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Kaluza-Klein Induced Weyl Invariant Effective Theory

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All dimensional parameters in most gravitational models can be promoted to dimensional field variables which can be embedded in some higher dimensional Kaluza-Klein vielbein. An elegant method of computing dimensional reduction process via vielbein and differential form is introduced. We compute the reduced lower dimensional effective scale invariant action after the dimensional reduction takes place. Some applications to the inflationary universe are also discussed.

Kaluza-Klein theory[1] has been an attractive and promising candidate for unifying gravitation and gauge interactions. In this approach, gauge connections are considered as off block-diagonal components of some higher dimensional spin connections. These gauge fields assume their low energy effective shapes immediately after the dimensional reduction takes place. The dimensional reduction process is generally believed to be active beyond the Planck scale under a (still) unknown mechanism. It is, however, suspected that the mechanism of the dimensional reduction might be similar to the mysterious spontaneous symmetry breaking algorithm induced by the Higgs mechanism. Therefore, it is important to study the Kaluza-Klein theory in greater details.

Moreover, the scale invariant effective theory[2] has been shown to indicate that the scale symmetry should be a manifest symmetry in many aspects. It is found that we can embed the dimension of all the dimensional parameters via the dimension-one scalars fields ϕ_i in $g_{\mu\nu}$ such that the general scalar measure $d^Dx\sqrt{g}$ is made dimensionless. In that case, $\dim g_{\mu\nu} = 2$. Therefore, $ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu$ becomes a dimensionless measure and hence all the dimensional parameters can be induced dynamically. The scale invariance can thus be introduced in a natural way (see below).

Our approach also differs from previous works[3] that imposes conformal frame in order to reproduce a $\phi^2 R$ interaction mainly by our motivation and its intrinsic physics that resolves the dimensional mystery and reproduces the scale invariant action simultaneously in a more convincing way. For example, the action $S = -\int\sqrt{g}\,(\phi^2 R + 1/2\nabla_\mu\phi\nabla^\mu\phi)$, with $\nabla_\mu \equiv \partial_\mu - S_\mu$, is a well-known conformal invariant action. It is also noted that the conventional scalar metric measure $g_{\mu\nu}dx^\mu dx^\nu$ should be modified[2] as $ds^2 \equiv \phi^2 g_{\mu\nu}dx^\mu dx^\nu$ in order to preserve local scale invariance. To be more specifically, we will interprete the conformal metric $G_{\mu\nu} = \phi^2 g_{\mu\nu}$ in this conformal invariant theory as a product of a scalar field ϕ and the conventional Riemannian metric $g_{\mu\nu}$.

In what follows,[4] the lower case latin indices a, b, c from the beginning and l, m, n from the middle of the alphabet will denote flat and curved internal space indices respectively. Also, the greek indices a, β, γ from the beginning and μ, ν, λ from the middle denote flat and curved space-time indices respectively. We use the coordinate $Z^M = (x^\mu, y^m)$ with x and y denoting the space-time and internal space coordinate respectively. Let ' s denote also D and d as the dimension of x-space andy-space respectively. We will write $N = D + d$ for convenience, i.e. we are considering an N-dimensional Kaluza-Klein theory (with M^N denoting its base manifold) which undergoes a dimensional reduction process to a D-dimensional Einstein-Yang-Mills theory. Also all hatted variables (e.g. $\hat{R}$) denotes N-dimensional variables and un-hatted variables (e.g. R) denote D-dimensional variables respectively whenever it is necessary.

