight vector $\boldsymbol{w}$. If the matrix $\boldsymbol{X}$ and vector $\boldsymbol{y}$ grow in time, then the problem of minimizing the norm of the residual vector recursively becomes the RLS problem. Until recently, it appears that only Givens and modified Gram–Schmidt methods have been considered for RLS computations. Some recent RLS problems based upon the use of Householder transformation have appeared [3], [15]. In [18], McWhirter showed that a QRD RLS systolic array, which was based on the Gentleman–Kung array, can be designed without first computing the weight vector of the RLS problem. This approach is useful for high throughput applications in various modern signal processing problems such as adaptive filtering and beamforming since optimal residuals are of direct interest while the weight vector needs not be computed. The basic idea of the QRD RLS systolic array in [18] is to update the $p \times p$ matrix $\boldsymbol{R}$ using a sequence of Givens rotation matrices when a new row of data arrives. Suppose we have the QRD of the data matrix $\boldsymbol{X}$ at time m and expressed as $\boldsymbol{X}(m) = \boldsymbol{Q}(m)\boldsymbol{R}(m)$. Define

$$\overline{\boldsymbol{Q}}^T(m) = \left[\begin{array}{c:c} \boldsymbol{Q}^T(m) & \boldsymbol{0} \\ \hdashline \boldsymbol{0}^T & 1 \end{array}\right].$$

When a new row of data arrives, we then have

$$\overline{\boldsymbol{Q}}^T(m)\boldsymbol{X}(m+1) = \left[\begin{array}{c} \boldsymbol{R}(m) \\ \hline \boldsymbol{0} \\ \hline x_1, x_2, \cdots, x_p \end{array}\right].$$

This new row of data can be zeroed out by applying a sequence of Givens rotations

$$\boldsymbol{G} = \boldsymbol{G}_p \cdots \boldsymbol{G}_2 \boldsymbol{G}_1$$

where the $(m + 1) \times (m + 1)$ transformation matrix $\boldsymbol{G}_i$ is defined by

$$\boldsymbol{G}_i = \left[\begin{array}{c:c:c:c} \boldsymbol{I}_{i-1} & 0 & \boldsymbol{0} & 0 \\ \hdashline 0 & c_i & 0 & s_i \\ \hdashline \boldsymbol{0} & 0 & \boldsymbol{I}_{m-i} & 0 \\ \hdashline 0 & -s_i & 0 & c_i \end{array}\right]$$

with

$$c_i = \frac{a}{\sqrt{a^2 + b^2}}, \qquad s_i = \frac{b}{\sqrt{a^2 + b^2}}$$

where a and b are elements of vectors in the ith and $(m + 1)$th rows under rotation.

Fig. 1(a) shows the systolic array proposed by McWhirter in [18]. It consists of a QRD triarray and a linear response array (RA). The rotation parameters are propagated from the boundary cells to the right for internal cells to update their contents, and the cosine parameters are also cumulated and propagated down diagonal boundary cells. Each skewed input row of data is zeroed out by the QRD triarray. The optimal residual is then obtained by the multiplication of the cumulated cosines and the rotated output of the desired response at the response array (see Fig. 1(a)) [18].

III. Systolic Block Householder Transformation

The Givens rotation method discussed above is a rank-1 update approach since each input consists of one row of data. For the systolic block Householder transformation (SBHT), we need a block data formulation. Denote the data matrix as

$$\boldsymbol{X}(n) = \begin{bmatrix} \boldsymbol{X}_1^T \\ \boldsymbol{X}_2^T \\ \vdots \\ \boldsymbol{X}_n^T \end{bmatrix} = \left[\begin{array}{c} \boldsymbol{X}(n-1) \\ \hline \boldsymbol{X}_n^T \end{array}\right] \in \Re^{nk \times p} \tag{1}$$

and the desired response vector as

$$\boldsymbol{y}(n) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \left[\begin{array}{c} \boldsymbol{y}(n-1) \\ \hline y_n \end{array}\right] \in \Re^{nk} \tag{2}$$

where $\boldsymbol{X}_i^T$ is the $k \times p$ ith data block matrix

$$\boldsymbol{X}_i^T = \begin{bmatrix} \boldsymbol{x}_{(i-1)k+1}^T \\ \boldsymbol{x}_{(i-1)k+2}^T \\ \vdots \\ \boldsymbol{x}_{ik}^T \end{bmatrix} = [\boldsymbol{x}_{i,1} \quad \boldsymbol{x}_{i,2} \quad \cdots \quad \boldsymbol{x}_{i,p}]$$

$$= \begin{bmatrix} x_{(i-1)k+1,1} & x_{(i-1)k+1,2} & \cdots & x_{(i-1)k+1,p} \\ x_{(i-1)k+2,1} & x_{(i-1)k+2,2} & \cdots & x_{(i-1)k+2,p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{ik,1} & x_{ik,2} & \cdots & x_{ik,p} \end{bmatrix} \in \Re^{k \times p} \tag{3}$$

(4)

and $\boldsymbol{y}_i$ is the $k \times 1$ ith desired response block vector

$$\boldsymbol{y}_i = \begin{bmatrix} y_{(i-1)k+1} \\ y_{(i-1)k+2} \\ \vdots \\ y_{ik} \end{bmatrix} \in \Re^k \tag{5}$$

where k is the block size and p is the order (i.e., number of columns) of the system.

For a rank-k update QR decomposition, suppose we have

$$\boldsymbol{Q}(n-1)\boldsymbol{X}(n-1) = \left[\begin{array}{c} \boldsymbol{R}(n-1) \\ \hline 0 \end{array}\right]. \tag{6}$$

Denote

$$\bar{\boldsymbol{Q}}_k(n-1) = \left[\begin{array}{c:c} \boldsymbol{Q}(n-1) & \boldsymbol{0} \\ \hdashline \boldsymbol{0}^T & \boldsymbol{I}_k \end{array}\right] \tag{7}$$

then we have

$$\bar{\boldsymbol{Q}}_k(n-1)\boldsymbol{X}(n) = \left[\begin{array}{c} \boldsymbol{R}(n-1) \\ \hline \boldsymbol{0} \\ \hline \boldsymbol{X}_n^T \end{array}\right]. \tag{8}$$

If we can find a matrix $\boldsymbol{H}(n)$ such that

$$\boldsymbol{H}(n)\bar{\boldsymbol{Q}}_k(n-1)\boldsymbol{X}(n) = \left[\begin{array}{c} \boldsymbol{R}(n) \\ \hline 0 \end{array}\right] \tag{9}$$

then the new $\boldsymbol{Q}(n)$ is

$$\boldsymbol{Q}(n) = \boldsymbol{H}(n)\bar{\boldsymbol{Q}}_k(n-1). \tag{10}$$

An $n \times n$ Householder transformation matrix $\boldsymbol{T}$ is of the form

$$\boldsymbol{T} = \boldsymbol{I} - \frac{2zz^T}{\|z\|^2} \tag{11}$$

where $z \in \Re^n$ [5]. When a vector $\boldsymbol{x}$ is multiplied by $\boldsymbol{T}$, it is reflected in the hyperplane defined by $span\{z\}^{\perp}$. Choosing $z = \boldsymbol{x} \pm \|\boldsymbol{x}\|_2 \boldsymbol{e}_1$, where $\boldsymbol{e}_1 = [1, 0, 0, \cdots, 0] \in \Re^n$, then $\boldsymbol{x}$ is reflected onto $\boldsymbol{e}_1$ by $\boldsymbol{T}$ as

$$\boldsymbol{T}\boldsymbol{x} = \pm\|\boldsymbol{x}\|_2 \boldsymbol{e}_1. \tag{12}$$

That is, all of the energy of $\boldsymbol{x}$ is reflected onto the unit vector $\boldsymbol{e}_1$ after the transformation. We can zero out $\boldsymbol{X}_n^T$ by applying successive Householder transformations as

follows:

$$\boldsymbol{H}^{(i)}(n)\begin{bmatrix}\boldsymbol{R}^{(i-1)}(n-1)\\ \hline \mathbf{0}\\ \hline 0,\cdots,0,\boldsymbol{x}_{n,i}^{(i-1)},\cdots,\boldsymbol{x}_{n,p}^{(i-1)}\end{bmatrix} = \begin{bmatrix}\boldsymbol{R}^{(i)}(n-1)\\ \hline \mathbf{0}\\ \hline 0,\cdots,0,0,\boldsymbol{x}_{n,i+1}^{(i)},\cdots,\boldsymbol{x}_{n,p}^{(i)}\end{bmatrix}$$

for $i = 1, \cdots, p$, where $\boldsymbol{x}_{n,i}^{(0)} = \boldsymbol{x}_{n,i}$, $\boldsymbol{R}^{(0)}(n-1) = \boldsymbol{R}(n-1)$, and the resultant matrix $\boldsymbol{H}(n)$ is

$$\boldsymbol{H}(n) = \boldsymbol{H}^{(p)}(n)\boldsymbol{H}^{(p-1)}(n)\cdots\boldsymbol{H}^{(1)}(n) \tag{13}$$

where each $\boldsymbol{H}^{(i)}(n)$ represents a Householder transformation which zeros out the ith column of the updated $\boldsymbol{X}_n^T$, i.e., $\boldsymbol{x}_{n,i}^{(i-1)}$.

