Improved Strapdown Coning Algorithms

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Three improved algorithms for strapdown attitude computation utilizing accumulated gyro increments from the previous and current update are developed and evaluated analytically under classical coning motion. The error criterion of Miller [6] is now derived directly from the rotation vector concept. The accuracy of updating rotation vector estimation can be improved at least two orders of magnitude as compared with those of conventional algorithms. The proposed algorithm is equivalent to increasing the number of gyro samples used in the conventional method and it requires less computer loading also.

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I. INTRODUCTION

The calculation for vehicle attitude parameters, e.g., direction cosines or quaternions, is a must for all strapdown inertial navigation systems. The strapdown gyros measure vehicle rotation by mounting directly to the host vehicle. Usually the outputs of gyros represent incremental angular velocity of the vehicle. The navigation computer is required to process these body angular increments appropriately to obtain an accurate vehicle attitude information. Hence designing a suitable attitude updating algorithm for a strapdown system is a matter of great importance.

The noncommutativity of finite rotations is one of the major error sources in numerical solutions of the attitude equation. It is also inevitable to update the attitude incorporated in digital data processing. In general, the commutativity error can be reduced by increasing the number of computation updates with an efficient algorithm. This requires a high-speed computer which is inherently limited by exploiting advances in microelectronic technology. However, it is impossible to increase the computer speed without bound. Thus designing efficient algorithms has become the attractive research topic.

Many algorithms have been developed and analyzed under pure coning motion. Most of them are derived through the rotation vector concept. It is conceptually clear that increasing gyro samples may improve the accuracy of the algorithm, since less commutativity error is quite easy to account for. Therefore, many algorithms result in a twofold computing scheme [1, 2, 3, 8]. The rotation vector is updated at a fast rate, while the attitude parameters are updated at a slow rate.

Jordan [2] developed a "preprocessor" algorithm using two gyro samples in each fast updating loop. Savage [3] also showed an algorithm which depends on the current and previous gyro samples assuming that the gyro outputs follow a linear function of time. An attitude updating algorithm developed by Mckern [4] can be used to compensate the low-order coning effect in which only one gyro sample is used for updating. This algorithm was also derived from a geometric point of view [5]. A coning algorithm using three gyro samples per update was presented by Miller [6]. By extension of Miller's technique, Lee et al. [7] developed an algorithm in which four gyro samples were used. Ignagni [8] provided a summary and generalization of strapdown attitude integration algorithms. The algorithms given by Savage [3], Mckern [4], and Mckern and Musoff [5] used the accumulated gyro outputs of the previous updating.

We propose a type of strapdown coning algorithm in which both the current and the previous accumulated gyro outputs are used. Three low-order algorithms are analyzed under the classical coning motion. It is shown that the coning error per update can be reduced more than two orders in magnitude when the previous accumulated gyro output is utilized. This is the same effect of increasing the number of gyro samples as used in conventional algorithm. Also it requires less computer loading.

II. ROTATION VECTOR AND CONING MOTION

Since the noncommutativity (or coning) rate is obviously contained in the rotation vector differential equation, it can be used to improve the accuracy of strapdown attitude algorithms effectively. The rotation vector differential equation can be written as [1, 10]

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} + \frac{1}{\phi^2} \times \left(1 - \frac{\phi \sin\phi}{2(1-\cos\phi)}\right) \boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega}) \quad (1)$$

where $\boldsymbol{\phi}$ represents the rotation vector with magnitude $\phi = (\boldsymbol{\phi}^T\boldsymbol{\phi})^{1/2}$ and $\boldsymbol{\omega}$ represents the angular velocity vector. The last two terms in (1) are referred as noncommutativity rate vector. For purposes of designing strapdown algorithms, let ϕ be small enough. Then (1) can be approximated by

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} + \tfrac{1}{12}\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega}). \quad (2)$$