In this paper, we will show that the proposed N-dimensional vielbein[1]

$$E_M^A = \begin{pmatrix} Ae_\mu^\alpha, & BA_\mu^m R_m^a \\ 0, & CR", \end{pmatrix} \tag{1}$$

can be considered as a consistent ansatz. Here we have assumed $E_M{}^A(Z) = E_M{}^A(x)$ and assumed further that all variables hidden in the vielbein *(1)* are y-independent. If the internal y-space general coordinate transformation is to be interpreted as $[U_1]^d$ abelian gauge transformation, one can show that B = C/e with e denoting the coupling constant associated with the gauge field A_μ. Assuming dim $E_M{}^A$ = 1, or equivalently dim A = dim C = 1 and dim $e_\mu{}^\alpha$ = dim $R_m{}^a$ = dim $B = 0$, *we* can derive the reduced action and solve for an inflationary solution. Therefore, we will consider the N-dimensional pure gravitational action given below:

$$\tilde{S} = \int d^N Z \sqrt{G}\,(-\hat{R} - \Lambda) \tag{2}$$

Here A is a dimensionless cosmological constant and G_{MN} denotes the N-dimensional (pseudo-) Riemannian metric tensor. Note that dim G_{MN} = 2 implies dim $\tilde{S}$ = 0. It is straightforward to show that an N-dimensional general coordinate transformation $Z'^M = Z^M + \varepsilon^M$ will induce the following D-dimensional transformation

$$\delta A = \epsilon^\mu \partial_\mu A \tag{3}$$

$$\delta B = \epsilon^\mu \partial_\mu B \tag{4}$$

$$\delta C = \epsilon^{\mu}\partial_{\mu}C \tag{5}$$

$$\delta e^{\alpha}_{\mu} = \epsilon^{\nu}\partial_{\nu}e^{\alpha}_{\mu} + \partial_{\mu}\epsilon^{\nu}e^{\alpha}_{\nu} \tag{6}$$

$$\delta R^{a}_{m} = \epsilon^{\mu}\partial_{\mu}R^{a}_{m} \tag{7}$$

$$\delta A^{m}_{\mu} = \epsilon^{\nu}\partial_{\nu}A^{m}_{\mu} + \partial_{\mu}\epsilon^{\nu}A^{m}_{\nu} + \frac{C}{B}\partial_{\mu}\epsilon^{m}. \tag{8}$$

Here we have assumed $E_M{}^A(Z) = E_M{}^A(x)$ and $\varepsilon^M(Z) = (\varepsilon^\mu(x), \varepsilon^m(x))$. Note that if $B = C/e$, the transformation (8) can be interpreted as a D-dimensional general coordinate transformation on a vecter field plus a $[U_1]^d$ gauge transformation. Furthermore, (3)-(7) show that A, B, C and $R_m{}^a$ *are* to be interpreted as D-dimensional general scalar and $e_\mu{}^\alpha$ is to be interpreted as D-dimensional vielbein.

It is known that $E_M{}^A$ given by (1) signifies the zero mode expansion of action (2). We need to show, however, that (1) is indeed a consistent ansatz to the field equation. The proof can be completed by noting that the Euler-Lagrange equaiton for the N-dimensional theory reads

$$\delta(\sqrt{G}\tilde{\mathcal{L}}) = \sqrt{G}(\hat{R}^{MN} - \frac{1}{2}G^{MN}(\hat{R} + \Lambda))\delta G_{MN} = 0. \tag{9}$$

Note also that $\delta G_{MN} = \delta G_{MN}/\delta A\ \delta A + \delta G_{MN}/\delta C\ \delta C + \delta G_{MN}/\delta A_\mu{}^m\ \delta A_\mu{}^m + \delta G_{MN}/\delta g_{mn}\ \delta g_{mn} + \delta G_{MN}/\delta g_{\mu\nu}\ \delta g_{\mu\nu}$. Here $G_{MN} \equiv E_M{}^A E_{NA}$, $g_{mn} \equiv R_m{}^a R_{na}$ and $g_{\mu\nu} \equiv e_\mu{}^\alpha e_{\nu\alpha}$ with $E_M{}^A$ given by (1).

It is thus straightforward to show that $\delta(\sqrt{g}\mathrm{L}) = 0$, the Euler-Lagrange equation for the D-dimensional Kaluza-Klein reduced effective theory described by the effective reduced action $S \equiv \int \sqrt{G}\tilde{\mathrm{L}}(E_M{}^A(A, C, A_\mu{}^m, g_{\mu\nu})) = \int \sqrt{g}\mathrm{L}(A, C, A_\mu{}^m, g_{\mu\nu})$, is exactly the same as (9) *once* we adopt the metric tensor of the form given by (1). Hence we show that the Kaluza-Klein viel-bein (1) is indeed compatible with the dimensional reduction process specified by the Kaluza-Klein vieibein (1). Thus (1) can be considered as a consistent ansatz.