To obtain $\boldsymbol{H}^{(1)}(n)$, denote

$$z_1 = [r_{11} - \sigma_1 \vdots \mathbf{0}_{(n-1)k-1}^T \vdots \boldsymbol{x}_{n,1}^T]^T$$

where r_{11} is the (1, 1) element of $\boldsymbol{R}(n-1)$, $\sigma_1^2 = r_{11}^2 + \|\boldsymbol{x}_{n,1}\|^2$. Then from (11)

$$\boldsymbol{H}^{(1)}(n) = \begin{bmatrix}h_{11}^{(1)}(n) & \vdots & \mathbf{0}^T & \vdots & \boldsymbol{h}_{12}^{(1)T}(n)\\ \hline \mathbf{0} & \vdots & \boldsymbol{I}_{(n-1)k-1} & \vdots & \mathbf{0}\\ \hline \boldsymbol{h}_{21}^{(1)}(n) & \vdots & \mathbf{0} & \vdots & \boldsymbol{H}_{22}^{(1)}(n)\end{bmatrix} \tag{14}$$

where $h_{11}^{(1)}(n)$ is a scalar, $\boldsymbol{h}_{12}^{(1)}(n)$ is a $k \times 1$ vector, $\boldsymbol{h}_{21}^{(1)}(n) = \boldsymbol{h}_{12}^{(1)}(n)$, and $\boldsymbol{H}_{22}^{(1)}(n)$ is a $k \times k$ matrix given by

$$\boldsymbol{H}_{22}^{(1)}(n) = \boldsymbol{I}_k - \frac{2\boldsymbol{x}_{n,1}\boldsymbol{x}_{n,1}^T}{\sigma_{z_1}^2} \tag{15}$$

with $\sigma_{z_1}^2 = \|z_1\|_2^2 = 2(\sigma_1^2 - \sigma_1 r_{11})$. Define $\psi_1 = \sigma_1^2 - \sigma_1 r_{11}$, (15) can be rewritten in a form without multiplication of the number 2 as

$$\boldsymbol{H}_{22}^{(1)}(n) = \boldsymbol{I}_k - \frac{\boldsymbol{x}_{n,1}\boldsymbol{x}_{n,1}^T}{\psi_1}.$$

In general,

$$\boldsymbol{H}^{(m)}(n) = \begin{bmatrix}\boldsymbol{H}_{11}^{(m)}(n) & \vdots & \mathbf{0} & \vdots & \boldsymbol{H}_{12}^{(m)}(n)\\ \cdots & \vdots & \cdots & \vdots & \cdots\\ \mathbf{0} & \vdots & \boldsymbol{I}_{(n-1)k-p} & \vdots & \mathbf{0}\\ \cdots & \vdots & \cdots & \vdots & \cdots\\ \boldsymbol{H}_{21}^{(m)}(n) & \vdots & \mathbf{0} & \vdots & \boldsymbol{H}_{22}^{(m)}(n)\end{bmatrix} \tag{16}$$

where $\boldsymbol{H}_{11}^{(m)}(n) \in \mathfrak{R}^{p\times p}$ is an identity matrix except for the mth diagonal entry; $\boldsymbol{H}_{12}^{(m)}(n) \in \mathfrak{R}^{p\times k}$ is a zero matrix except for the mth row; $\boldsymbol{H}_{21}^{(m)}(n) = \boldsymbol{H}_{12}^{(m)T}(n)$; and

$$\boldsymbol{H}_{22}^{(m)}(n) = \boldsymbol{I}_k - \frac{\boldsymbol{x}_{n,m}^{(m-1)}\boldsymbol{x}_{n,m}^{(m-1)T}}{\psi_m} \in \mathfrak{R}^{k\times k} \tag{17}$$

is symmetric with $\psi_m = \sigma_m^2 - \sigma_m r_{mm}$, where $\sigma_m^2 = r_{mm}^2 + \|\boldsymbol{x}_{n,m}^{(m-1)}\|^2$. It can be easily seen that $\boldsymbol{H}_{12}^{(i)}(n)\boldsymbol{H}_{21}^{(j)}(n) = \mathbf{0}$, $\boldsymbol{H}_{21}^{(i)}(n)\boldsymbol{H}_{12}^{(j)}(n) = \mathbf{0}$, for $\forall i \neq j$. Thus we have the following lemma.

Lemma 1: The Householder transformation matrix, $\boldsymbol{H}(n) \in \mathfrak{R}^{nk\times nk}$, is orthogonal and is of the form

$$\boldsymbol{H}(n) = \begin{bmatrix}\boldsymbol{H}_{11}(n) & \vdots & \mathbf{0} & \vdots & \boldsymbol{H}_{12}(n)\\ \cdots & \vdots & \cdots & \vdots & \cdots\\ \mathbf{0} & \vdots & \boldsymbol{I}_{(n-1)k-p} & \vdots & \mathbf{0}\\ \cdots & \vdots & \cdots & \vdots & \cdots\\ \boldsymbol{H}_{21}(n) & \vdots & \mathbf{0} & \vdots & \boldsymbol{H}_{22}(n)\end{bmatrix} = \boldsymbol{H}^{(p)}(n)\boldsymbol{H}^{(p-1)}(n)\cdots\boldsymbol{H}^{(1)}(n) \tag{18}$$

with

$$\boldsymbol{H}_{11}(n) = \boldsymbol{H}_{11}^{(p)}(n)\cdots\boldsymbol{H}_{11}^{(2)}(n)\boldsymbol{H}_{11}^{(1)}(n)$$

$$\boldsymbol{H}_{22}(n) = \boldsymbol{H}_{22}^{(p)}(n)\cdots\boldsymbol{H}_{22}^{(2)}(n)\boldsymbol{H}_{22}^{(1)}(n). \quad \square \tag{19}$$

For the block size of $k = 1$ the Givens rotation method reduces to the special case of the rank-1 update Householder transformation [5], and the $\boldsymbol{H}$ matrix in Lemma 1 becomes a Givens rotation matrix $\boldsymbol{G}$ of the form [18]

$$\boldsymbol{G}(n) = \begin{bmatrix}\boldsymbol{K}(n) & \mathbf{0} & \boldsymbol{h}(n)\\ \mathbf{0} & \boldsymbol{I}_{n-p-1} & \mathbf{0}\\ \boldsymbol{h}^H(n) & \mathbf{0} & \gamma(n)\end{bmatrix}$$

where $\boldsymbol{K}(n)$ is a $p \times p$ matrix, $\boldsymbol{h}(n)$ is a $p \times 1$ vector, and $\gamma(n)$ is a scalar given by $\gamma(n) = \Pi_{i=1}^{p} c_i(n)$, $n \geq p$ where $c_i(n)$ is the cosine parameter associated with the ith Givens rotation.