Without loss of the generality, let the rotation vector describing the classical coning motion be [6, 7]

$$\boldsymbol{\phi} = \begin{bmatrix} 0 \\ \phi\cos(\omega t) \\ \phi\sin(\omega t) \end{bmatrix} \quad (3)$$

where ϕ denotes the coning half-apex angle and ω denotes the coning frequency. Then the angular velocity describing the coning motion can be written as

$$\boldsymbol{\omega} = \begin{bmatrix} 2\omega\sin^2\left(\frac{\phi}{2}\right) \\ -\omega\sin(\phi)\sin(\omega t) \\ \omega\sin(\phi)\cos(\omega t) \end{bmatrix}. \quad (4)$$

Note from (3) and (4) that the angular velocity $\boldsymbol{\omega}$ is orthogonal to the rotation vector $\boldsymbol{\phi}$, i.e., $\boldsymbol{\phi}^T\boldsymbol{\omega} = 0$. Therefore (2) can be readily simplified as

$$\dot{\boldsymbol{\phi}} = \left(1 - \tfrac{1}{12}\phi^2\right)\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega}. \quad (5)$$

This implies that the noncommutativity rate vector has a maximum under the coning motion. Hence it follows that the classical coning motion is the worse case input for testing strapdown attitude algorithms.

III. A MODIFIED ALGORITHM

It is evident that the main difficulties in designing a strapdown attitude updating algorithm arise from the noncommutativity effect. Since only the gyro samples over a fixed time interval are available, a correction for commutativity error must resort to these gyro outputs. Usually, the coning correction is updated at a fast rate so that the high-frequency motion can be tracked accurately. However, the attitude parameters can be computed at a slower rate without difficulty.

Applying the rotation vector concept, a general algorithm for updating the rotation vector $\boldsymbol{\phi}_n$ at a fast rate can be approximated by [2, 3, 8, 9]

$$\boldsymbol{\phi}_{n+1} = \boldsymbol{\phi}_n + \tfrac{1}{2}\boldsymbol{\phi}_n \times \boldsymbol{\theta} + \Delta\hat{\boldsymbol{\phi}} \quad (6)$$

where $\boldsymbol{\theta}$ represents the incremental angle accumulated by strapdown gyros over the current update interval and $\Delta\hat{\boldsymbol{\phi}}$ denotes an estimated updating rotation vector which is computed from gyro output samples. If N gyro samples per update are taken, then $\boldsymbol{\theta}$ can be written as

$$\boldsymbol{\theta} = \sum_{i=1}^{N} \boldsymbol{\theta}_i \quad (7)$$

where $\boldsymbol{\theta}_i$ represents the ith gyro sample. Note that $\boldsymbol{\phi}_n$ must be reset to zero at the end of every slow loop. The second term in (6) can be inferred from (2) or (5). The key feature in different algorithms can be deduced from the estimated updating rotation vector $\Delta\hat{\boldsymbol{\phi}}$ which is determined by utilizing the gyro samples. It can be written as [6–8]

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + \sum_{i=1}^{N-1}\sum_{j>i}^{N} K_{ij}(\boldsymbol{\theta}_i \times \boldsymbol{\theta}_j). \quad (8)$$

It is obvious that the desired accuracy of the estimated rotation vector can be achieved by increasing the number of gyro samples as usual, but the computer storage and computing time must be increased inevitably. Since the accumulated incremental angle in the previous update, denoted as $\boldsymbol{\theta}'$, is available already, we propose a modified algorithm:

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + \sum_{i=1}^{N-1}\sum_{j>i}^{N} K_{ij}(\boldsymbol{\theta}_i \times \boldsymbol{\theta}_j) + G(\boldsymbol{\theta}' \times \boldsymbol{\theta}) \quad (9)$$

where the last term $G(\boldsymbol{\theta}' \times \boldsymbol{\theta})$ is the additive compensation term.

The problem now becomes to determine the weighting coefficients, K_{ij} and G, subject to the classical coning motion, such that the magnitude of estimation error contained in $\Delta\hat{\boldsymbol{\phi}}$ is minimized.

