Probability Distribution for Benefit/Cost Ratio and Net Benefit

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Abstract: Benefit/cost ratio and net benefit are the two most commonly used criteria for evaluating the economic merit of public development projects. Due to the existence of uncertainty in cost and benefit estimations, the benefit/cost ratio and net benefit cannot be quantified with absolute certainty. In most probabilistic benefit/cost analysis the probability distributions of the benefit/cost ratio are chosen arbitrarily. The intent of this paper is to present the results from a numerical experiment in attempting to identify the appropriateness of various commonly used probability distributions in describing the random behavior of the benefit/cost ratio and net benefit of an economic development project.

Introduction

In many public economic development projects, the benefit/cost (B/C) ratio has been commonly used as the criterion for evaluating the feasibility and the relative merit of competitive projects. It is not uncommon that, when assessments of benefits and costs are made, only few benefit and cost components in economic analysis could be quantified with reasonable certainty. In water-resource project B/C analysis, significant uncertainties in quantifying benefits and costs are attributed to a number of factors including, but not limited to: (1) The randomness of hydrologic events that affect the benefits to be generated from irrigation, flood control, and other aspects of a multipurpose water development project; (2) the uncertainty of projected future population growth, which affects the estimate of benefits due to water supply, recreation, and flood control; (3) the uncertainty in the community demand functions for various water usages; (4) limited data records; (5) uncertainty in engineering cost estimation due to variations in site physical conditions, delays in constructions, and variable productivity; and (6) uncertainty in economic factors including interest rates, inflation, and project life (Dandy 1985; Howe 1971; James and Lee 1971).

Project evaluation in water resource developments using performance criteria such as the B/C ratio is no longer a trivial exercise when risk and uncertainty are involved. Under such circumstances, the performance criteria proposed for project evaluation are also subject to uncertainty. One common practice in performing the B/C analysis in the presence of uncertainty is to inflate the anticipated cost and to deflate the benefit. This conservative way of performing economic analysis does not remove the subjectivity in the process of determining suitable inflation and deflation factors. Furthermore, the conservatism of such a practice could potentially mask the real merit of a proposed project development. For better evaluation of the true economic merit of a project, probabilistic economic analysis

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should be taken in that probability distributions of economic criteria are the essentially required information.

Several studies have employed a probabilistic approach to B/C analysis of water resource projects. The probability distributions used for benefits and costs or the B/C ratio were chosen primarily on the basis of computational simplicity and mathematical tractability. Mercer and Morgan (1975) used a Monte Carlo simulation in deriving the probability distribution of the B/C ratio. Weibull distribution was chosen for its analytical flexibility. Goicoechea et al. (1982) analytically derived the probability distribution of the B/C ratio when all the benefit and cost elements have independent normal or gamma distributions. The derivation of the distribution for the B/C ratio, in effect, is a problem of finding the frequency distributions of ratios (Curtis 1941; Hayya et al. 1975; Hinkley 1969; Marsaglia, 1965). Park (1984) developed a probabilistic B/C analysis procedure based on the condition that both benefit and cost are correlated normal random variables. The resulting distribution for the B/C ratio is a function of a bivariate normal distribution.

The use of an analytically derived distribution, in general, requires performing numerical integration or using some special probability tables for probability computation. The condition that all benefits and costs have normal or gamma distribution is neither realistic nor flexible in describing the random characteristics of benefit and cost components. Alternatively, Dandy (1985) proposed an approximation to a probabilistic B/C analysis in that the mean and variance of the B/C ratio are estimated by the first-order second-moment (FOSM) method (Benjamin and Cornell 1970). Then, a commonly used analytical distribution was assumed to describe the random characteristics of the B/C ratio. Dandy (1985) performed a limited investigation on the appropriateness of normal, lognormal, and gamma distributions in describing the random behavior of the B/C ratio, with its statistical moments estimated by the FOSM method. The candidate distributions were compared against the exact distribution of the B/C ratio derived by Goicoechea et al. (1982) under normal and independent conditions. On the basis of percentage error of the B/C ratio at 0.05, 0.50,and 0.95 levels, Dandy concluded that the gamma distribution best fits the exact distribution derived by Goicoechea et al. (1982).

The main purpose of this paper is to present a more comprehensive experiment to examine (1) The appropriateness of the commonly used probability distributions in describing the random nature of the B/C ratio and net benefit criteria; and (2) the sensitivity of the distribution of economic performance criteria to the probability distribution of benefit and cost components. The statistical properties of economic evaluation criteria such as the mean, variance, skew coefficient, and kurtosis are estimated by the FOSM method. It is hoped that the results would provide some justifications for the selection of a probability distribution for the B/C ratio and net benefit for probabilistic economic analyses.

Estimation of Statistical Properties of the B/C Ratio and Net Benefit

Consider a project involving M benefit components and N cost components. Assume also that all benefit and cost components have been converted to the present or annual values. The B/C ratio and net benefit of the project can be expressed as

$$U = \frac{\sum_{i=1}^{M} B_i}{\sum_{j=1}^{N} C_j} \quad (1)$$

$$V = \sum_{i=1}^{M} B_i - \sum_{j=1}^{N} C_j \quad (2)$$

in which U and V = the B/C ratio and net benefit, respectively; B_i = the ith benefit component; and C_j = the jth cost component. In reality, the estimations of B_i and C_j cannot be made without uncertainty. Therefore, those benefit and cost components should be treated as random variables, and so should the B/C ratio, U, or its variations and the net benefit, V.

In a probabilistic economic analysis, statistical properties of the B/C ratio and net benefit must be expressed as functions of those of the benefit and cost components. Ideally, the complete statistical properties of random net benefit and B/C ratio are their respective probability distributions. It is generally difficult, if not impossible, to derive analytically the exact probability distribution of the random B/C ratio and net benefit from the probability distributions of benefit and cost components.