A. Vectorized SBHT QRD Systolic Array

Now we propose a vectorized systolic array to implement the QRD based on the SBHT. Similar to the *QR* triarray of Gentleman–Kung [4], this array has both boundary and internal cells. The boundary cell takes an input of block size k from the above internal processor or directly from the input port, updates its content and generates the reflection vector, and sends it to the right for the internal cell processing (see Fig. 2(a)). Define

$$\bar{\boldsymbol{x}}_{n,i}^{(i-1)T} = [\mathbf{0}_{i-1} \vdots r_{ii} \vdots \mathbf{0}_{(n-1)k-i} \vdots \boldsymbol{x}_{n,i}^{(i-1)T}],$$

$$i = 1, \cdots, p$$

and $z_i = \bar{\boldsymbol{x}}_{n,i}^{(i-1)} - \sigma_i \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is a zero vector except for a unity at the ith position. When an internal cell receives the reflection vector, instead of forming the matrix $z_i z_i^T$ and performing matrix arithmetics, it performs an inner product operation to update its content r_{ii} by doing

$$\boldsymbol{H}^{(i)}(n)\bar{\boldsymbol{x}}_{n,j}^{(i-1)} = \bar{\boldsymbol{x}}_{n,j}^{(i-1)} - \frac{\boldsymbol{z}_i}{\psi_i}(z_i^T \cdot \bar{\boldsymbol{x}}_{n,j}^{(i-1)})$$

$$j = i+1, \cdots, p \tag{20}$$

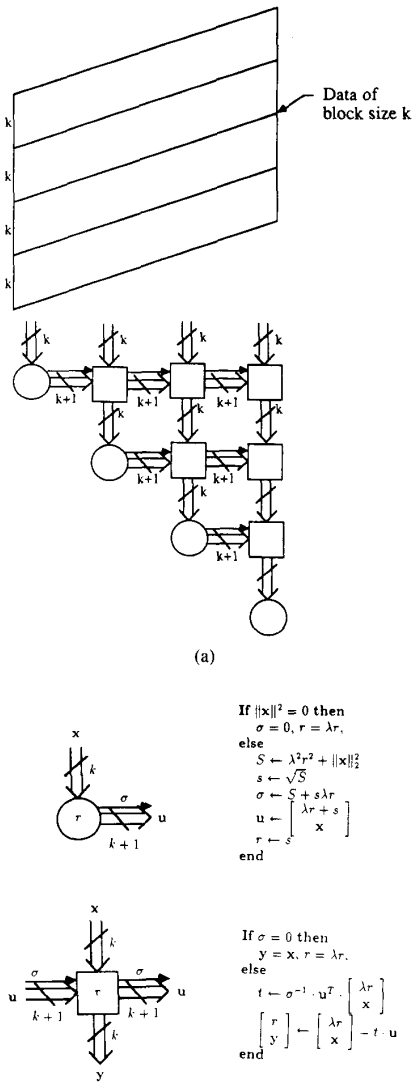


Fig. 2. (a) SBHT QRD systolic array. (b) Processing cells of the SBHT QRD systolic array.

and sends the reflected data $x_{n,j}$ downward for further processing. Fig. 2 shows the SBHT QRD array architecture and the processing cells. When the block size is $k = 1$, this vectorized array degenerates to the Gentleman–Kung's Givens rotation triarray.

IV. SBHT RLS Algorithm

The LS problem is to choose a weight vector $\boldsymbol{w}(n) \in \mathcal{R}^p$, such that the block-forgetting norm of

$$\boldsymbol{\varepsilon}(n) = \begin{bmatrix} \boldsymbol{e}_1(n) \\ \boldsymbol{e}_2(n) \\ \vdots \\ \boldsymbol{e}_n(n) \end{bmatrix} = \boldsymbol{X}(n)\boldsymbol{w}(n) - \boldsymbol{y}(n) \tag{21}$$

is minimized. The optimal weight vector $\hat{\boldsymbol{w}}(n)$ satisfies

$$\min_{\boldsymbol{w}} \|\boldsymbol{\varepsilon}(n)\|_{\Lambda_k} = \|\boldsymbol{X}(n)\hat{\boldsymbol{w}}(n) - \boldsymbol{y}(n)\|_{\Lambda_k} \tag{22}$$

where

$$\|\boldsymbol{\varepsilon}(n)\|_{\Lambda_k} = \|\boldsymbol{\Lambda}_k(n)\boldsymbol{\varepsilon}(n)\|_2 = \sqrt{\sum_{i=1}^{n} \lambda^{2(n-i)} \cdot \|\boldsymbol{e}_i(n)\|_2^2}$$
$$0 < \lambda \le 1. \tag{23}$$

$\boldsymbol{\Lambda}_k(n)$ is a block-diagonal exponential weighting matrix of the form

$$\boldsymbol{\Lambda}_k(n) = \begin{bmatrix} \lambda^{n-1}\boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \lambda\boldsymbol{I}_k & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I}_k \end{bmatrix} \in \mathcal{R}^{nk \times nk} \tag{24}$$

and $\|\cdot\|_2$ is the Euclidean norm,

$$\|\boldsymbol{e}_i(n)\|_2^2 = \sum_{j=1}^{k} |e_{(i-1)k+j}(n)|^2. \tag{25}$$

The exponential forgetting weighting λ is incorporated in the RLS filtering scheme to avoid overflow in the processors as well as to facilitate nonstationary data updating.

The QRD of the weighted augmented data matrix at time n (in the block sense, it is equivalent to nk snapshots), is given by

$$\boldsymbol{\Lambda}_k(n)[\boldsymbol{X}(n) \vdots \boldsymbol{y}(n)] = [\boldsymbol{Q}_1^T(n) \vdots \boldsymbol{Q}_2^T(n)] \begin{bmatrix} \boldsymbol{R}(n) & \vdots & \boldsymbol{u}(n) \\ \boldsymbol{0} & \vdots & \boldsymbol{v}(n) \end{bmatrix} \tag{26}$$

where

$$\boldsymbol{Q}(n) = \begin{bmatrix} \boldsymbol{Q}_1(n) \\ \boldsymbol{Q}_2(n) \end{bmatrix}$$

constitutes an orthogonal transformation matrix with $\boldsymbol{Q}_1(n) \in \mathcal{R}^{p \times nk}$ spanning the column space of the weighted data matrix $\boldsymbol{\Lambda}_k(n)\boldsymbol{X}(n)$ and $\boldsymbol{Q}_2(n) \in \mathcal{R}^{(nk-p) \times nk}$ spanning the null space, $\boldsymbol{R}(n) \in \mathcal{R}^{p \times p}$ is an upper triangular matrix and

$$\boldsymbol{Q}(n)\boldsymbol{y}(n) = \begin{bmatrix} \boldsymbol{u}(n) \\ \boldsymbol{v}(n) \end{bmatrix}.$$

The optimal weight vector can be obtained by solving

$$\boldsymbol{R}(n)\hat{\boldsymbol{w}}(n) = \boldsymbol{u}(n). \tag{27}$$

Obviously, $\boldsymbol{\Lambda}_k(n)\boldsymbol{X}(n) = \boldsymbol{Q}_1^T(n)\boldsymbol{R}(n)$. As a result, the weighted optimal residual of (21) is

$$\begin{aligned} \boldsymbol{\Lambda}_k(n)\hat{\boldsymbol{\varepsilon}}(n) &= \boldsymbol{Q}_1^T(n)\boldsymbol{R}(n)\hat{\boldsymbol{w}}(n) - \boldsymbol{Q}_1^T(n)\boldsymbol{u}(n) - \boldsymbol{Q}_2^T(n)\boldsymbol{v}(n) \\ &= -\boldsymbol{Q}_2^T(n)\boldsymbol{v}(n) \end{aligned} \tag{28}$$

which lies in the null space of the weighted data matrix.