IV. ERROR CRITERION

Before the "best" values of K_{ij} and G are evaluated, a criterion must be selected. We select the error criterion presented by Miller [6] and Lee et al. [7] except that we generate the error criterion from updating rotation vector itself rather than from the associated quaternion.

Since the classical coning motion has been chosen as base motion, the true updating rotation vector can be easily obtained. The error criterion is naturally defined as the difference between the true updating rotation vector $\Delta\boldsymbol{\phi}$ and its estimate $\Delta\hat{\boldsymbol{\phi}}$. However, as shown in the coning motion (4), there is solely a non-zero average error along the coning axis. Thus the criterion can be reduced to the non-zero average component of the difference between these two vectors, i.e.,

$$\phi_\epsilon = \Delta\phi_1 - \Delta\hat{\phi}_1 \tag{10}$$

where $\Delta\phi_1$ represents the first component of $\Delta\boldsymbol{\phi} = [\Delta\phi_1, \Delta\phi_2, \Delta\phi_3]^T$ and similarly, $\Delta\hat{\phi}_1$, the first component of $\Delta\hat{\boldsymbol{\phi}}$.

To show the vadility of (10), consider the rotation vector described the coning motion in (3). The corresponding quaternion $Q(t)$ can be written as

$$Q(t) = \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ 0 \\ \sin\left(\frac{\phi}{2}\right)\cos(\omega t) \\ \sin\left(\frac{\phi}{2}\right)\sin(\omega t) \end{bmatrix}. \tag{11}$$

Then the true updating quaternion $q(h)$ which represents the coning motion during the updating interval from time t to $(t+h)$ can be derived as [6, 7]

$$q(h) = Q^{-1}(t) * Q(t+h) \tag{12}$$

where $*$ denotes quaternion multiplication. After manipulations,

$$q(h) = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 - 2\sin^2\left(\frac{\phi}{2}\right)\sin^2\left(\frac{\omega h}{2}\right) \\ -\sin^2\left(\frac{\phi}{2}\right)\sin(\omega h) \\ -\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\sin\left[\omega\left(t+\frac{h}{2}\right)\right] \\ \sin(\phi)\sin\left(\frac{\omega h}{2}\right)\cos\left[\omega\left(t+\frac{h}{2}\right)\right] \end{bmatrix}. \tag{13}$$

By the definition of quaternion, there exists a true updating rotation vector $\Delta\boldsymbol{\phi}$ associated with $q(h)$. The components of $q(h)$ can be expressed as

$$q_0 = \cos\left(\frac{\Delta\phi}{2}\right) \tag{14}$$

$$q_i = \sin\left(\frac{\Delta\phi}{2}\right)\frac{\Delta\phi_i}{\Delta\phi}, \qquad i = 1,2,3 \tag{15}$$

where $\Delta\phi$ represents the magnitude of $\Delta\boldsymbol{\phi}$. Assuming small $\Delta\phi$ and using small angle approximation $\sin(\Delta\phi/2) \approx \Delta\phi/2$, it can be found from (15) that

$$\Delta\phi_i = 2q_i, \qquad i = 1,2,3. \tag{16}$$

Substituting (13) into (16), we have

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \end{bmatrix} = \begin{bmatrix} -\sin^2\left(\frac{\phi}{2}\right)\sin(\omega h) \\ -\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\sin\left[\omega\left(t+\frac{h}{2}\right)\right] \\ \sin(\phi)\sin\left(\frac{\omega h}{2}\right)\cos\left[\omega\left(t+\frac{h}{2}\right)\right] \end{bmatrix}. \tag{17}$$

