Graphs and Combinatorics 8, 119–123 (1992)

Graphs and Combinatorics

Total Colorings of Graphs of Order $2n$ Having Maximum Degree $2n - 2$*

Bor-Liang Chen and Hung-Lin Fu

Department of Applied Mathematics National Chiao-Tung University, Hsin-Chu, Taiwan, Republic of China

Abstract. Let $\chi_t(G)$ and $\Delta(G)$ denote respectively the total chromatic number and maximum degree of graph G. Yap, Wang and Zhang proved in 1989 that if G is a graph of order p having $\Delta(G) \geq p - 4$, then $\chi_t(G) \leq \Delta(G) + 2$. Hilton has characterized the class of graph G of order $2n$ having $\Delta(G) = 2n - 1$ such that $\chi_t(G) = \Delta(G) + 2$. In this paper, we charactarize the class of graphs G of order $2n$ having $\Delta(G) = 2n - 2$ such that $\chi_t(G) = \Delta(G) + 2$.

1. Introduction

Throughout of this paper, all graphs are finite, simple, and undirected. Let G be a graph, and its vertex set, edge set, chromatic index, complementary graph, and the maximum degree be denoted by $V(G)$, $E(G)$, $\chi'(G)$, G^c, and $\Delta(G)$ respectively. For convenience, without mentioning otherwise, if H is isomorphic to a subgraph of G, we will simply call H is a subgraph of G. Other terms and notation not defined in this paper can be found in [3].

A *total coloring* π of a graph G is a mapping $\pi: V(G) \cup E(G) \to \{1, 2, \ldots\}$ such that no two adjacent vertices receive the same color, no two edges incident with the same vertex receive the same color, and no edge receives the same color as either of the vertices it is incident with. Define a k-total coloring of G, the total chromatic number $\chi_t(G)$ of a graph G is the smallest integer k such that G has a total coloring having image set $\{1, 2, \ldots, k\}$. From the definition of $\chi_t(G)$, it is clear that $\chi_t(G) \geq \Delta(G) + 1$. Behzad [2] and Vizing [8] made the following conjecture.

Total Coloring Conjecture (TCC). For any graph G, $\chi_t(G) \leq \Delta(G) + 2$.

The TCC has been verified for several classes of graphs [6, 7, 10], especially those graphs with very low or very high degree. Thus, similar to the argument of the chromatic index $\chi'(G)$ of G, we classify those graphs which satisfy the TCC.

Definition 1.1. A graph G is said to be of *type one* if $\chi_t(G) = \Delta(G) + 1$ and it is of *type two* if $\chi_t(G) = \Delta(G) + 2$.

* Research supported by National Science Council of the Republic of China (NSC 79-0208-M009-15)

In [5], Hilton proves the following theorem.

Theorem 1.1. *Let J be a subgraph of K_{2n}, let $e = |E(J)|$ and let $j(J)$ be the maximum size of a matching in J. Then $\chi_t(K_{2n} - E(J)) = 2n + 1$ if and only if $e + j \leq n - 1$.*

From this theorem, he deduces

Corollary 1.2. *If $H = K_{2n} - E(J)$, then H is of type one provided that $2n - 1 \geq e + j \geq n$, and H is of type two whenever $e + j \leq n - 1$.*

This corollary gives a complete classification of graphs of order $2n$ having maximum degree $2n - 1$. Since the graphs of order $2n$ with $\Delta(G) = 2n - 2$ also satisfy the TCC, we shall now classify these graphs. Our main result is as follows.

Theorem 1.2. *Let G be a graph of order $2n$ and $\Delta(G) = 2n - 2$. Then G is of type two if and only if G^c is a disjoint union of an edge and a star having $2n - 3$ edges.*

2. Proof of Theorem 1.2

Let S_x be a star having x edges. In what follows, two graphs are said to be disjoint provided that they have no vertex in common. The union of disjoint k stars $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ is denoted by $S(n_1, n_2, \ldots, n_k)$. For $2n = k + \sum_{i=1}^{k} n_i$, we let $G(n_1, n_2, \ldots, n_k) = K_{2n} - E(S(n_1, n_2, \ldots, n_k))$, i.e. $G(n_1, n_2, \ldots, n_k) = S(n_1, n_2, \ldots, n_k)^c$. It is easy to see that $\Delta(G(n_1, n_2, \ldots, n_k)) = 2n - 2$.

Lemma 2.1. *Let G be a graph of order $2n$ having $\Delta(G) = 2n - 2$. Then G is a subgraph of $G(n_1, n_2, \ldots, n_k)$ for some n_i, $i = 1, 2, \ldots, k$, such that $k + \sum_{i=1}^{k} n_i = 2n$.*

Proof. Since $\Delta(G) = 2n - 2$, G^c contains a spanning forest F. In F, by deleting those edges which are incident with two vertices of degrees greater than one, we obtain a spanning star $S(n_1, n_2, \ldots, n_k)$ forest of G^c, where $k + \sum_{i=1}^{k} n_i = 2n$. This implies that G is a subgraph of $G(n_1, n_2, \ldots, n_k)$. □

Now we are ready to work on the total coloring of G. Suppose $\chi_t(G) = g$ and π is a g-total coloring of G. We say that the color c_i *occurs* on the vertex v if either $\pi(v) = c_i$ or there is an edge e which is incident with v and $\pi(e) = c_i$. For each color c_i, let r_i be the number of vertices for which c_i occurs. It is easy to see that

$$\sum_{i=1}^{g} r_i = |V(G)| + 2|E(G)|. \tag{1}$$

We will use this fact to prove the following lemma.

Lemma 2.2. $G(1, 2n - 3)$ *is of type two.*

Proof. Assume that $G(1, 2n - 3)$ is of type one, i.e., $\chi_t(G(1, 2n - 3)) = 2n - 1$. Let $\{u, v\}$ be the independent edge which forms S_1 in $G(1, 2n - 3)^c$. First, suppose u and

v are colored with the same color, c_j. Since the center w of the deleted star S_{2n-3} is adjacent to u and v, $\pi(w) = c_1$, $\pi(\{w,u\}) = c_2$, and $\pi(\{w,v\}) = c_3$ are not c_j. Hence $r_j \leq 2n-2$. ($\pi(y) \neq c_j$ if $y \in V(G)\setminus\{u,v,w\}$.) For color c_i, $i \neq 1, 2, 3, j$, $r_i \leq 2n-1$. In total, we have $\sum_{i=1}^{g} r_i \leq (2n-2) + 3(2n) + (2n-5)\cdot(2n-1) = 4n^2 - 4n + 3 < |V(G(1,2n-3))| + 2|E(G(1,2n-3))|$. Thus $2n-1$ colors are not enough. Next, suppose u and v are colored with different colors. There exists a vertex in $V(G)\setminus\{u,v,w\}$ which is colored with color $c_1 = \pi(w)$. Similarly, $\sum_{i=1}^{g} r_i = r_1 + \sum_{i=2}^{g} r_i \leq 2n + (2n-2)(2n-1) < |V(G(1,2n-3))| + 2|E(G(1,2n-3))|$. Again, $2n-1$ colors are not enough. Hence $G(1,2n-3)$ is type two. □

Later we shall prove that $G(1,2n-3)$ is *critical* in the sense that without lowering the maximum degree of $G(1,2n-3)$, if we delete one edge from $G(1,2n-3)$, then the new graph obtained is of type one.

Before we prove the above fact, let us consider the type one graphs first. A perfect matching of K_{2n} is called a 1-factor of K_{2n}. It is well-known that K_{2n} can be decomposed into $2n-1$ 1-factors. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$ be such a collection of 1-factors. $\mathcal{F}$ is said to be a 1-factorization of K_{2n}. The following theorem is a result of Andersen and Mendelsohn.

Theorem 2.3. [1] *Let D be a set of edges of K_{2m}, where $|D| \leq 2m-2$. Then K_{2m} has a 1-factorization with all edges of D in distinct 1-factors if and only if D is not the edge-set of the graph H_{2m}, or, if $m = 3$, of H_5' or H_6'.* (Fig. 1.)