Now, suppose we have the data matrix up to time $n-1$ and the QRD of $\Lambda_k(n-1)[X(n-1) \vdots y(n-1)]$. The recursive LS problem here is to compute efficiently the optimum residual at time n from the results we have at time $n-1$. In particular, we are interested in the new nth block of the optimal residual

$$\hat{e}_n(n) = X_n^T \hat{w}(n) - y_n. \tag{29}$$

From (8), (9), and (18), (26) can be expressed as

$$\begin{bmatrix} R(n) & \vdots & u(n) \\ \cdots & \cdot & \cdots \\ 0 & \vdots & v(n) \end{bmatrix} = \begin{bmatrix} H_{11}(n) & \vdots & 0 & \vdots & H_{12}(n) \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ 0 & \vdots & I_{(n-1)k-p} & \vdots & 0 \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ H_{21}(n) & \vdots & 0 & \vdots & H_{22}(n) \end{bmatrix} \cdot \begin{bmatrix} \lambda R(n-1) & \vdots & \lambda u(n-1) \\ \cdots & \cdot & \cdots \\ 0 & \vdots & \lambda v(n-1) \\ \cdots & \cdot & \cdots \\ X_n^T & \vdots & y_n \end{bmatrix}.$$

$$\Lambda_k(n)\hat{\varepsilon}(n) = \begin{bmatrix} \hat{\varepsilon}(n-1|n) \\ \hat{e}_n(n) \end{bmatrix} = \begin{bmatrix} -\lambda Q_2^T(n-1)v(n-1) - Q_1^T(n-1)H_{21}^T(n)v_n \\ -H_{22}^T(n)v_n \end{bmatrix} \tag{34}$$

By recursion on n, we relate $Q(n)$ and $Q(n-1)$ using (10) and have

$$Q(n) = \begin{bmatrix} H_{11}(n) & \vdots & 0 & \vdots & H_{12}(n) \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ 0 & \vdots & I_{(n-1)k-p} & \vdots & 0 \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ H_{21}(n) & \vdots & 0 & \vdots & H_{22}(n) \end{bmatrix} \cdot \begin{bmatrix} Q_1(n-1) & \vdots & 0 \\ \cdots & \cdot & \cdots \\ Q_2(n-1) & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & I_k \end{bmatrix}.$$

$$= \begin{bmatrix} H_{11}(n)Q_1(n-1) & \vdots & H_{12}(n) \\ \cdots & \cdot & \cdots \\ Q_2(n-1) & \vdots & 0 \\ \cdots & \cdot & \cdots \\ H_{21}(n)Q_1(n-1) & \vdots & H_{22} \end{bmatrix}. \tag{30}$$

We can see that $Q_2(n)$ is updated from $Q_1(n-1)$ and $Q_2(n-1)$ by

$$Q_2(n) = \begin{bmatrix} Q_2(n-1) & \vdots & 0 \\ \cdots & \cdot & \cdots \\ H_{21}(n)Q_1(n-1) & \vdots & H_{22} \end{bmatrix}. \tag{31}$$

On the other hand, the updated $[u^T(n), v^T(n)]^T$ is

$$\begin{bmatrix} u(n) \\ \cdots \\ v(n) \end{bmatrix} = Q(n)\Lambda_k(n) \begin{bmatrix} y(n-1) \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda H_{11}(n)u(n-1) + H_{12}(n)y_n \\ \cdots \\ \lambda v(n-1) \\ \cdots \\ v_n \end{bmatrix} \tag{32}$$

where

$$v_n = \lambda H_{21}(n)u(n-1) + H_{22}(n)y_n. \tag{33}$$

Therefore, from (28), (31), and (32), the weighted optimal residual vector can be obtained from parameters at time $n-1$ by

where $\hat{\varepsilon}(m|n)$ denotes the estimate of $\hat{\varepsilon}$ at time m, $m \le n$, given all of the data up to time n. The new nth block of the optimal residual is then obtained as

$$\hat{e}_n(n) = -H_{22}^T(n)v_n = -H_{22}^{(1)}(n)H_{22}^{(2)}(n) \cdots H_{22}^{(p)}(n)v_n. \tag{35}$$

For the block size of $k = 1$, all vector parameters in (35) become scalars and can be expressed as

$$e_n(n) = -\prod_{i=1}^{p} c_i v_n \tag{36}$$

which was first shown by McWhirter in [18]. Note that there are some differences between the optimal residuals estimated by SBHT and Givens rotation methods. To be specific, the optimal residual vector in (35) is given by

$$\hat{e}_n(n) = \begin{bmatrix} e_{(n-1)k+1}((n-1)k+1|nk) \\ \vdots \\ e_{nk-1}(nk-1|nk) \\ e_{nk}(nk|nk) \end{bmatrix} \tag{37}$$

while, the optimal residual estimated by the Givens rotation method in (36) is

$$\hat{e}_n(n) = e_n(n|n). \tag{38}$$

In this sense, the SBHT RLS gives a better estimate of the residual since it uses more data samples to estimate the optimal residual. As an example, consider $k = 2$. Then the optimal residuals obtained from the SBHT RLS and Givens methods are $[e_{2n-1}((2n-1)|2n), e_{2n}(2n|2n)]$ and $[e_{2n-1}((2n-1)|(2n-1), e_{2n}(2n|2n)]$, respectively. It is clear now that the SBHT RLS method gives a better estimate for the previous residual than the Givens rotation method because the former makes use of the future data sample at time $2n$ to estimate the residual at time $2n-1$, while the latter does not.

A. Vectorized SBHT RLS Array

In order to obtain the RLS filtering residual vector in the systolic array, we can use two possible approaches. The first approach is to generalize the architecture of McWhirter's Givens rotation approach [18]. A SBHT QRD array with a RA based on this approach is shown in Fig. 3. Since the $\boldsymbol{v}_n$ in (33) results from the reflection computation in (32), therefore $\boldsymbol{v}_n$ is obtained naturally from the output of RA. Each boundary cell then forms the matrix $\boldsymbol{H}_{22}^{(\cdot)}$ and propagates it down the diagonal boundary cells. Since $\boldsymbol{H}_{22}^{(i)}$ is generated earlier than $\boldsymbol{H}_{22}^{(j)}$ for $i < j$, (35) has to be computed from left to right involving matrix–matrix multiplications. As a result, each boundary cell performs the matrix multiplication to accumulate $\boldsymbol{H}_{22}^{(\cdot)}$ when it is propagated down diagonal boundary cells. The matrix multiplications needed in the boundary cells in this approach are objectionable since they not only slow down the throughput but also increase the complexity of the boundary cells. We note, McWhirter's original approach based on Givens rotation worked well since only scalars need to be propagated down the diagonal boundary cells and the order of multiplications for the scalars is irrelevant.

Obviously, we prefer to compute (35) from right to left such that only inner product computations are performed. Instead of forming the matrix $\boldsymbol{H}_{22}^{(\cdot)}$ and propagating it down, another approach is to use the facts that $\boldsymbol{H}_{22}^{(\cdot)}$ can be expressed by using (17) and the reflection vectors are sent to the right from boundary cells as described in Section III-A. From these observations, (35) can be computed in a manner similar to the internal cell operation. A new architecture shown in Fig. 4 is thus introduced to circumvent this problem. A column array of internal cells called backward propagation array (BPA) is added at the right-hand side to perform the backward propagation of $\boldsymbol{v}_n$. Each row, say the ith one, needs $2(p-i)$ delayed buffers as shown in Fig. 4. The $\boldsymbol{v}_n$ obtained at the output of RA is then backward propagated through the BPA. From (17), each cell of this array performs the operation

$$\boldsymbol{H}_{22}^{(i)}(n)\tilde{\boldsymbol{v}}_n = \tilde{\boldsymbol{v}}_n - \frac{\boldsymbol{x}_{n,i}^{(i-1)}}{\psi_i}\left(\boldsymbol{x}_{n,i}^{(i-1)T}\tilde{\boldsymbol{v}}_n\right)$$

$$i = p, \cdots, 2, 1 \qquad (39)$$

where $\tilde{\boldsymbol{v}}_n$ is an updated $\boldsymbol{v}_n$. This is a subset of the operations performed by the internal cell shown in (20). The residual vector is obtained from the top of the newly appended column array.

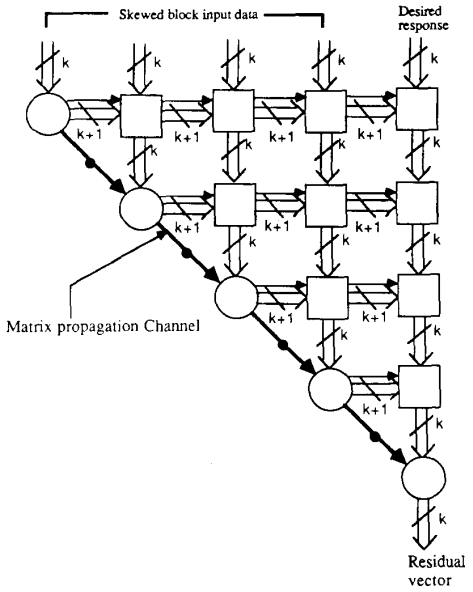


Fig. 3. SBHT RLS systolic array obtained by direct generalization of the Givens rotation array.