On the other hand, assuming that there are N gyro samples per update, then the ith gyro sample $\boldsymbol{\theta}_i$ can be obtained by integrating (4) as

$$\boldsymbol{\theta}_i = \int_{t+((i-1)/N)h}^{t+(i/N)h} \boldsymbol{\omega}(\tau)\,d\tau, \qquad i = 1,2,\ldots,N \tag{18}$$

$$= \begin{bmatrix} -\frac{2}{N}(\omega h)\sin^2\left(\frac{\phi}{2}\right) \\ -2\sin(\phi)\sin\left(\frac{\omega h}{2N}\right)\sin\left[\omega\left(t+\frac{2i-1}{2N}h\right)\right] \\ 2\sin(\phi)\sin\left(\frac{\omega h}{2N}\right)\cos\left[\omega\left(t+\frac{2i-1}{2N}h\right)\right] \end{bmatrix}. \tag{19}$$

Also the accumulated gyro outputs in the updating interval from t to $(t+h)$ is therefore

$$\boldsymbol{\theta} = \sum_{i=1}^{N}\boldsymbol{\theta}_i = \begin{bmatrix} -2(\omega h)\sin^2\left(\frac{\phi}{2}\right) \\ -2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\sin\left[\omega\left(t+\frac{h}{2}\right)\right] \\ 2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\cos\left[\omega\left(t+\frac{h}{2}\right)\right] \end{bmatrix}. \tag{20}$$

Similarly, the accumulated previous gyro output $\boldsymbol{\theta}'$ can be readily obtained by replacing t by $(t-h)$ into (20), i.e.,

$$\boldsymbol{\theta}' = \begin{bmatrix} -2(\omega h)\sin^2\left(\frac{\phi}{2}\right) \\ -2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\sin\left[\omega\left(t-\frac{h}{2}\right)\right] \\ 2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\cos\left[\omega\left(t-\frac{h}{2}\right)\right] \end{bmatrix}. \tag{21}$$

Substituting (19)–(21) into (9), the estimated updating rotation vector can be expressed as

$$\begin{bmatrix}\Delta\hat{\phi}_1\\ \Delta\hat{\phi}_2\\ \Delta\hat{\phi}_3\end{bmatrix} = \begin{bmatrix} -2(\omega h)\sin^2\left(\frac{\phi}{2}\right) \\ -2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\sin\left[\omega\left(t+\frac{h}{2}\right)\right] \\ 2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\cos\left[\omega\left(t+\frac{h}{2}\right)\right]\end{bmatrix} + \sum_{i=1}^{N-1}\sum_{j>i}^{N} K_{ij}$$
$$\times \begin{bmatrix} 4\sin^2(\phi)\sin^2\left(\frac{\omega h}{2N}\right)\sin\left(\frac{j-i}{N}\omega h\right) \\ -\frac{8}{N}(\omega h)\sin^2\left(\frac{\phi}{2}\right)\sin(\phi)\sin\left(\frac{\omega h}{2N}\right)\sin\left(\frac{j-i}{2N}\omega h\right)\sin\left[\omega\left(t+\frac{j+i-1}{2N}h\right)\right] \\ \frac{8}{N}(\omega h)\sin^2\left(\frac{\phi}{2}\right)\sin(\phi)\sin\left(\frac{\omega h}{2N}\right)\sin\left(\frac{j-i}{2N}\omega h\right)\cos\left[\omega\left(t+\frac{j+i-1}{2N}h\right)\right]\end{bmatrix}$$
$$+ G\begin{bmatrix} 4\sin^2(\phi)\sin^2\left(\frac{\omega h}{2}\right)\sin(\omega h) \\ -8(\omega h)\sin^2\left(\frac{\phi}{2}\right)\sin(\phi)\sin^2\left(\frac{\omega h}{2}\right)\sin\omega t \\ 8(\omega h)\sin^2\left(\frac{\phi}{2}\right)\sin(\phi)\sin^2\left(\frac{\omega h}{2}\right)\cos\omega t\end{bmatrix}. \tag{22}$$

estimation error ϕ_ϵ approaches zero as closely as possible.

It is demostrated, from (17) and (22), that only the first component of $\Delta\boldsymbol{\phi}$ and $\Delta\hat{\boldsymbol{\phi}}$ are nonperiodic. Thus an equivalent gyro drift is induced along this coning axis. We are intrested in the non-zero average estimation error. Therefore the error criterion is defined as

$$\phi_\epsilon = \Delta\phi_1 - \Delta\hat{\phi}_1. \tag{23}$$

Substituting (16) into (23), yields

$$\phi_\epsilon = 2q_1 - \Delta\hat{\phi}_1. \tag{24}$$

This is identical to the criterion of Miller [6] and Lee, et al. [7].

Using the first component of (17) and (22) with small ϕ assumption, the estimation error in (24) can be written as

$$\phi_\epsilon = \frac{1}{2}\phi^2\left[(\omega h) - \sin(\omega h) - \sum_{i=1}^{N-1}\sum_{j>i}^{N} 8K_{ij} \times \sin^2\left(\frac{\omega h}{2N}\right)\sin\left(\frac{j-i}{N}\omega h\right) - 8G\sin^2\left(\frac{\omega h}{2}\right)\sin(\omega h)\right]. \tag{25}$$

Consequently, whenever the number of gyro samples per update N is given, the algorithm (9) and its error (25) are well defined. The objective is to determine the values of K_{ij} and G such that the magnitude of

V. ALGORITHMS AND ACCURACIES

It is obvious that the estimation error in updating the rotation vector depends on the number of gyro samples per update. Following (25), the optimal values of K_{ij} and G can be determined by setting some of the Taylor expansion coefficients in (ωh) to zero by obsevation. Three low-order algorithms are generated as follows.

Algorithm 1. In case only one gyro sample per update is taken, i.e., $N = 1$, K_{ij} is arbitary. The algorithm for $\Delta\hat{\boldsymbol{\phi}}$ in (9) can be written as

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + G(\boldsymbol{\theta}' \times \boldsymbol{\theta}). \tag{26}$$

The associated estimation error in (27) becomes

$$\phi_\epsilon = \frac{1}{2}\phi^2\left[(\omega h) - \sin(\omega h) - 8G\sin^2\left(\frac{\omega h}{2}\right)\sin(\omega h)\right]. \tag{27}$$

Expanding the sine terms in (27) into Taylor power series, it can be found that

$$\phi_\epsilon = \tfrac{1}{2}\phi^2\left[\left(\tfrac{1}{6} - 2G\right)(\omega h)^3 + \left(\tfrac{1}{2}G - \tfrac{1}{120}\right)(\omega h)^5 + \text{h.o.t.}\right] \tag{28}$$

By setting the leading coefficient in (28) to zero, it is readily found that

$$G = \tfrac{1}{12}. \tag{29}$$

Therefore inserting (29) into (26), the one-sample algorithm is

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + \tfrac{1}{12}(\boldsymbol{\theta}' \times \boldsymbol{\theta}). \tag{30}$$

This algorithm had been shown by Mckern [4] and Mckern and Musoff [5]. Substituting (29) into (28), the estimation error is

$$\phi_\epsilon = \tfrac{1}{60}\phi^2(\omega h)^5. \tag{31}$$

It should be noted, from (27), that if

$$G = \frac{(\omega h) - \sin(\omega h)}{4[1-\cos(\omega h)]\sin(\omega h)} \tag{32}$$

then $\phi_\epsilon = 0$. This might be practical on condition that the coning frequency ω is precisely known. It is also evident that the value of G in (29) can be obtained by taking a limiting case, i.e.,

$$G = \lim_{\omega h \to 0} \frac{(\omega h) - \sin(\omega h)}{4[1-\cos(\omega h)]\sin(\omega h)} = \frac{1}{12}. \tag{33}$$

If $\boldsymbol{\theta}'$ is not used, i.e., $G = 0$, then the algorithm (26) is reduced to

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} \tag{34}$$

and the associated estimation error can be found from (28) to be

$$\phi_\epsilon = \tfrac{1}{12}\phi^2(\omega h)^3. \tag{35}$$

Comparing (31) with (35), it is seen that the magnitude of estimation error is reduced by two orders when the gyro output $\boldsymbol{\theta}'$, from the previous update interval, is used.

Algorithm 2. In case two gyro samples per update are used, i.e., $N = 2$, the algorithm for $\Delta\hat{\boldsymbol{\phi}}$ in (9) can be written as

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + K_{12}(\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_2) + G(\boldsymbol{\theta}' \times \boldsymbol{\theta}). \tag{36}$$

The associated estimation error in (25) becomes

$$\phi_\epsilon = \frac{1}{2}\phi^2\left[(\omega h) - \sin(\omega h) - 8K_{12}\sin^2\left(\frac{\omega h}{4}\right) \times \sin\left(\frac{\omega h}{2}\right) - 8G\sin^2\left(\frac{\omega h}{2}\right)\sin(\omega h)\right]. \tag{37}$$

Expanding the sine terms in (37), after manipulations, yields

$$\phi_\epsilon = \frac{1}{2}\phi^2\left[\left(\frac{1}{6} - \frac{K_{12}}{4} - 2G\right)(\omega h)^3 + \left(-\frac{1}{120} + \frac{K_{12}}{64} + \frac{G}{2}\right)(\omega h)^5 + \left(\frac{1}{5040} - \frac{K_{12}}{2560} - \frac{G}{20}\right)(\omega h)^7 + \text{h.o.t.}\right]. \tag{38}$$

Setting the coefficients of $(\omega h)^3$ and $(\omega h)^5$ to zero, the optimal values of K_{12} and G can be solved from the following equations:

$$K_{12} + 8G = \tfrac{2}{3} \tag{39}$$

$$K_{12} + 32G = \tfrac{8}{15}. \tag{40}$$

It is easily found:

$$K_{12} = \tfrac{32}{45}; \qquad G = -\tfrac{1}{180}. \tag{41}$$

Substituting (41) into (36), the two-sample algorithm is

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + \tfrac{32}{45}(\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_2) - \tfrac{1}{180}(\boldsymbol{\theta}' \times \boldsymbol{\theta}). \tag{42}$$

Using (41) in (38), the estimation error can be obtained as

$$\phi_\epsilon = \frac{1}{10{,}080}\phi^2(\omega h)^7. \tag{43}$$

If $\boldsymbol{\theta}'$ is not used, i.e., $G = 0$, the algorithm is readily reduced to

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + \tfrac{2}{3}(\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_2). \tag{44}$$

This is Jordan's "preprocessor" algorithm [2]. The associated estimation error is

$$\phi_\epsilon = \tfrac{1}{960}\phi^2(\omega h)^5. \tag{45}$$

It is obvious, from (43) and (45), that the magnitude of estimation error is reduced by two orders when the gyro output $\boldsymbol{\theta}'$, from the previous update interval, is used.

Algorithm 3. In case three gyro samples per update are used, i.e., $N = 3$, the algorithm for $\Delta\hat{\boldsymbol{\phi}}$ in (9) can be written as

$$\Delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta} + K_{12}(\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_2) + K_{13}(\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_3) + K_{23}(\boldsymbol{\theta}_2 \times \boldsymbol{\theta}_3) + G(\boldsymbol{\theta}' \times \boldsymbol{\theta}). \tag{46}$$

The associated estimation error in (25) becomes

$$\phi_\epsilon = \frac{1}{2}\phi^2\left[(\omega h) - \sin(\omega h) - 8(K_{12} + K_{23})\sin^2\left(\frac{\omega h}{6}\right) \times \sin\left(\frac{\omega h}{3}\right) - 8K_{13}\sin^2\left(\frac{\omega h}{6}\right)\sin\left(\frac{2\omega h}{3}\right) - 8G\sin^2\left(\frac{\omega h}{2}\right)\sin(\omega h)\right]. \tag{47}$$

Expanding the sine terms in (47) and setting the coefficients of $(\omega h)^3$, $(\omega h)^5$, and $(\omega h)^7$ to zero, after manipulations, it can be found that the optimal values of K_{12}, K_{13}, K_{23}, and G are governed by

$$\begin{bmatrix} 1 & 2 & 27 \\ 1 & 6 & 243 \\ 3 & 46 & 6561 \end{bmatrix} \begin{bmatrix} K_{12} + K_{23} \\ K_{13} \\ G \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ \frac{81}{20} \\ \frac{729}{28} \end{bmatrix}. \tag{48}$$

Hence, it follows that

$$K_{12} + K_{23} = \tfrac{1539}{1120}; \qquad K_{13} = \tfrac{243}{560}; \qquad G = \tfrac{1}{3360} \tag{49}$$

Note that the values of K_{12} and K_{23} are not unique. Different algorithms can be obtained by chosing different values of K_{12} and K_{23}. What is relevant to the accuracy of the algorithm is merely their summation. The estimation error can be shown as

$$\phi_\epsilon = \frac{1}{3{,}674{,}160}\phi^2(\omega h)^9. \tag{50}$$

If θ' is not used, i.e., $G = 0$ in (46), then this algorithm is reduced to the Miller's result [6]:

$$\Delta\hat{\phi} = \theta + K_{12}(\theta_1 \times \theta_2) + K_{13}(\theta_1 \times \theta_3) + K_{23}(\theta_2 \times \theta_3) \tag{51}$$

in which

$$K_{12} + K_{23} = \tfrac{27}{20}; \qquad K_{13} = \tfrac{9}{20} \tag{52}$$

and the associated estimation error is

$$\phi_\epsilon = \frac{1}{204{,}120}\phi^2(\omega h)^7. \tag{53}$$

It is seen, from (50) and (53), that the magnitude of estimation error is reduced by two orders when the gyro output θ', from the previous update interval, is used.

Finally, for comparison, the estimation error associated with the four-sample algorithm without using the previous gyro outputs is given by [7]

$$\phi_\epsilon = \frac{1}{82{,}575{,}360}\phi^2(\omega h)^9. \tag{54}$$

Comparing (31) with (45), (43) with (53), and (50) with (54), it is evident that the effect of using the previous accumulated gyro outputs is almost equal to the increasing one more gyro sample. And the proposed algorithms also can reduce computer loading as compared with the conventional algorithms.

VI. CONCLUSIONS

A set of new coning algorithms for strapdown inertial navigation systems are developed which reduce the estimation error by two orders of magnitude as compared with conventional methods. The new algorithms utilize accumulated gyro increments from the previous and current update interval. The weighting coefficients are optimized under classical coning motion in which maximum commutativity error is contained. It is straightforward to define the error criterion to be the difference between the true and the estimated updating rotation vector along the non-zero average coning axis.

As a general rule, accuracy can be improved by increasing the number of gyro samples. The effect of utilizing the previous accumulated gyro outputs is found equivalent to that of increasing the number of gyro samples. With the new algorithms, the computer loading can also be reduced. It is also possible to force the estimated coning error vanished using one-sample algorithm if the coning frequency is precisely known.

Since the algorithm and error criterion are presented in a generalized form, it is easy for extension to high-performance algorithms.

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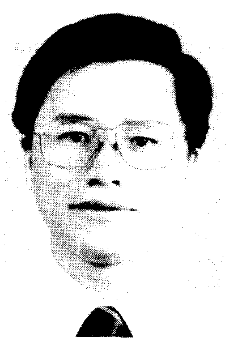

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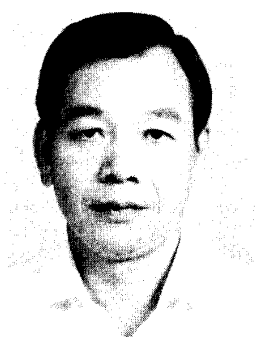

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