One remarks here that there may exist nontrivial constraint that excludes many unphysical ansatze by above consistency check. For example, it has been shown in ref. [5] that the the standard Kaluza-Klein ansatz for massless gauge fields is in general inappropriate. Exceptions are the compactifications from an eleven dimensional manifold M^{11} to M^4 x S^7 in $d = 11$ supergravity. Although it is possible that the inconsistency may be resolved if we understand the mechanism of the compactification better, it is, however, important to make *sure* that our ansatz does survive the consistency check at this stage. This will enable us to treat the compactified lower dimensional action as effective theory.

In order to perform a consistency check,' we will in general need the detailed form for the Ricci tensor R_{MN}. This can be easily derived by using differential form formalism. Let us define the curvature 2-form as $\hat{C}_{AB} \equiv 1/2\, \hat{R}_{ABMN} dZ^M \wedge dZ^N = d\omega_{AB} + \omega_{AC} \wedge \omega_{CB}$. Here $\omega_{AB} = \omega_{ABM} dZ^M$ is the connection 1-form, and $\omega_{ABM} \equiv E_{AN} D_M E_B{}^N$ is the connection for the corresponding base Riemannian manifold. Note that the Ricci tensor $\hat{R}_{MN} \equiv \hat{R}_{MNP}{}^P$ and the curvature 2-form $\hat{C}_{MN} = E_M{}^A E_N{}^B \hat{C}_{AB}$ are defined differently with different symmetries under permutation with respect to M and N.

We will writie $\phi = A$, $\varphi = C$ and $h_{mn} = \varphi^2 g_{mn}$. Note that we can set $\phi = 1$ for convenience and restore the ϕ field afterwards. Note that restoring the ϕ field is rather straightforward by observing that the compatibility of g_{mn} $(D_\xi g_{mn} = 0)$ implies the compatibility of $\bar{g}_{\mu\nu}$ $(\equiv \phi^2 g_{\mu\nu})$. Note that $\phi^2 R(\bar{g}_{\mu\nu}) = R(g_{\mu\nu}) + 2(D-1) D_\mu \partial^\mu \ln \phi + (D-1)(D-2)\partial_\mu \ln \phi \partial_\mu \ln \phi$. This will simplify our computations greatly. By setting $\phi = 1$, one derives the following expressions for the curvature 2-form $\hat{C}_{MN}$:

$$
\begin{aligned}
\hat{C}_{\alpha\beta} = \;& C_{\alpha\beta} - \frac{1}{4} dZ^{\gamma\lambda}(F_{\alpha\beta a} F_{\gamma\lambda a} + F_{\alpha\gamma a} F_{\beta\lambda a}) \\
& - \frac{1}{2} dZ^{\gamma a}(D_\gamma F_{\alpha\beta a} + F_{\alpha\gamma b} A_{\beta b a} - F_{\beta\gamma b} A_{\alpha b a} + 2F_{\alpha\beta b} A_{\gamma a b}) \\
& - dZ^{ab}\left(\frac{1}{4} F_{\alpha\gamma a} F_{\beta\gamma b} + A_{\alpha c a} A_{\beta c b}\right)
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\hat{C}_{\alpha a} = \;& -\frac{1}{2} dZ^{\beta\gamma}(D_\beta F_{\alpha\gamma a} - F_{\alpha\beta b} A_{\gamma a b} \cdot I\, F_{\beta\gamma b} A_{\alpha a b}) \\
& + dZ^{\beta b}\left(\frac{1}{4} F_{\alpha\gamma b} F_{\beta\gamma b} - D_\beta A_{\alpha a b} + 2A_{\alpha a c} A_{c b \beta} - 2A_{\alpha c b} A_{a c \beta} - A_{\alpha c b} A_{\beta a c}\right) \\
& + \frac{1}{2} dZ^{bc} F_{\alpha\beta b} A_{\beta a c}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
C_{ab} = \;& -dZ^{\alpha\beta}\left(\frac{1}{4} F_{\alpha\gamma a} F_{\beta\gamma b} + A_{\alpha c a} A_{\beta c b}\right) + dZ^{\alpha c}\left(\frac{1}{2} A_{\gamma a c} F_{\gamma\alpha b} - \frac{1}{2} A_{\gamma b c} F_{\gamma\alpha a}\right) \\
& - dZ^{cd} A_{\alpha a c} A_{\alpha b d}.
\end{aligned} \tag{12}
$$

Here we have defined $A_{\alpha ab} \equiv 1/2\, \partial_\alpha h_{mn} \bar{R}_a{}^m \bar{R}_b{}^n$ and $A_{ab\alpha} \equiv 1/2\, h_{mn} \bar{R}_a{}^m \partial_\alpha \bar{R}_b{}^n$ while $\bar{R}_a{}^m \equiv \varphi^{-1} R_a{}^m$. Also, we have written $dZ^M \wedge dZ^N$ as dZ^{MN} for simplicity. Note that all components of the curvature tensor $\hat{R}_{ABCD}$ can be read off directly from (10)-(12). After some algebra, one derives

$$
\begin{aligned}
\tilde{S} = & \int \sqrt{gh}\,\phi^{D-2}\{-R + (D-1)\partial_\mu \ln h \partial^\mu \ln \phi + (D-1)(D-2)\partial_\mu \ln \phi \partial^\mu \ln \phi \\
& - \frac{1}{4\phi^2} F^m_{\mu\nu} F^{\mu\nu n} h_{mn} + \frac{1}{4}\partial_\mu \ln h \partial^\mu \ln h + \frac{1}{4}\partial_\mu h_{mn} \partial^\mu h^{mn} - \Lambda\phi^2\},
\end{aligned} \tag{13}
$$

after a long but straightforward calculation. Here $h \equiv \det h_{mn}$. Note that we can reproduce the result in Ref. 1 by setting $g_{mn} = \delta_{mn}$, $\phi = \delta^\gamma$ and $\delta = \varphi^d = \sqrt{h}$.

Note that (13) has many different applications if h_{mn} is chosen differently. In what follows, we will give a few well-known example shortly. Note that the scale symmetry can be introduced by observing that the scale invariance can be guaranteed if the vielbein $E_M{}^A$ is kept fixed while varying its embedded physical fields $\phi, e_\mu{}^\alpha, \varphi$ and $R_m{}^a$ accordingly.

For example, the well-known local scale (Weyl) transformation

$$\phi' = s\phi \tag{14}$$

$$e'^{\alpha}_{\mu} = s^{-1} e^{\alpha}_{\mu} \tag{15}$$

will simply imply $E'_\mu{}^\alpha = E_\mu{}^\alpha$. Here $s = s(x)$ is the local scale parameter. Therefore the invariance of the action under (14) and (15) is apparently correct. Note that additional scale transformations can also be introduced by requiring φ and $R_m{}^a$ transform accordingly. Note also that there is, however, no room for $A_\mu{}^m$ to transform in this approach. This is expected and well-known result. We hence generated a whole class of scale invariant theories given by (13) which are derivable from Kaluza-Klein action (2).

For example, if we take $h_{mn} = \varphi^2\delta_{mn}$, one has

$$\begin{aligned}\tilde{S} = \int \sqrt{g}\phi^{D-2}\varphi^d\{&-R + (D-1)(D-2)\partial_\mu \ln\phi\partial^\mu \ln\phi + d(d-1)\partial_\mu \ln\varphi\partial^\mu \ln\varphi \\ &-\frac{\varphi^2}{4\phi^2}F^m_{\mu\nu}F^{\mu\nu n}\delta_{mn} + 2d(D-1)\partial_\mu \ln\varphi\partial^\mu \ln\phi - \Lambda\phi^2\}\end{aligned} \tag{16}$$