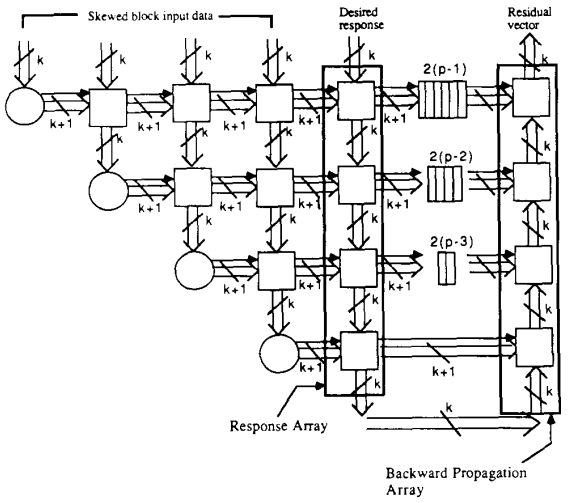


Fig. 4. New matrix-multiplication-free SBHT RLS systolic array.

The costs for this proposed architecture are: an increased latency time from $(2p+1)t_s$ of McWhirter's Givens method to $3pt_v$, where t_s represents the processing time for the scalar operations used in the Givens rotation method and t_v is the processing time for vector operations used in the SBHT method; the number of delay elements needed increases from p to $\Sigma_1^p 2(p-i) = p(p-1)$; and p additional internal processing cells. The operations of the boundary and internal cells are still given in Fig. 2(b).

These results clearly show that HT can be implemented simply on a systolic array to achieve massive parallel processing with vector operations. This provides an efficient method to obtain a high throughput rate for recursive LS filtering by using the HT method.

V. Two-Level Pipelined Implementations

The SBHT QRD array and the RLS array discussed in the above sections are derived using the conventional Householder transformation as shown in (11). Due to the vector processing nature of the conventional method, the cells of both arrays perform vector operations such as inner products. This means the complexity of each cell is high and the I/O bandwidth is large in order to achieve an effective vector data communication. Each cell, due to the complexity of vector processing, may require a large processor. Clearly, this is not desirable for VLSI implementation. Thus, we are motivated to find a suitable algorithm to pipeline the data down to the word level such that the I/O bandwidth as well as the complexity can be reduced. In addition, we still wish to achieve a high throughput which is needed in many modern signal processing applications.

The conventional approach in computing Householder transformation, $\boldsymbol{y} = \boldsymbol{T}\boldsymbol{q}$, based on (11) is to form first z and $\|z\|^2$ from $\boldsymbol{x}$ and then $z^T\boldsymbol{q}/\|z\|^2$ and $\boldsymbol{q} - 2z(z^T\boldsymbol{q}/\|z\|^2)$ as considered before. It can be stated in the following form:

HT Algorithm (Conventional):

Step 1. $S_{xx} = \|\boldsymbol{x}\|^2$.

Step 2. If $S_{xx} = 0$, then $\boldsymbol{y} = \boldsymbol{q}$.

Step 3. If $S_{xx} \neq 0$ then

(1) $s = \sqrt{S_{xx}}$, $z = \boldsymbol{x} + [s, 0, 0, \cdots, 0]^T$,

(2) $\phi = S_{xx} + sx_1$, $S_{zq} = \boldsymbol{z}^T\boldsymbol{q}$,

(3) $d = S_{zq}/\phi$, $\boldsymbol{y} = \boldsymbol{q} - dz$.

In [25], Tsao pointed out that by skipping the computation of ϕ and avoiding the cumbersome intermediate steps of forming vector z for further computations, a modified algorithm with smaller round off error and fewer operations can be obtained. Only step 3 of the conventional algorithm is modified as follows.

Modified HT Algorithm [25]:

Step 3. If $S_{xx} \neq 0$ then

(1) $s = \sqrt{S_{xx}}$, $\sigma = x_1 + s$,

(2) $S_{xq} = \boldsymbol{x}^T\boldsymbol{q}$,

(3) $y_1 = -S_{xq}/s$, $d = (q_1 - y_1)/\sigma$, $y_i = q_i - dx_i$, $i = 2, \cdots, n$.

With this algorithm, the operations of the cells of the vectorized systolic arrays can be modified as shown in Fig. 5. As we can see, for the boundary cell, the vector $\boldsymbol{u}$, which consists of the weighted diagonal element of the upper triangular matrix and one column of the input data block (updated or not), can be sent out immediately when the input $\boldsymbol{x}$ is available, without waiting for any computations as required in the implementation using the conventional algorithm. Due to this advantage and the modified operations in the internal cells, we can then pipeline the vector operations down to the word level such that each cell only performs scalar operations, which will significantly reduce its complexity.

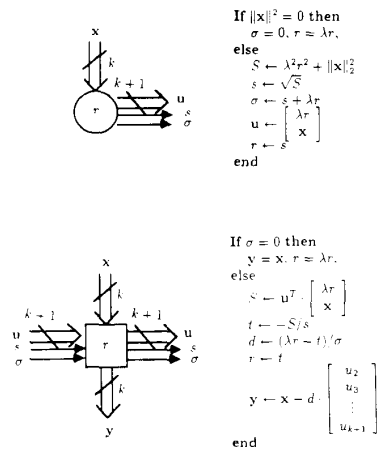
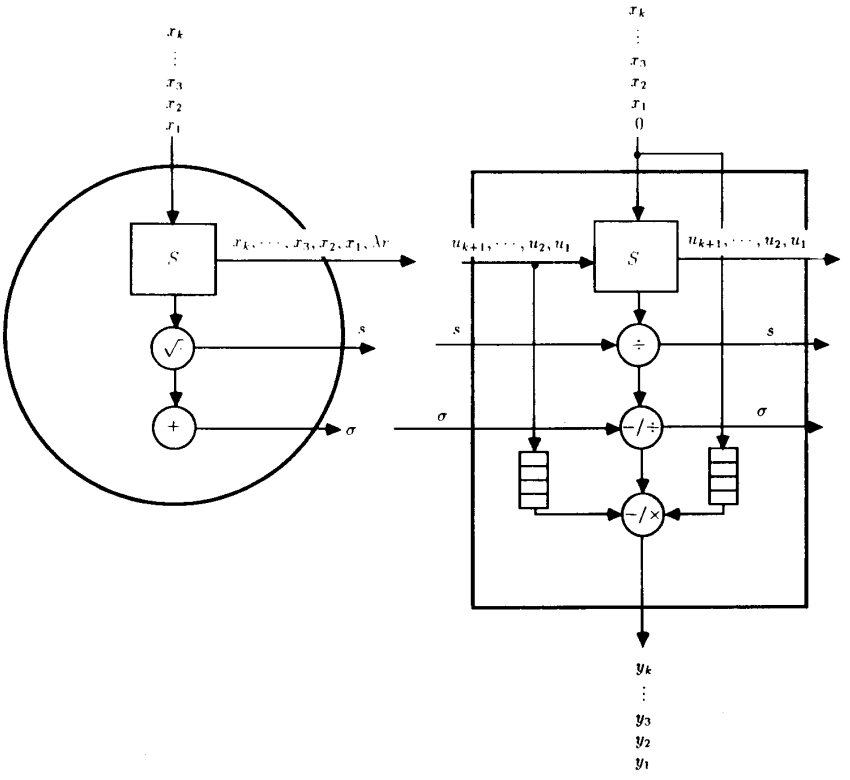
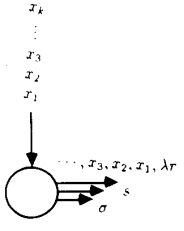


Fig. 5. Operations of the processing cells by using the modified Householder transformation.

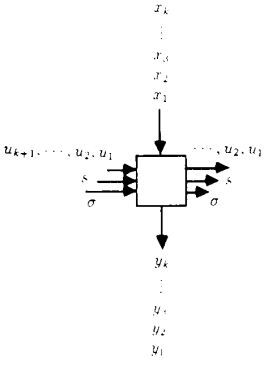
A two-level pipelined implementation of the modified HT algorithm is given in Fig. 6(a). The boundary cell performs three major functions: square and accumulate, square root, and addition. For each data block, the boundary cell fetches one data sample, accumulates the squares of the samples, and sends the data to the right for internal cell. When all the data of the block are processed, the content of S is then sent down for square-root operation. The resultant s is sent to the right for internal cell as well as sent down to obtain σ, which is then sent to the right when available. At the same time, when an internal cell receives a u_i, it multiplies u_i with an input x_i and accumulates all these products to obtain S. When S is available, it is sent down for division operation with s, which arrives at the same time, to obtain t; then t is sent down and σ again arrives at the same time to compute d. To compute y_i of (3) in step 3, we need registers to store u_i and x_i temporarily. Since data from the next block are continuously being sent into the system, each internal cell needs $2(k + 3)$ registers to store u_i and x_i as indicated in Fig. 6(a). When d is available, the y_i are then obtained one by one and sent down for further processing. Data from the next block undergo the same processing. When a new d is available in the internal cell, the corresponding x_i and u_i are already waiting in the registers. Therefore the vector operations are successfully pipelined down to the word level. This means that by using the modified HT algorithm, we have not only pipelined the SBHT arrays at the vector level but also at the word level. The input data is now skewed in the word level as shown in Fig. 6(a) rather than in the vector level as shown in Fig. 4. The functional descriptions of the processing cells for two-level pipelined implementation are given in Fig. 6(b).