As a practical alternative, statistical moments of U and V, such as the mean, standard deviation, skew coefficient, etc., can be approximated by the FOSM method. Once the statistical properties of U or V are estimated, they can be used along with an assumption of the probability density function for U or V to assess the probability that U or V would exceed a specified threshold value.

In assessing the uncertainty of a benefit or cost component, it is practical to specify the "pessimistic," "most likely," and "optimistic" values of the benefit or cost. The "pessimistic" and "optimistic" values define more or less the confidence interval for the true value. Dandy (1985) proposed the use of the following equation to estimate reasonable standard deviations for the benefit and cost components based on the "pessimistic" and "optimistic" values;

$$S_x = \frac{X_{op} - X_{pe}}{K} \quad (3)$$

where S_x = the standard deviation of benefit or cost component; X_{pe} and X_{op} = the "pessimistic" and "optimistic" values of the benefit or cost component, respectively; and K = a constant. The value of K can be determined if the confidence level associated with the interval defined by X_{pe} and X_{op} is specified for a given probability density function. This approach has been widely used in quantifying the uncertainty of project activity duration in critical path/project evaluation review technique (CMP/PERT) analysis (Hillier and Lieberman 1986; Taha 1987).

Estimation of Statistical Moments of Net Benefit

If the probability density function for each individual benefit or cost component is assumed, along with the specifications of the "pessimistic" value, X_{pe}, the "most likely" value, X_{mo}, and the "optimistic" value, X_{op}, the mean and the standard deviation of each benefit and cost components can be determined. Then the mean and variance of the total benefit can then be obtained as

$$\bar{B} = \sum_{i=1}^{M} \overline{B_i} \quad (4)$$

$$S_B^2 = \sum_{i=1}^{M} S_{B_i}^2 + 2 \sum_{i} \sum_{i'} \rho_{B_i B_{i'}} S_{B_i} S_{B_{i'}} \quad (5)$$

in which $\bar{B}$ and $\bar{B}_i$ = the means of total benefit and the ith benefit component, respectively; S = the standard deviation; and ρ = the correlation coefficient between benefit items. Similarly, replacing B by C, the mean and variance of the total cost can be obtained from (4) and (5). In reality, components in the benefit and/or cost might not be entirely independent. In such cases, the analyst must make assessment of the covariance or correlation between each pair of benefit and/or cost components.

For the most commonly used probability density functions, the distributional characteristics are often completely defined by the first two moments. In this study, a distribution using Fisher-Cornish (FC) (Fisher and Cornish 1960) asymptotic expansion is also included for the purpose of comparison. The probability function defined by the FC method requires knowledge of the skew coefficient and/or kurtosis, depending on the number of terms in the expansion series. The main advantage of using FC asymptotic expansion for the distribution is that it frees analysts from making strong reliance on any parametric distribution. The price to pay is that one has to estimate higher-order moments such as skew coefficient and/or kurtosis. Considering only the first two moments reduces the FC asymptotic expansion to the standard normal distribution.

Calculations of skew coefficient and kurtosis for correlated benefit and cost components require knowledge about higher-order product moments, which are difficult to obtain in practice. Under the assumption of independence of benefit and cost items, the skew coefficient and kurtosis of the total benefit could be obtained as

$$\gamma_B = \frac{\sum_{i=1}^{M} S_{B_i}^3 \gamma_{B_i}}{S_B^3} \quad (6)$$

$$\kappa_B = \frac{\sum_{i=1}^{M} S_{B_i}^4 \kappa_{B_i} + 6 \sum_{i} \sum_{i'} S_{B_i}^2 S_{B_{i'}}^2}{S_B^4} \quad (7)$$

where γ_X and κ_X = the skew coefficient and kurtosis of the random variable X, respectively.

Finally, by putting all the aforementioned formulas together, the mean, variance, skew coefficient, and kurtosis of the net benefit can be obtained as

$$\bar{V} = \bar{B} - \bar{C} \quad (8)$$

$$S_V^2 = S_B^2 + S_C^2 \quad (9)$$

$$\gamma_V = \frac{S_B^3 \gamma_B - S_C^3 \gamma_C}{S_V^3} \quad (10)$$

$$\kappa_V = \frac{S_B^4 \kappa_B + 6 S_B^2 S_C^2 + S_C^4 \kappa_C}{S_V^4} \quad (11)$$

TABLE 1. Estimated Range of Values for Halstead Flood Protection Project

Component (1)	Optimistic value ($) (2)	Most Likely Value ($)			Pessimistic value ($) (6)
		$\gamma > 0$ (3)	$\gamma = 0$ (4)	$\gamma < 0$ (5)	
Costs					
Interests and amortization	197,250	241,083	263,000	284,917	328,750
Operation and maintenance	14,175	17,325	18,900	20,475	23,625
Benefits					
Inundation reduction	411,585	340,005	357,900	375,795	304,215
Location	51,810	45,530	47,100	48,670	42,390
Affluence	26,840	23,587	24,400	48,670	21,960
Floodproofing costs prevented	16,000	5,333	8,000	10,667	0
Employment	39,800	13,267	19,900	26,533	0
Recreation	7,500	2,500	3,750	5,000	0