If $\varphi = u$ (a constant), $A_\mu{}^m = 0$ and $D = 4$, we will have

$$\tilde{S} = \int \sqrt{g}\xi(-\phi^2 R + 6\partial_\mu\phi\partial^\mu\phi - \Lambda\phi^4). \tag{17}$$

Here $\xi = u^d \int d^d y$. We can further write $\bar{\phi} = \sqrt{12\xi}\phi$ and $\lambda = \Lambda/16\xi$ such that

$$\tilde{S} = \int \sqrt{g}\left(-\frac{1}{2}\epsilon\bar{\phi}^2 R + \frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} - \frac{\lambda}{8}\bar{\phi}^4\right) \tag{18}$$

Here $\epsilon = 1/6$. Note that the action (18) is, however, not stable due to the negative kinetic energy term for $\bar{\phi}$. This is a general feature for the action (16) that all dimensional *one* fields ϕ and φ tend to be unstable. In general, there are a few ways out of this trouble. The first choice is that the Weyl symmetry has to be broken by imposing an asymptotic boundary condition on $\bar{\phi}$ in action (18), namely,

$$\bar{\phi}(r \to \infty) = v. \tag{19}$$

To be more specific, $\bar{\phi} = \bar{\phi}(t) = v$ is the only stable configuration to (18). Or equivalently, if the Kaluza-Klein induced Weyl invariant effective theory is suggested ' to play an important role in the inflationary universe, the stability of the action (18) favors the constant configuration $\bar{\phi}$ = v. Therefore, the action (18) reduced to

$$\tilde{S} = \int \sqrt{g}\left(-\frac{1}{2}\bar{\epsilon}R - \frac{\lambda}{8}v^4\right). \tag{20}$$

Here $\bar{\epsilon} = \epsilon v^2$. If (20) has something to do with the inflationary process in the very early universe, it has to admit a Robertson-Walker type inflationary solution. The Robertson-Walker metric can be read off directly from the expression: $ds^2 = -dt^2 + a^2(t)(dr^2/1-kr^2 + r^2d\Omega)$. Here $d\Omega$ is the solid angle $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$, and $k = 0, \pm 1$ stand for a flat, closed or open universe respectively.

Indeed, the equation of motion for (20) can be shown[2] to be

$$\frac{(a')^2 + k}{a^2} = \frac{\lambda v^2}{4} \tag{21}$$

$$2\frac{a''}{a} + \frac{(a')^2 + k}{a^2} = \frac{3\lambda v^2}{4} \tag{22}$$

Note that (21) and (22) have an inflationary solution, $a = a_0 e^{\sqrt{\frac{\lambda v^2}{4}}t} + \frac{k}{\lambda v^2 a_0}e^{-\sqrt{\frac{\lambda v^2}{4}}t}$. Similar argument as shown in Ref. 2 can be applied to show that above solution indicates a small cosmological constant observed today.

Also, the other choice out of the stability trouble is that at least one of the ϕ and φ fields must be Weyl transformed to other fields in order to reverse the negative kinetic term. For example, by requiring $\phi^{D-2}\varphi^d = 1$ and $\phi^D\varphi^\delta = \bar{\phi}^D$, one should write $\phi = \bar{\phi}^{D/2}$ and $\varphi = \bar{\phi}^{\frac{2D-D^2}{2d}}$. Therefore, (16) becomes

$$\tilde{S} = \int \sqrt{g}\left(-R - \frac{p}{2}\partial_\mu \ln\bar{\phi}\,\partial^\mu \ln\bar{\phi} - \Lambda\bar{\phi}^D\right). \tag{23}$$

Here $p = D^2(D-2)/2d\,(D+d-2) \geq 0$ for $D \geq 2$. Note that there are other feasible combinations considered previously.[3]

In summary, we have shown that a whole class of Weyl invariant effective theories can be generated from a higher dimensional Kaluza-Klein theory by promoting all dimensional parameters to dimensional field variables which can be embedded in the Kaluza-Klein vielbein in a natural way. We have also performed a simple consistency check to show that the vielbein ansatz (1) is indeed a proper ansatz to (2). We are now studying the generalization to non-abelian embedded Kaluza-Klein theories which are also very interesting.

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