(a)



send_right λr
$S \leftarrow \lambda^2 r^2$
do $i = 1, k$
  fetch x_i; $S \leftarrow S + x_i^2$; send_right x_i,
end do
$r' \leftarrow \lambda r$; $s \leftarrow \sqrt{S}$; send_right s,
$r \leftarrow s$; $\sigma \leftarrow r' + s$; send_right σ.



$S = 0$.
do $i = 1, K + 1$
fetch u_i.
if $i = 1$ **then**
$S \leftarrow S + u_i \lambda r$; send_right u_i.
else
fetch x_{i-1}; $S = S + u_i x_{i-1}$; send_right u_i.
end if
fetch s; $r' \leftarrow \lambda r$; $t \leftarrow -S/s$.
fetch σ; $r \leftarrow t$; $d \leftarrow (r' - t)/\sigma$.
end do
do $i = 1, k$
$y_i = x_i - d u_{i+1}$; send_down y_i.
end do

(b)

Fig. 6. (a) Architectures of the processing cells for two-level pipelined implementation. (b) Functional descriptions of processing cells for two-level pipelined implementation.

TABLE I
COMPARISONS OF THE SBHT AND GIVENS ROTATION METHODS

	Givens Rotation	SBHT
Number of cells	$(p^2 + 3p)/2$	$(p^2 + 5p)/2$
Number of delay elements	p	$p(p - 1)$
Number of registers	0	$(p^2 + 3p)(k + 3)$
System latency	$2p + 1$	$2p(k + 4)$
Cell complexity	less	higher
Numerical stability	good	better

Since the most time consuming operation of this two-level pipelined implementation is the square root operation which is also the critical operation in McWhirter's Givens rotation implementation, the throughput of this two-level pipelined implementation is as fast as that of the McWhirter's Givens array. However, with a longer pipeline, a longer system latency for the SBHT method is obtained. This is due to the fact that the registers of the internal cells have to be filled before we can obtain the residual vector. For the SBHT RLS systolic array of order p, we have $(p^2 + 3p)/2$ internal cells, including the BPA. Thus, there are a total of $(p^2 + 3p)(k + 3)$ registers for the whole system. The system latency is given by $t_s = 2p(k + 4)$, which is linearly proportional to p and k. However, for the Givens rotation method, the system latency is only $t_s = 2p + 1$. Comparisons of both RLS arrays based on the SBHT and Givens rotation are summarized in Table I. In general, the throughput of the SBHT RLS systolic array is as fast as the Givens rotation method. Of course, while the cell complexity of the SBHT array is higher, it does offer better numerical stability [26]. A detailed backward error analysis carried out by Wilkinson showed that for an $n \times n$ matrix A, after $n(n - 1)/2$ Givens rotations, the roundoff error in the upper triangular matrix is in the order of $\mathcal{O}(\kappa_g n^{3/2} \mu \|A\|)$ [26, p. 138], while a series of $(n - 1)$ HT gives $\mathcal{O}(\kappa_h n \mu \|A\|)$ [26, p. 160], with κ_g and κ_h being constants and μ a machine floating point computation precision.

VI. CONSTRAINED RLS PROBLEMS

In the above sections, we have dealt with an unconstrained RLS problem. The RLS systolic array considered there was motivated originally by the sidelobe canceller beamformation problem [18]. Other practical motivation could have come from the adaptive filtering problem [6]. However, there are other signal processing applications which are modeled by a constrained RLS problem. The MVDR beamformation constitutes such an example [19], [20], [23]. It is interesting to determine whether a systolic array for an unconstrained RLS problem can also be used for a constrained RLS problem. In [19], McWhirter and Shepherd showed an extension of the unconstrained RLS array to the MVDR beamforming problem. Based on their approach, we shall also demonstrate the implementation of a MVDR beamformation problem using a SBHT RLS array.

The MVDR beamforming problem is to minimize

$$\xi^{(l)}(n) = \|\boldsymbol{X}(n)\boldsymbol{w}^{(l)}(n)\|_\Lambda, \qquad l = 1, \cdots, L \tag{40}$$

subject to the linear constaints of

$$\boldsymbol{c}^{(l)T}\boldsymbol{w}^{(l)}(n) = \beta^{(l)}, \qquad l = 1, \cdots, L \tag{41}$$

where L is the number of constraints. We are interested in the *a posteriori* residual vector

$$\hat{\boldsymbol{e}}_n^{(l)}(n) = \boldsymbol{X}_n^T \hat{\boldsymbol{w}}^{(l)}(n). \tag{42}$$

The optimal solution of the weight vector is known [19] to be given by

$$\hat{\boldsymbol{w}}^{(l)}(n) = \frac{\beta^{(l)}\boldsymbol{M}^{-1}(n)\boldsymbol{c}^{(l)}}{\boldsymbol{c}^{(l)T}\boldsymbol{M}^{-1}(n)\boldsymbol{c}^{(l)}} = \frac{\beta^{(l)}\boldsymbol{R}^{-1}(n)\boldsymbol{a}^{(l)}(n)}{\|\boldsymbol{a}^{(l)}(n)\|^2} \tag{43}$$

where $\boldsymbol{M} = \boldsymbol{X}^T(n)\boldsymbol{\Lambda}_k^2(n)\boldsymbol{X}(n)$ is the weighted covariance matrix, $\boldsymbol{R}(n)$ is the upper triangular matrix resulted from the QRD of the weighted data matrix $\boldsymbol{\Lambda}_k\boldsymbol{X}(n)$, and

$$\boldsymbol{a}^{(l)}(n) = \boldsymbol{R}^{-T}(n)\boldsymbol{c}^{(l)}. \tag{44}$$

Therefore the optimal residual vector at time n is

$$\hat{\boldsymbol{e}}_n^{(l)}(n) = \frac{\beta^{(l)}}{\|\boldsymbol{a}^{(l)}(n)\|^2} \cdot \boldsymbol{X}_n^T\boldsymbol{R}^{-1}(n)\boldsymbol{a}^{(l)}(n). \tag{45}$$

A crucial step needed is for the efficient recursive updating of $\boldsymbol{a}^{(l)}(n)$. A novel approach was proposed for performing this updating [19]. Specifically, from (8), (9), and (44)

$$\begin{aligned}\boldsymbol{c}^{(l)} &= \boldsymbol{R}^T(n-1)\boldsymbol{a}^{(l)}(n-1) \\ &= \lambda^{-2}[\lambda\boldsymbol{R}^T(n-1) \,\vdots\, \boldsymbol{0}^T \,\vdots\, \boldsymbol{X}_n]\begin{bmatrix}\lambda\boldsymbol{a}^{(l)}(n-1) \\ \hline \lambda\boldsymbol{b}^{(l)}(n-1) \\ \hline \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{46}$$

where $\boldsymbol{b}^{(l)}(n-1)$ is an arbitrary $((n-1)k - p) \times 1$ vector. Then from Lemma 1, (8), and (9), we have

$$\begin{aligned}\boldsymbol{c}^{(l)} &= \lambda^{-2}[\lambda\boldsymbol{R}^T(n-1) \,\vdots\, \boldsymbol{0}^T \,\vdots\, \boldsymbol{X}_n]\boldsymbol{H}^T(n)\boldsymbol{H}(n) \\ &\quad \cdot \begin{bmatrix}\lambda\boldsymbol{a}^{(l)}(n-1) \\ \hline \lambda\boldsymbol{b}^{(l)}(n-1) \\ \hline \boldsymbol{0}\end{bmatrix} \\ &= \boldsymbol{R}^T(n) \cdot \lambda^{-2}(\lambda\boldsymbol{H}_{11}(n)\boldsymbol{a}^{(l)}(n-1)).\end{aligned} \tag{47}$$