Estimation of Statistical Moments of the B/C Ratio

Based on (1), the statistical moments of U can be approximated by the FOSM method in which only the first-order terms in the Taylor expansion series of the B/C ratio, U, are retained. Then, the expectation and variance operators are applied to the truncated expansion series for the B/C ratio, U, to obtain approximations of the mean and variance of U. The resulting approximations are

$$\bar{U} = U_o(1 + \Omega_C^2 - \rho_{BC}\Omega_B\Omega_C) \quad \text{(12)}$$

$$S_U^2 = U_o^2(\Omega_B^2 + \Omega_C^2 - 2\rho_{BC}\Omega_B\Omega_C) \quad \text{(13)}$$

in which Ω_B and Ω_C = the coefficients of variation of the total benefit and total cost, respectively; $U_o = \bar{B}/\bar{C}$, and ρ_{BC} is the correlation coefficient of the total benefit and total cost. To use FC asymptotic expansion, the skew coefficient and kurtosis of the B/C ratio, under the assumption of independence between and within benefit and cost items, can be obtained as

$$\gamma_U = \frac{U_o^3(\Omega_B^3\gamma_B - \Omega_C^3\gamma_C)}{S_U^3} \quad \text{(14)}$$

$$\kappa_U = \frac{\Omega_B^4\kappa_B + 6\Omega_B^2\Omega_C^2 + \Omega_C^4\kappa_C}{S_U^4} \quad \text{(15)}$$

Similar to the net benefit case, a probability distribution is assigned to the B/C ratio U along with the statistical moments of U estimated from the FOSM method. The probability that the B/C ratio would exceed a certain threshold value can then be calculated.

Investigation of Appropriate Distribution for B/C Ratio and Net Benefit

In a probabilistic economic analysis, evaluation of the relative merit of economic projects generally involves computations of the probability of a selected performance criterion being better than a certain threshold level. As a result, the probability distribution of the selected economic performance criterion, in addition to its statistical moments, must be known. In reality, the probability distribution of an economic performance criterion is a function of the probability distributions of random benefit and cost components which could be a mixture of distribution of various forms. Except under some very special but rather rare cases, the exact probability distri-

TABLE 2. Probability Distributions of Benefit and Cost Components for Various Cases and Subcases Examined in Empirical Experiment

Case number (1)	Skew coefficient (2)	Subcase Number 1 (3)	2 (4)	3 (5)	4 (6)	5 (7)	6 (8)
1	$\gamma_B < 0$	All B	All T	All B	(B,T,T,B,B,T)		
	$\gamma_C < 0$	All B	All T	All T	(T,B)		
2	$\gamma_B < 0$	All B	All B	All B	All T	All T	(T,B,B,T,B,T)
	$\gamma_C = 0$	All N	All B	All T	All N	All B	(B,T)
3	$\gamma_B < 0$	All B	All B	All T	All T	All T	(T,B,B,T,T,B)
	$\gamma_C > 0$	All LN	All G	All W	All G	All B	(LN,G)
4	$\gamma_B = 0$	All N	All N	All B	All B	All T	(N,B,T,B,T,N)
	$\gamma_C < 0$	All T	All B	All B	All T	All B	(B,T)
5	$\gamma_B = 0$	All N	All B	All T	All N	All B	(B,N,T,B,T,N)
	$\gamma_C = 0$	All N	All B	All T	All B	All T	(N,B)
6	$\gamma_B = 0$	All N	All N	All B	All B	All T	(T,N,B,T,B,N)
	$\gamma_C > 0$	All LN	All G	All LN	All W	All LN	(LN,G)
7	$\gamma_B > 0$	All LN	All LN	All G	All B	All T	(LN,G,B,T,W,G)
	$\gamma_C < 0$	All B	All T	All B	All B	All T	(T,B)
8	$\gamma_B > 0$	All LN	All W	All G	All B	All T	(LN,G,B,T,W,G)
	$\gamma_C = 0$	All B	All N	All T	All T	All T	(T,B)
9	$\gamma_B > 0$	All LN	All G	All B	All T	All W	(LN,G,B,T,W,LN)
	$\gamma_C > 0$	All LN	All G	All B	All T	All W	(B,G)

Note: B = Beta; G = Gamma; LN = Lognormal; N = Normal; T = Triangular; and W = Weibull.

butions of economic performance criteria for project evaluation are difficult, if not impossible, to derive analytically.

The following describes the procedure used in this study to evaluate the appropriateness of five candidate probability models (including normal, lognormal, gamma, Weibull, and FC asymptotic expansion) for the economic performance criteria when the probability distribution of benefit and cost components take a variety of forms (including beta, gamma, lognormal, normal, triangle, and Weibull). Since the true or exact probability density function of the net benefit or B/C ratio could not be analytically derived for most of the situations, the true quantities of the economic performance criteria are obtained from a Monte Carlo simulation.

Monte Carlo Experiment

The data set used in this Monte Carlo simulation was adopted from the Halstead Flood Protection Project considered by Goicoechea et al. (1982). The project involves the construction of flood control levee and channel improvements to reduce potential flood damage around the city of Halstead, Kansas. The six benefit and two cost items involved in the project with their optimistic and pessimistic values [taken from Goicoechea et al. (1982)] are given in Table 1.

In this investigation, the probability distributions for the benefit and cost components were categorized into three types: positively skewed, symmetric, and negatively skewed. For a positively skewed benefit or cost component, the most likely value was made to be located at one-third of the range, defined by $|X_{op} - X_{pe}|$, to the right of the lower-bound values; while for a negatively skewed distribution, the most likely value was located at one-third of the range to the left of the upper bound. The most likely value, for a symmetric distribution, was the mean of the pessimistic and optimistic values. The most likely values, under different skewed conditions, for benefit and cost items in the example are given in Table 1.