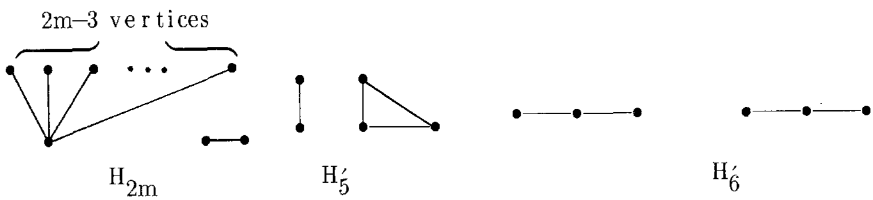

Fig. 1

Now we can prove

Lemma 2.4. *Let $\sum_{i=1}^{k} n_i = 2n - k$ and $G(n_1, n_2, \ldots, n_k)$ be a graph which is not isomorphic to $G(1,2n-3)$. Then $G(n_1, n_2, \ldots, n_k)$ is of type one.*

Proof. Obviously, $G(n_1)$ is a complete graph of order $2n-1$ which is type one. Now consider $k \geq 2$ and $n \geq 4$. Since $G(n_1, n_2, \ldots, n_k)^c$ is not isomorphic to H_{2m} and $G(n_1, n_2, \ldots, n_k)^c$ has at most $2n-2$ edges, $e_1, e_2, \ldots, e_{2n-k}$, $k \geq 2$, by Theorem 2.3 there exists a 1-factorization of K_{2n} with all edges of $G(n_1, n_2, \ldots, n_k)^c$ in distinct 1-factors. Without loss of generality, let the 1-factorization be $\{F_1, F_2, \ldots, F_{2n-1}\}$ and $e_i \in F_i$, $i = 1, 2, \ldots, 2n-k$; let the edges $e_1, e_2, \ldots, e_{n_1}$ form the star S_{n_1}; $e_{n_1+1}, \ldots, e_{n_1+n_2}$ from the star S_{n_2} and so on. Now define a mapping π from the vertex set and edge set of $G(n_1, n_2, \dots, n_k)$ to $\{1, 2, \dots, 2n\}$ as follows: (i) for each $e \in F_i \backslash e_i$, $\pi(e) = i$, and (ii) for each star S_{n_j} in $G(n_1, n_2, \dots, n_k)^c$ with $V(S_{n_j}) = \{v_j^{(0)}, v_j^{(1)}, \dots, v_j^{(n_j)}\}$, we color $v_j^{(0)}$ and $v_j^{(1)}$ with the same color $\sum_{i=1}^{j-1} n_i + 1; j = 1, 2, \dots, k$, we color $v_j^{(h)}$ with $\sum_{i=1}^{j-1} n_i + h; h = 2, 3, \dots, n_j$. It is easy to see that π is a total coloring of $G(n_1, n_2, \dots, n_k)$ using $2n - 1$ colors. Hence $G(n_1, n_2, \dots, n_k)$ is of type one provided that $n \geq 4$. For the case $n = 3$, a total coloring each of $G(1, 1, 1)$ and $G(2, 2)$ can be obtained by the following color-tables where the (i, i) entry is the color of the vertex v_i and (i, j) entry is the color of the edge $\{v_i, v_j\}$ (see Fig. 2.). The case $n \leq 2$ is easy to settle. □

Now we can show that $G(1, 2n - 3)$ is critical in the sense of total coloring.

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	1	x	4	3	5	2
v_2	x	1	3	5	2	4
v_3	4	3	2	x	1	5
v_4	3	5	x	2	4	1
v_5	5	2	1	4	3	x
v_6	2	4	5	1	x	3

A 5–total coloring of G(1,1,1).

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	2	x	x	3	1	4
v_2	x	1	4	2	5	3
v_3	x	4	2	5	3	1
v_4	3	2	5	4	x	x
v_5	1	5	3	x	4	2
v_6	4	3	1	x	2	5

A 5–total coloring of G(2,2).