Thus, $\boldsymbol{a}^{(l)}(n) = \lambda^{-2}(\lambda\boldsymbol{H}_{11}(n)\boldsymbol{a}^{(l)}(n-1))$ can be obtained by updating $\boldsymbol{a}^{(l)}(n-1)$ in a way similar to that $\boldsymbol{u}(n)$ is obtained by updating $\boldsymbol{u}(n-1)$ using (32). The only differences are the input for updating $\boldsymbol{a}^{(l)}(n-1)$ is a zero vector and a scaling factor λ^{-2}. Due to the structure of $\boldsymbol{H}$ in Lemma 1, the vector $\boldsymbol{b}^{(l)}(\cdot)$ plays no role in the updating of $\boldsymbol{a}^{(l)}(\cdot)$. Furthermore, from (27) and (29), we have

$$\hat{\boldsymbol{e}}_n(n) = \boldsymbol{X}_n^T\boldsymbol{R}^{-1}(n)\boldsymbol{u}(n) - \boldsymbol{y}_n. \tag{48}$$

From (32), we see that $\boldsymbol{u}(n)$ results from the update of

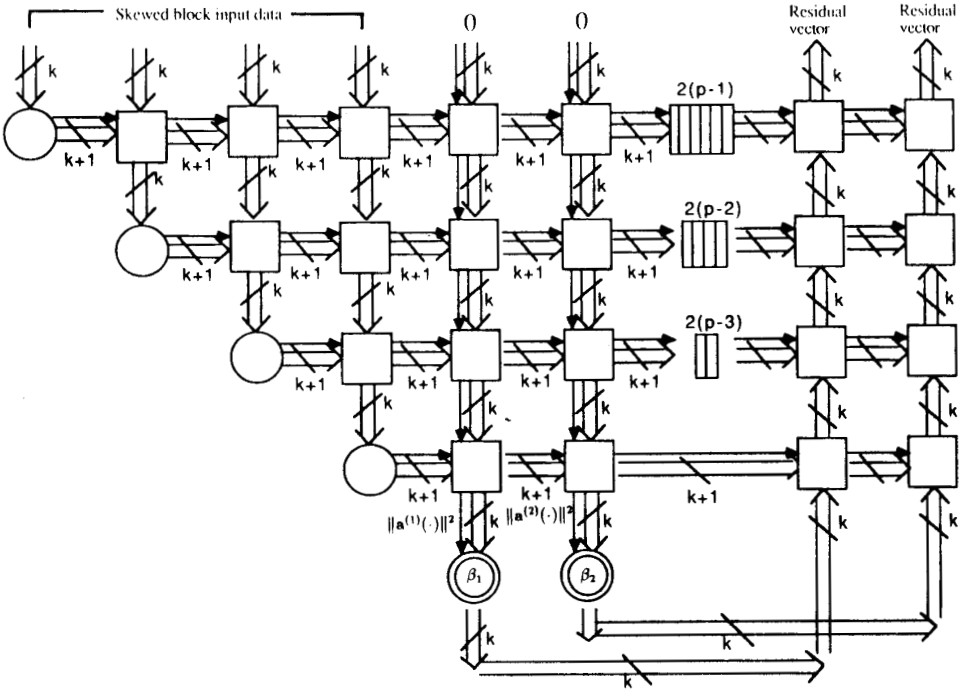


Fig. 7. MVDR beamforming using the SBHT systolic array.

$[y(n-1) \vdots y_n]$, where y_n is the new input. Now replacing $\boldsymbol{u}(n)$ with $\boldsymbol{a}^{(l)}(n)$ and y_n with a zero vector, we have

$$\hat{e}_n(n) = \boldsymbol{X}_n^T \boldsymbol{R}^{-1}(n)\boldsymbol{a}^{(l)}(n) \tag{49}$$

and from (45), we then obtain

$$\hat{e}_n^{(l)}(n) = \frac{\beta^{(l)}}{\|\boldsymbol{a}^{(l)}(n)\|^2} \cdot \hat{e}_n(n). \tag{50}$$

This equation reveals that by the proper scaling of $\hat{e}_n(n)$, which can be obtained from the SBHT RLS systolic array, we can obtain the *a posteriori* residual vector, $\hat{e}_n^{(l)}(n)$, of the MVDR beamformation. Fig. 7 shows an extension of the SBHT RLS array for the new problem. Now one more data channel is needed for the RA to pipeline cumulation of $\|\boldsymbol{a}^{(l)}(n)\|^2$, and the scaling of the residual vector is done at the bottom of the RA when a new $\|\boldsymbol{a}^{(l)}(\cdot)\|^2$ is available. Each RA/BPA pair in Fig. 7 represents one of the K constraints. The optimal *a posteriori* residual vector of each linear constraint is obtained at the output of the corresponding backward propagation array.

As pointed out in [19], there are two ways to initialize the array. One method is to set $\boldsymbol{R}(0) = \delta \boldsymbol{I}$, where δ is a small scalar, and thus from (44), $\boldsymbol{a}^{(l)}(0) = \delta^{-1}\boldsymbol{c}^{(l)}$, $l = 1, \cdots, L$. Another method is to obtain $\boldsymbol{R}(n)$ to some time n, then use (44) to obtain $\boldsymbol{a}^{(l)}(n)$. The details of a two-mode operation required for this initialization procedure are also considered in [19].

VII. Systolic Array for Hyperbolic Householder Transformation

In some applications, the fixed window approach is preferred to the exponentially weighted window approach. The updating of new data and downdating of old data must then be considered. Rader and Steinhardt [22] in 1986 proposed a hyperbolic Householder transformation (HHT) to simultaneously perform up/downdating. Let us define a J-hyperbolic Householder matrix H_J as follows:

$$H_J = J - 2\boldsymbol{h}\boldsymbol{h}^T/\|\boldsymbol{h}\|_J^2 \tag{51}$$

where $\boldsymbol{h}$ is a column vector, J is a pseudoidentity matrix

$$J = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & -I_k \end{bmatrix}$$

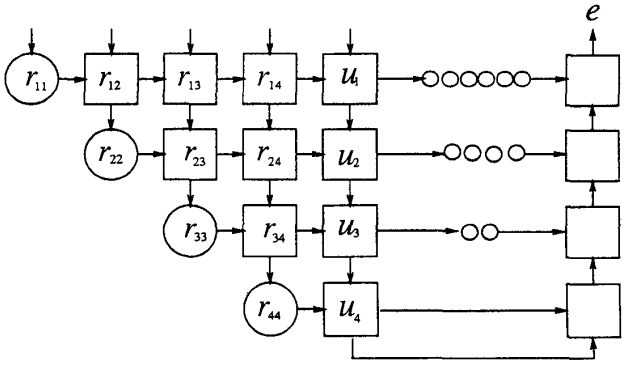
with I_p representing preserving the previous Cholesky factor, I_k incorporating the new data for updating, $-I_k$ discarding the old data for downdating, and

$$\|h\|_J^2 = \sum_{i=1}^{p} h_i^2 + \sum_{i=p+1}^{p+k} h_i^2 - \sum_{i=p+k+1}^{p+2k} h_i^2$$

is the J-pseudo vector norm.