The suitable probability distributions applicable in simulation, when a benefit or cost component is positively skewed, are beta, gamma, lognormal,

TABLE 3. Frequency of Ranks of Candidate Probability Models Considered for Net Benefit Based on MAE% and RMSE%

Rank (1)	Normal (2)	Lognormal (3)	Gamma (4)	Weibull (5)	Fisher/Cornish (6)
	Candidate Probability Models				
(a) Based on MAE%					
1	9	0	2	0	41
2	32	1	11	0	8
3	9	2	38	2	1
4	1	44	0	6	1
5	1	5	1	46	1
(b) Based on RMSE%					
1	9	0	2	0	41
2	33	1	9	2	7
3	7	4	41	0	0
4	2	44	0	4	2
5	1	3	0	46	2

TABLE 4. Frequency of Ranks of Candidate Probability Models Considered for B/C Ratio Based on MAE% and RMSE%

Rank (1)	Normal (2)	Lognormal (3)	Gamma (4)	Weibull (5)	Fisher/Cornish (6)
	(a) Based on MAE%				
1	0	12	3	0	37
2	0	33	11	0	8
3	48	6	38	0	2
4	4	1	0	3	2
5	0	0	0	49	3
	(b) Based on the RMSE%				
1	0	25	3	0	24
2	0	20	23	0	9
3	11	6	26	0	9
4	41	1	0	3	7
5	0	0	0	49	3

Candidate Probability Models: columns (2)–(6).

triangle, and Weibull; when negatively skewed, are beta and triangle; when symmetric, are beta, normal, and triangle. Since the total benefit and cost could be positively skewed, negatively skewed, and symmetric, a total of nine cases with several subcases in each case (see Table 2) were considered in this simulation study. There were a total of 52 subcases with different combinations of skewness and distribution considered to investigate the appropriateness of the five candidate distributions in describing the random behaviors of the net benefit and B/C ratio.

Before random benefits and costs are generated in the Monte Carlo simulation, parameters in the probability density function for each benefit and cost component are determined from the knowledge of pessimistic, most likely, and optimistic values. In this study, the most likely value was considered as the mode of the distribution. The standard deviation was estimated from (3) with K being set to 6 in order to be consistent with Goicoechea et al. (1982). Under the normality condition, this implies that the pessimistic and optimistic values of benefits and costs correspond to the 99.7% confidence limits. However, when the distribution of benefits and costs are not normal, the confidence level specified by the pessimistic and optimistic values will be different. For all the six basic distributions utilized (i.e. gamma, normal, lognormal, beta, triangle, and Weibull) for the benefit and cost components, the distributional parameters can be determined when the mode and standard deviation are specified (Hastings and Peacock 1974; Patel et al. 1976). For some distributions, solving a small system of nonlinear equations is required to obtain distributional parameters for simulation.

In the Monte Carlo simulation, benefit and cost items were assumed to be independent of each other. For every subcase listed in Table 2, 2,000 sets of random realizations for each benefit and cost item were generated and the corresponding economic performance criteria were calculated. Further, the simulated economic performance criteria were ranked in either ascending or descending order for purposes of estimating the true quantiles at different probability levels. For the purpose of comparison, the quantiles

TABLE 5. Quantile Values of Net Benefit and Percentage Error (in Parentheses) at Different Probability Levels under Various Distribution Models Considered for Case 13

p (1)	Normal (2)	Lognormal (3)	Gamma (4)	Weibull (5)	Fisher/Cornish (6)	Simulation (7)
0.010	0.115 (−2.7%)	0.128 (7.7%)	0.124 (4.5%)	0.094 (−20.6%)	0.120 (1.2%)	0.119
0.025	0.128 (0.4%)	0.136 (7.0%)	0.133 (4.8%)	0.111 (−13.1%)	0.129 (1.8%)	0.127
0.050	0.138 (0.3%)	0.144 (4.2%)	0.142 (2.8%)	0.125 (−9.4%)	0.139 (0.6%)	0.138
0.100	0.150 (1.4%)	0.153 (3.1%)	0.152 (2.4%)	0.141 (−4.6%)	0.150 (1.1%)	0.148
0.150	0.159 (1.3%)	0.159 (1.8%)	0.159 (1.5%)	0.153 (−2.6%)	0.158 (0.7%)	0.157
0.200	0.165 (1.3%)	0.165 (1.1%)	0.165 (1.1%)	0.161 (−1.2%)	0.164 (0.7%)	0.163
0.300	0.176 (1.5%)	0.174 (0.5%)	0.175 (0.8%)	0.175 (0.9%)	0.175 (0.9%)	0.173
0.400	0.185 (1.5%)	0.182 (0.1%)	0.183 (0.5%)	0.186 (2.1%)	0.184 (1.0%)	0.182
0.500	0.193 (1.4%)	0.191 (−0.1%)	0.192 (0.4%)	0.196 (2.8%)	0.193 (1.1%)	0.191
0.600	0.202 (0.8%)	0.199 (−0.6%)	0.200 (−0.1%)	0.206 (2.7%)	0.202 (0.7%)	0.200
0.700	0.211 (0.1%)	0.209 (−1.1%)	0.210 (−0.6%)	0.216 (2.4%)	0.211 (0.2%)	0.211
0.750	0.216 (−0.2%)	0.214 (−1.2%)	0.215 (−0.8%)	0.221 (2.1%)	0.217 (−0.0%)	0.217
0.800	0.222 (−0.4%)	0.220 (−1.0%)	0.221 (−0.7%)	0.227 (1.9%)	0.222 (−0.1%)	0.223
0.850	0.228 (−0.7%)	0.228 (−0.9%)	0.228 (−0.7%)	0.234 (1.6%)	0.229 (−0.3%)	0.230
0.900	0.234 (−0.9%)	0.238 (−0.4%)	0.238 (−0.5%)	0.242 (1.2%)	0.238 (−0.5%)	0.239
0.950	0.249 (−1.7%)	0.253 (0.1%)	0.252 (−0.4%)	0.253 (−0.0%)	0.250 (−1.2%)	0.253
0.975	0.259 (−1.4%)	0.267 (1.6%)	0.265 (0.6%)	0.262 (−0.4%)	0.260 (−1.0%)	0.263
0.990	0.272 (0.7%)	0.285 (5.5%)	0.280 (3.8%)	0.272 (1.0%)	0.272 (1.0%)	0.270
BIAS%	0.55%	0.56%	0.55%	0.05%	0.50%	
MAE%	1.03%	1.21%	0.97%	2.71%	0.72%	
RMSE%	1.16%	1.96%	1.39%	4.08%	0.82%	

Note: Numbers in columns 2–6 represent quantile of net benefit at order p (in \$1,000,000); with percentage error compared with simulated values in parentheses.

of the two economic performance criteria from the simulation were taken to be the "true" values.

Criteria for Comparison

The relative performance of the candidate probability models considered for the net benefit and B/C ratio was measured by three performance criteria: (1) Percentage biasness; (2) percentage mean absolute error; and (3) percentage root mean squared error. Each of the three criteria was used simultaneously in an attempt to identify the best probability model for describing the random characteristics of the net benefit and B/C ratio. These criteria are mathematically defined as

1. Percentage biasness (BIAS%)

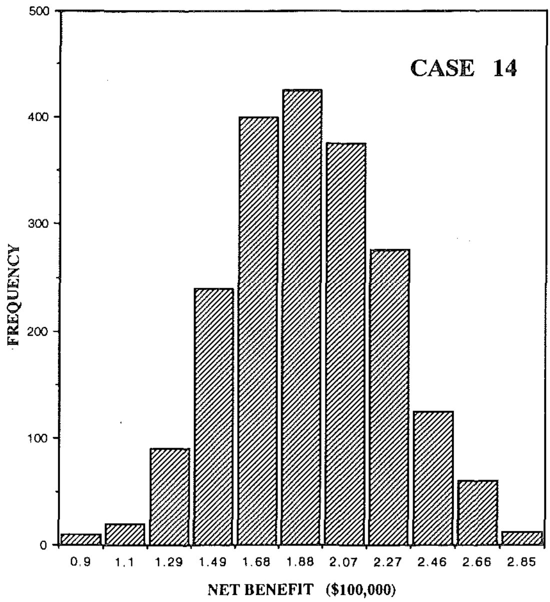

FIG. 1. Histogram of Simulated Net Benefit for Case 14

$$\text{BIAS\%} = \int_0^1 \left(1 - \frac{x_{p,f}}{x_p}\right) dp \quad \text{(16)}$$

2. Percentage mean absolute error (MAE%)

$$\text{MAE\%} = \int_0^1 \left|1 - \frac{x_{p,f}}{x_p}\right| dp \quad \text{(17)}$$

3. Percentage root mean squared error (RMSE%)

$$\text{RMSE\%} = \left[\int_0^1 \left(1 - \frac{x_{p,f}}{x_p}\right)^2 dp\right]^{0.5} \quad \text{(18)}$$

where x_p and $x_{p,f}$ = the "true" value from the simulation and the estimate of the pth-order quantile determined from the assumed probability model, f, in conjunction with the mean and variance from the FOSM method, respectively. In computing the values of the three model performance criteria, numerical integration was performed at probability levels p = (0.01, 0.025, 0.05, 0.10, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.80, 0.85, 0.90, 0.95, 0.975, 0.99), where the quantiles x_{pf} from the assumed distribution model were also computed. The value of the pth-order quantile for the four parametric distributions could be easily calculated. For the FC model, the algebra involved for computing the quantiles of any random variable with known first four moments are given in the Appendix I.

TABLE 6. Comparisons of Skew Coefficient and Kurtosis for Net Benefit Method and Simulation

Case number (1)	Method (2)	Subcase Number 1 (3)	2 (4)	3 (5)	4 (6)	5 (7)	6 (8)
		(a) Skew Coefficient					
1	S	0.058	0.077	0.101	0.102		
	F	0.042	0.058	0.117	0.117		
2	S	−0.103	−0.067	−0.071	−0.077	−0.041	−0.073
	F	−0.072	−0.072	−0.050	−0.100	−0.100	−0.101
3	S	−0.130	−0.182	0.111	−0.025	−0.210	−0.190
	F	−0.187	−0.147	0.039	−0.154	−0.193	−0.184
4	S	0.178	0.146	0.278	0.126	0.121	0.170
	F	0.165	0.129	0.130	0.165	0.097	0.126
5	S	0.014	−0.000	0.026	0.062	−0.044	0.024
	F	0.000	0.008	0.009	0.000	0.001	0.002
6	S	−0.044	0.046	−0.174	0.192	−0.033	−0.083
	F	−0.115	−0.075	−0.115	0.350	−0.085	−0.088
7	S	0.196	0.043	0.236	0.312	0.207	0.201
	F	0.182	0.110	0.164	0.204	0.201	0.193
8	S	0.062	−0.165	0.027	0.086	0.094	0.056
	F	0.053	−0.082	0.024	0.051	0.076	0.029
9	S	−0.071	0.064	−0.074	−0.023	0.094	−0.054
	F	−0.063	−0.041	−0.055	−0.054	0.153	−0.086
		(b) Kurtosis					
1	S	2.763	2.563	2.664	2.733		
	F	2.562	2.727	2.652	2.653		
2	S	2.873	2.760	2.672	2.648	2.624	2.629
	F	2.855	2.683	2.654	2.867	2.756	2.745
3	S	3.011	2.898	2.880	2.760	2.752	2.826
	F	2.897	2.870	1.806	2.877	2.851	2.894
4	S	2.574	2.743	2.868	2.678	2.692	2.876
	F	2.736	2.671	2.594	2.691	2.650	2.685
5	S	2.774	2.656	2.559	2.732	2.676	3.057
	F	3.000	2.750	2.727	2.828	2.695	2.924
6	S	3.081	2.897	2.930	3.076	2.896	2.939
	F	3.042	3.015	2.965	3.246	2.900	2.897
7	S	2.825	2.801	2.875	3.041	2.560	2.649
	F	2.686	1.273	2.676	2.660	2.727	2.741
8	S	2.718	3.091	2.670	2.690	2.567	2.669
	F	2.843	4.791	2.746	2.736	2.730	2.747
9	S	3.173	2.993	2.713	2.574	3.228	2.966
	F	3.057	3.020	2.964	2.727	3.042	2.981

Note: S = by simulation; and F = by FOSM method.

Results and Analylsis

Judging on the basis of BIAS%, there is no single probability model among the five candidates, in the majority of the 52 subcases, dominating the others. However, with regard to MAE% and RMSE%, a best model could clearly be identified. Tables 3 and 4 tabulate the frequency that each candidate probability model for the net benefit and B/C ratio take on the various ranks. Rank 1 represents the best associated with the smallest value of MAE% or RMSE%.

From Tables 3 and 4 one observes that, in the great majority of the 52 subcases, the FC asymptotic expansion model best described the random characteristics of both the net benefit and B/C ratio based on the performance criteria MAE% and RMSE%. The Weibull distribution clearly is

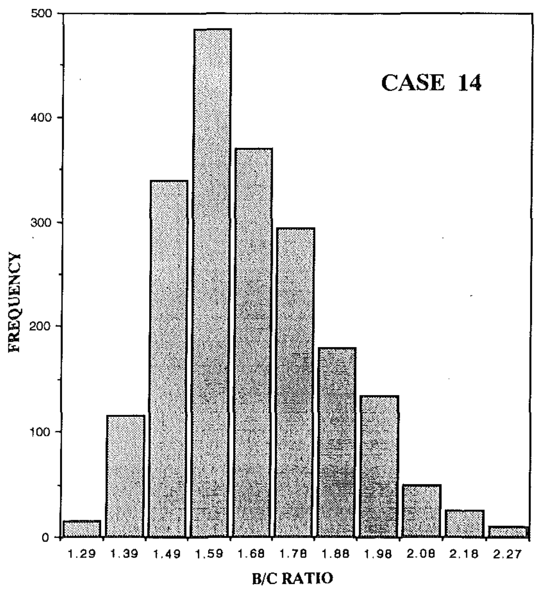

FIG. 2. Histogram of Simulated Benefit/Cost Ratio for Case 14

the worst distribution model to use for the two economic criteria considered herein.

As far as the net benefit is concerned, the normal distribution, among the four parametric distributions (i.e., normal, lognormal, gamma, and Weibull), yielded a better fit than the others when compared on the basis of MAE% and RMSE%. Although the normal distribution is dominated by the FC asymptotic expansion on the basis of ranking, the differences in MAE% and RMSE% between the two distributions are quite small (see Table 5). A histogram of the net benefit from the simulation representative of the 52 subcases is shown in Fig. 1. This histogram strongly suggests the existence of symmetry. A possible explanation for the normal distribution being outperformed by the FC model is that a symmetric distribution might not be normal; although the normal distribution is symmetric. This could be observed by comparing the skew coefficient = 0 for a symmetric distribution and kurtosis of the simulated net benefits with those associated with the assumed probability models (see Table 6). The use of the FC model takes into account the effect of skewness and kurtosis, which made the approximation better.

With regard to the B/C ratio, the lognormal distribution performed better than the other parametric distributions on the basis of MAE% and RMSE% (see Table 4). The gamma distribution followed as a close second. The histogram for the simulated B/C ratio representative of the 52 subcases considered is shown in Fig. 2. Examining Fig. 2 and Table 7, one observes that there indeed existed a positive skewness in the histogram. Although it is observed that a lognormal distribution describes the random behavior of the B/C ratio better than a gamma as far as the overall rank is concerned, the differences in the magnitude of the performance criteria between the two distributions are not significant (see Table 8).

TABLE 7. Comparisons of Skew Coefficient and Kurtosis for B/C Ratio Obtained by First-Order Analysis and Simulation

Case number (1)	Method (2)	Subcase Number 1 (3)	2 (4)	3 (5)	4 (6)	5 (7)	6 (8)
		(a) Skew Coefficient					
1	S	0.405	0.569	0.626	0.585		
	F	0.515	0.551	0.598	0.595		
2	S	0.398	0.327	0.389	0.372	0.354	0.267
	F	0.390	0.301	0.377	0.348	0.274	0.277
3	S	0.296	0.212	1.114	0.451	0.100	0.205
	F	0.263	0.316	−0.173	0.295	0.191	0.249
4	S	0.619	0.532	0.697	0.630	0.492	0.579
	F	0.607	0.427	0.427	0.607	0.399	0.425
5	S	0.436	0.292	0.397	0.403	0.393	0.482
	F	0.410	0.324	0.386	0.323	0.391	0.408
6	S	0.363	0.475	0.180	−0.203	0.354	0.270
	F	0.284	0.336	0.284	1.227	0.274	0.285
7	S	0.530	0.588	0.636	0.704	0.601	0.645
	F	0.446	0.572	0.437	0.457	0.602	0.610
8	S	0.367	0.415	0.393	0.458	0.418	0.452
	F	0.344	0.365	0.399	0.409	0.415	0.400
9	S	0.308	0.448	0.129	0.218	1.182	0.169
	F	0.303	0.347	0.248	0.235	0.742	0.237
		(b) Kurtosis					
1	S	3.089	2.881	3.011	2.951		
	F	2.343	2.622	2.556	2.557		
2	S	3.270	2.832	2.766	3.071	2.854	2.680
	F	2.967	2.627	2.554	2.960	2.695	2.686
3	S	3.134	2.879	5.874	3.420	2.676	3.007
	F	3.063	3.006	0.086	2.993	2.925	3.039
4	S	2.894	3.049	3.533	3.136	3.011	3.329
	F	2.587	2.412	2.387	2.576	2.485	2.428
5	S	3.217	2.713	2.688	2.896	2.867	3.617
	F	3.000	2.654	2.622	2.675	2.573	2.979
6	S	3.238	3.486	2.965	5.446	3.188	3.069
	F	3.086	3.030	3.069	4.502	3.038	3.039
7	S	3.165	3.095	3.432	3.725	2.906	3.015
	F	2.447	2.139	2.445	2.435	2.634	2.604
8	S	2.819	3.679	2.804	2.901	2.714	2.812
	F	2.694	2.140	2.600	2.596	2.631	2.604
9	S	3.221	3.482	2.552	2.504	6.672	2.734
	F	3.086	3.031	2.950	2.620	4.658	2.954

Note: S = by simulation; and F = by first-order analysis.

The above discussions and observations are made from an overall view of the performance of the five distribution models for describing the random characteristics of the net benefit and the B/C ratio based on all 52 subcases considered. There is a case, i.e., case 7, that deserves some special attention. Note that in case 7 the total benefit is positively skewed while the total cost is negatively skewed. This type of configuration of benefit and cost corresponds to the general conservative practice in economic analysis when faced with uncertainty; i.e., the total project benefit is somewhat deflated while the total cost is inflated. For this particular case, the rankings of the five distribution models for the net benefit as well as for the B/C ratio are given in Table 9. As can be seen, in the majority of the six subcases in case 7, the gamma distribution (rather than the normal) dominated other para-

TABLE 8. Quantile Values of B/C Ratio and Percentage Error (in Parentheses) at Different Probability Levels under Various Distribution Modesl Considered for Case 13

p (1)	Normal (2)	Lognormal (3)	Gamma (4)	Weibull (5)	Fisher/Cornish (6)	Simulation (7)
0.010	1.284 (−5.0%)	1.322 (−2.2%)	1.310 (−3.1%)	0.906 (−33.0%)	1.191 (−11.9%)	1.352
0.025	1.347 (−2.6%)	1.372 (−0.8%)	1.364 (−1.3%)	1.040 (−24.7%)	1.305 (−5.6%)	1.382
0.050	1.401 (−1.7%)	1.416 (−0.6%)	1.411 (−0.9%)	1.156 (−18.8%)	1.405 (−1.3%)	1.424
0.100	1.463 (−0.4%)	1.470 (0.1%)	1.467 (−0.1%)	1.288 (−12.3%)	1.479 (0.8%)	1.468
0.150	1.505 (0.8%)	1.507 (0.9%)	1.506 (0.8%)	1.374 (−8.0%)	1.510 (1.1%)	1.493
0.200	1.538 (1.0%)	1.537 (0.9%)	1.537 (0.9%)	1.441 (−5.4%)	1.531 (0.5%)	1.523
0.300	1.593 (1.9%)	1.587 (1.5%)	1.589 (1.6%)	1.546 (−1.1%)	1.570 (0.4%)	1.563
0.400	1.639 (1.8%)	1.631 (1.4%)	1.634 (1.5%)	1.631 (1.4%)	1.614 (0.3%)	1.609
0.500	1.683 (2.1%)	1.674 (1.5%)	1.677 (1.7%)	1.708 (3.6%)	1.664 (0.9%)	1.649
0.600	1.726 (1.5%)	1.718 (1.0%)	1.720 (1.2%)	1.781 (4.8%)	1.719 (1.1%)	1.700
0.700	1.772 (0.9%)	1.765 (0.5%)	1.768 (0.7%)	1.855 (5.6%)	1.776 (1.1%)	1.756
0.750	1.798 (0.2%)	1.793 (−0.1%)	1.795 (0.0%)	1.895 (5.6%)	1.805 (0.6%)	1.795
0.800	1.827 (−0.3%)	1.823 (−0.5%)	1.825 (−0.4%)	1.937 (5.8%)	1.835 (0.2%)	1.832
0.850	1.860 (−1.2%)	1.860 (−1.2%)	1.860 (−1.2%)	1.986 (5.5%)	1.866 (−0.9%)	1.882
0.900	1.902 (−2.1%)	1.907 (−1.8%)	1.906 (−1.9%)	2.044 (5.3%)	1.903 (−2.0%)	1.942
0.950	1.964 (−3.9%)	1.978 (−3.2%)	1.974 (−3.4%)	2.126 (4.0%)	1.971 (−3.6%)	1.044
0.975	2.018 (−4.0%)	2.043 (−2.9%)	2.035 (−3.3%)	2.194 (4.3%)	2.063 (−1.9%)	2.103
0.990	2.081 (−4.1%)	2.120 (−2.3%)	2.107 (−2.9%)	2.268 (4.6%)	2.224 (2.5%)	2.169
BIAS%	0.28%	0.25%	0.26%	−0.35%	−0.07%	
MAE%	1.57%	1.15%	1.27%	5.93%	1.18%	
RMSE%	1.86%	1.35%	1.50%	8.10%	1.95%	

Note: Numbers in columns 2–6 represent quantile of B/C ratio at order p; with percentage error compared with simulated values in parentheses.

metric distributions for the net benefit while the lognormal distribution is dominant for the B/C ratio.

SUMMARY AND CONCLUSIONS

Probabilistic approaches to B/C analysis have been receiving more and more attention by researchers and practitioners. The success and conclusion of the analysis using such an approach could hinge on an accurate assessment of the probability density function of economic evaluation criteria and their statistical properties. Because the distribution of various benefit and cost components involved in a project could be different, on analytical derivation of the probability density function for the economic evaluation criteria could be difficult. As a practical alternative, probabilistic B/C analysis could be

TABLE 9. Ranking of Candidate Probability Models for Net Benefit and B/C Ratio Based on MAE% and RMSE% for Case 7

	SUBCASE NUMBER											
	1		2		3		4		5		6	
Rank (1)	MAE% (2)	RMSE% (3)	MAE% (4)	RMSE% (5)	MAE% (6)	RMSE% (7)	MAE% (8)	RMSE% (9)	MAE% (10)	RMSE% (11)	MAE% (12)	RMSE% (13)
(a) Net Benefit												
1	FC	FC	N	N	FC	FC	FC	FC	FC	FC	FC	FC
2	N	N	G	W	N	G	G	G	G	G	G	G
3	G	G	W	G	G	N	N	LN	N	N	N	N
4	LN	LN	LN	LN	LN	LN	LN	LN	LN	LN	LN	LN
5	W	W	FC	FC	W	W	W	W	W	W	W	W
(b) B/C Ratio												
1	FC	LN	LN	LN	FC	LN	FC	LN	FC	LN	FC	LN
2	LN	G	G	G	LN	G	LN	G	LN	G	LN	G
3	G	FC	N	N	G	FC	G	FC	G	FC	G	N
4	N	N	FC	FC	N	N	N	N	N	N	N	FC
5	W	W	W	W	W	W	W	W	W	W	W	W

Note: Fisher/Cornish; G = Gamma; LN = Lognormal; N = Normal; and W = Weibull.

carried out by first estimating the statistical moments of the net benefit and/or the B/C ratio using the FOSM method. Then, the estimated statistical moments could be utilized jointly with an appropriate probability distribution function. The issue here is which probability distribution is the most appropriate. Because the probability distributions of the net benefit and B/C ratio are functions of those of benefit and cost components, a practical question is how sensitive are the most appropriate distributions of the net benefit and the B/C ratio of the distributions of benefit and cost components.

This paper presents a study examining the appropriateness of some commonly used probability distributions as applied to describe the random characteristics of the net benefit and B/C ratio when the distributions of benefit and cost components may be different. The results indicate that, in the majority of the cases considered, the Fisher-Cornish (Fisher and Cornish 1960) asymptotic expansion is the best probability model for both net benefit and B/C ratio. The second best probability model for the net benefit is the normal distribution whereas the lognormal distribution is for the B/C ratio.

Although the FC asymptotic expansion turns out to be the best probability model in describing the random characteristics of the net benefit and B/C ratio, it was, however, a bit more cumbersome computationally as compared with the normal and lognormal distributions. This is because it requires the estimation of skew coefficient and kurtosis. From the practical viewpoint, the differences in the MAE and RMSE between the FC expansion and the second best distribution for the net benefit and for the B/C ratio are insignificant. Therefore, the adoption of normal distribution for the net benefit and of lognormal distribution for the B/C ratio in probabilistic B/C analysis should be acceptable.

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Appendix I. Fisher-Cornish Asymptotic Expansion

In Fisher-Cornish asymptotic expansion, the quantile of any nonnormal, standardized distribution is related to the standard normal quantile and higher-order moments. The pth-order quantile for any standardized random variable W can be approximated, using only the skew coefficient and kurtosis, as

$$w_p = z_p + \gamma_W \frac{H_2(z_p)}{6} + \kappa_W \frac{H_3(z_p)}{24} - \gamma_W^2 \frac{2H_3(z_p) + H_1(z_p)}{36} \quad \ldots\ldots (19)$$

in which z_p = the pth-order quantile from the standard normal distribution, and $H_1(z_p)$, $H_2(z_p)$, and $H_3(z_p)$ = Hermit polynomials, which can be computed by (Abramowitz and Stegun 1972)

$$H_r(z_p) = z_p^r - \frac{r^2}{2\cdot 1!} z_p^{r-2} + \frac{r^4}{2^2\cdot 2!} z_p^{r-4} - \ldots \quad \ldots\ldots (20)$$

Once the value of w_p by (19) is computed, the pth-order quantile of the original random variable Y can be obtained as

$$y_p = \hat{Y} + w_p S_Y \quad \ldots\ldots (21)$$

where $W = (Y - \bar{Y})/S_Y$, with $\bar{Y}$ and S_Y being the mean and standard deviation of random variable Y.

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Appendix III. Notations

The following symbols are used in this paper:

B_i = ith benefit item;
$\bar{B}$ = mean of total benefit;
$\bar{B}_i$ = mean of the ith benefit item;
C_j = jth cost item;
$\bar{C}$ = mean of total cost;
$\bar{C}_j$ = mean of jth cost item;
$H_r(\,)$ = rth order Hermit polynomial;
K = constant;
M = number of benefit items;

N = number of cost items;
S_x = standard deviation of random variable X;
U = benefit-cost ratio;
U_0 = $\bar{B}/\bar{C}$;
V = net benefit;
X_{mo} = estimation of most likely value;
X_{op} = estimation of optimistic value;
X_{pe} = estimation of pessimistic value;
x_p = pth order quantile of random variable X;
$x_{p,f}$ = estimated pth order quantile of random variable X based on distribution model f;
z_p = pth order quantile of standard normal random variable;
γ_X = skew coefficient of random variable X;
κ_X = kurtosis of random variable X;
ρ = correlation coefficient; and
Ω_X = coefficient of variation of random variable X.