Fig. 2

Lemma 2.5. *Let H be a proper subgraph of $G = G(1, 2n - 3)$ with $\Delta(H) = 2n - 2$. Then H is of type one.*

Proof. It suffices to show that for any edge $e \in E(G)$, $G - e$ is of type one. Let the vertex set of G be $\{v_0, v_1, v_2, \dots, v_{2n-2}, v_{2n-1}\}$, where v_{2n-2} is not adjacent to v_{2n-1} and v_0 is not adjacent to v_i, $i = 1, 2, \dots, 2n - 3$. Without loss of generality, we consider the following three cases: (i) $e = \{v_1, v_{2n-1}\}$, (ii) $e = \{v_1, v_2\}$, and (iii) $e = \{v_0, v_{2n-1}\}$. First, let $n \geq 4$. In Case (i), $G - e + \{v_0, v_1\}$ is actually a $G(2, 2n - 4)$ graph which by Lemma 2.4, is of type one. In Case (ii), $G \backslash e + \{v_0, v_1\} + \{v_0, v_2\}$ is a $G(1, 1, 2n - 5)$ graph which by Lemma 2.4 again, is of type one. In Case (iii), a $(2n - 1)$-total coloring π of G can be obtained by modifying a $(2n - 1)$-total coloring φ of K_{2n-1}: Let $p = 2n - 1$ and let the vertex set of K_p be $\{1, 2, \dots, p\}$. The p color classes of φ are $C(j) = \{\{j - i, j + i\} | i = 1, 2, \dots, n - 1\} \cup \{j\}, j = 1, 2, \dots, p$, where $j - i$ and $j + i$ are calculated modulo p. The color classes of π are $C(1) \cup \{2n\}$, $C(j)$ for $j = 2, 3, \dots, p - 1$ and $C(p) - \{n - 1, n\} \cup \{n, 2n\}$. Finally, for the case $n \leq 3$, it is not difficult to establish a $(2n - 1)$-total coloring for G. □

Combining Lemma 2.2, 2.4, and 2.5, we have proved Theorem 1.2, i.e., we have shown that a graph G of order $2n$ and $\Delta(G) = 2n - 2$ is of type two if and only if G^c is a disjoint union of an edge and a star with $2n - 3$ edges.

Remark and acknowledgement. The deficiency of a graph G, $def(G)$, is given by the summation of $(\Delta(G) - deg(G))$, for all $v \in V(G)$, i.e., $def(G) = \sum_{v \in V(G)} (\Delta(G) - deg(G))$. G is conformable if G has a vertex coloring ϕ with $\Delta(G) + 1$ colors such that

$$def(G) \geq |\{i: |\phi^{-1}(i)| \equiv |V(G)| + 1 \ (mod\ 2), 1 \leq i \leq \Delta(G) + 1\}|$$

In [4], Chetwynd and Hilton conjectured that a simple graph G with $\Delta(G) \geq \frac{1}{2}\lfloor |V(G)| + 1 \rfloor$ is type two if and only if G contains a non-conformable subgraph H with $\Delta(H) = \Delta(G)$. Now, it is not difficult to see that $G(1, 2n - 3)$ is conformable, thus this disproves the conjecture.

Finally, we would like to express our thanks to both Professor A.J.W. Hilton and Professor H.P. Yap for their helpful comments.

References

1. Andersen, L.D., Mendelsohn, E.: 1-factorizations of K_{2m} with given edges in distinct 1-factors. University Center of Aalborg (1985) (preprint)
2. Behzad, M., Chartrand, G., Cooper, J.: The colouring numbers of complete graphs. J. London Math. Soc. **42**, 226–228 (1967)
3. Bondy, J.A., Murty, U.S.R.: Graph theory with applications. Elsevier North Holland, Inc., 1976
4. Chetwynd, A.G., Hilton, A.J.W.: Some refinements of the total chromatic number cojecture. Congressus Numerantium **66**, 195–216 (1988)
5. Hilton, A.J.W.: A total-chromatic number analogue of Plenatholt's theorem. Discrete Math. **79**, 169–175 (1989/90)
6. Kostachka, A.V.: The total coloring of multigraph with maximal degree 4. Discrete Math. **17**, 161–163 (1977)
7. Rosenfeld, M.: On the total coloring of certain graphs. Israel J. Math. **9(3)**, 396–402 (1971)
8. Vizing, V.G.: On an estimate of the chromatic class of a p-graph (Russian). Diskret. Analiz **3**, 25–30 (1964)
9. Yap, H.P. Total colourings of graphs. Bull. London Math. Soc. **21**, 159–163 (1989)
10. Yap, H.P. Wang Jian-Fang, Zhang Zhongfu: Total chromatic number of graph of high degree. J. Aust. Math. Soc. (Series A) **47**, 445–452 (1989)

Received: July 17, 1990
Revised: June 4, 1991