We note that H_J is Hermitian and J-pseudo orthogonal, namely,

$$H_J = H_J^T \tag{52}$$



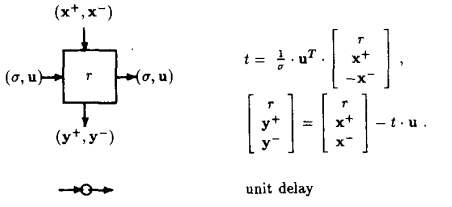


Fig. 8. Block hyperbolic Householder systolic array.

and

$$H_J^T J H_J = J. \tag{53}$$

We can compress all of the J-pseudo energy of a vector $\boldsymbol{a}$ into its jth entry by premultiplying it (performing pseudoorthogonal transformation) by H_J and choosing

$$\boldsymbol{h} = J\boldsymbol{a} + \alpha \boldsymbol{u}_j \tag{54}$$

with

$$\alpha = (\pm a_j / \|a_j\|)\|\boldsymbol{a}\|_J. \tag{55}$$

Here $\boldsymbol{u}_j$ is a unit vector with all zeros except for its jth entry. Then we have

$$H_J \boldsymbol{a} = -\alpha \boldsymbol{u}_j. \tag{56}$$

An algorithm using HHT to update the block data matrix $A = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p]$ and downdate the matrix $B = [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_p]$ from the Cholesky factor R is given below:

The HHT Up/Downdating Algorithm

For $i = 1, \cdots, p$, do
  $\tilde{r}_{ii} = \sqrt{r_{ii}^2 + \boldsymbol{a}_i^T \boldsymbol{a}_i - \boldsymbol{b}_i^T \boldsymbol{b}_i}$;
  if $r_{ii} < 0$, $\tilde{r}_{ii} = -\tilde{r}_{ii}$;
  For $j = (i+1), \cdots, p$, do
    $\tilde{r}_{ij} = (r_{ii} r_{ij} + \boldsymbol{a}_i^T \boldsymbol{a}_j - \boldsymbol{b}_i^T \boldsymbol{b}_j)/\tilde{r}_{ii}$;
    $\boldsymbol{a}_j = \boldsymbol{a}_j - (\tilde{r}_{ij} + r_{ij}/\tilde{r}_{ii} + r_{ii})\boldsymbol{a}_i$;
    $\boldsymbol{b}_j = \boldsymbol{b}_j - (\tilde{r}_{ij} + r_{ij}/\tilde{r}_{ii} + r_{ii})\boldsymbol{b}_i$;
    if $r_{ii} < 0$, $\boldsymbol{a}_j = -\boldsymbol{a}_j$; $\boldsymbol{b}_j = -\boldsymbol{b}_j$;
  End;
End.

Same as previous sections, it can be shown that $\hat{Q}^{\perp} = \hat{H}_J^{(1)\perp} \cdots \hat{H}_J^{(p)\perp}$, where $\hat{H}_J^{(j)\perp} \in \mathcal{R}^{2k \times 2k}$ is the lower right submatrix of the hyperbolic Householder reflection matrix

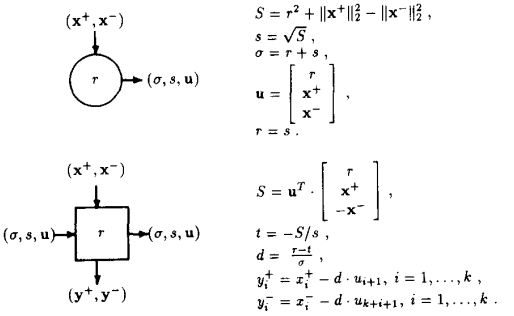


Fig. 9. Modified processing cells for the block HHT systolic array.

in zeroing out the jth column of appended data

$$\begin{bmatrix} X^+ & y^+ \\ X^- & y^- \end{bmatrix}$$

with $[X^+ \quad y^+]$ and $[X^- \quad y^-]$ representing the new and old data block to be up/downdated, respectively. The residual vector $\boldsymbol{e}$ therefore can be written as $\boldsymbol{e} = -\hat{H}_J^{(1)\perp} \cdots \hat{H}_J^{(p)\perp} \boldsymbol{v}$, which can be computed by a series of backward matrix-vector multiplications as given in previous sections. A block HHT systolic array for RLS filtering is given in Fig. 8. Fig. 9 depicts the modified boundary and regular processors based on Tsao's algorithm.

VIII. Conclusions

In this paper, we have shown that the Householder transformation can be implemented on a systolic array. By using a two-level pipelined implementation, the throughput of the SBHT RLS systolic array can be as fast as that of the original Givens array in [18]. While, the system latency is longer for the SBHT, it provides a better numerical stability than the Givens method. Clearly, the Givens array is a special case of the SBHT array with a block size of one. In general, the block size is an important variable. A larger block size results in a better numerical stability, while the system latency is increased. Many known properties of the Givens array are also applicable to the SBHT array. For example, the real-time algorithm-based fault-tolerant scheme proposed in [14] can also be easily incorporated into the SBHT RLS array. From the results described in this paper, it shows that the Householder transformation method is useful in real-time high throughput applications of modern signal processing as well as in VLSI implementation.

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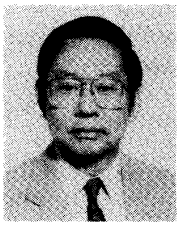
KuoJuey Ray Liu (S'86-M'90) received the B.S. degree in electrical engineering from National Taiwan University in 1983 and the M.S.E. degree in electrical engineering and computer science in 1987, both from the University of Michigan, Ann Arbor, and the Ph.D. degree in electrical engineering from the University of California, Los Angeles (UCLA), in June 1990.

From 1983 to 1985, he served in the Signal Corps, Taiwan, as a Communications Officer. Since 1985, he had been a Teaching Assistant and Research Assistant at the University of Michigan and UCLA. He is currently an Assistant Professor in the Department of Electrical Engineering and Systems Research Center of the University of Maryland, College Park. His research interests include parallel processing algorithms and architectures for signal/image processing and communications, adaptive signal processing, spectral estimation, video signal processing, fault-tolerant computing in VLSI systems, design automation for DSP VLSI systems, and fast algorithms.

Dr. Liu was awarded the President Research Partnership from the University of Michigan in 1987, and the University Fellowship and the Hortense Fishbaugh Memorial Scholarship from UCLA in 1987-1988 and 1989, respectively. He was also awarded the Outstanding Graduate Student Award in Science and Engineering from the Taiwanese-American Foundation.

Shih-Fu Hsieh (S'87-M'90) was born in Taichung, Taiwan, in 1962. He received the B.S. degree from the National Taiwan University, Taipei, Republic of China, in 1984, and the M.S., Engineer, and Ph.D. degrees from the University of California, Los Angeles, in 1987, 1989, and 1990, respectively, all in electrical engineering.

From 1984 to 1986 he was an Ensign Instructor at the Naval Communication and Electronics School, Taiwan. Since 1990 he has been with the National Chiao Tung University, Hsinchu, Taiwan, where he is currently an Associate Professor of Communication Engineering. His current research interests include adaptive signal processing, digital communications, and parallel algorithms and architectures.



Kung Yao (S'59-M'65-SM'91) was born in Hong Kong on November 24, 1938. He received the B.S.E. (highest honors), M.A., and Ph.D. degrees in electrical engineering from Princeton University, Princeton, NJ, in 1961, 1963, and 1965, respectively.

During the summers of his college years, he worked at the Princeton-Penn Accelerator in Penns Neck, NJ, Brookhaven National Laboratory in Upton, NY, and Bell Telephone Laboratories in Murray Hill, NJ. In 1965-1966, he was an NAS-NRC postdoctoral Research Fellow at the University of California, Berkeley. Since September 1966, he has been with the University of California, Los Angeles. Presently, he is a Professor in the Electrical Engineering Department. In the fall of 1969, he was a Visiting Assistant Professor at the Massachusetts Institute of Technology and a Visiting Senior Research Associate at the NASA Electronics Research Center, Cambridge, MA. In 1973-1974, he was a Visiting Associate Professor at Eindhoven Technical University in Eindhoven, the Netherlands. From 1985 to 1988, he served as an Assistant Dean of the School of Engineering and Applied Science at UCLA. His research interests include stochastic processes, digital communication theory, satellite communication systems, simulation, radar systems, systolic and VLSI algorithms and systems, and system identification. He is the coauthor of a book, *Processing and Algorithm in Communication and Radar Systems*, under preparation.

Dr. Yao is a member of Phi Beta Kappa, Sigma Xi, and the American Association for the Advancement of Science. He has served as Program Chairman, Secretary, and Chairman of the IEEE Information Theory Group in Los Angeles and served two terms as a member of the Board of Governors of the IEEE Information Theory Group. He was the Cochairman of the 1981 International Symposium on Information Theory held at Santa Monica, CA, and the Representative of the IT-BOG in the organization of the 1987 IEEE Information Theory Workshop held at Bellagio, Italy. He was also the Chair of the Technical Program of the 1990 IEEE Workshop on VLSI Signal Processing. He has served as an Associate Editor for Book Reviews of the IEEE TRANSACTIONS ON INFORMATION THEORY and was a member of the Editorial Board of the journal, *Probability in the Engineering and Informational Sciences*, published by Cambridge Press. Presently, he is an Associate Editor of VLSI Signal Processing for the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS.