Controller design for linear multivariable systems with periodic inputs

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Abstract: The authors propose a controller design method for linear multivariable systems with periodic inputs. The periodic inputs may have different periods in different input channels. The plant is assumed to be minimum phase but may be unstable. In addition to achieving closed-loop stability and input–output decoupling, the design is to satisfy prespecified bounds on relative steady-state tracking error and sensitivity function. The paper shows that, for minimum phase plants, it is possible to achieve arbitrarily small sensitivity over large bandwidth and arbitrarily small (integral square) tracking error for piecewise continuous periodic inputs in each channel. The paper proposes a design algorithm and gives an illustrative example.

1 Introduction

Periodic reference input signals are common in many practical servo control systems. For example, in robot control systems, it is typical for robot manipulation tasks to be repetitive. For a single input, single output feedback system to be able to track an arbitrary T-periodic command signal, the (forward) loop transfer function must contain infinitely many frequency modes (poles) at $\pm j(2\pi k/T)$, $k = 0, 1, \ldots$ [3]. One way to generate these infinitely many frequency modes proposed by Hara *et al.* [5, 6] is to use a time-delay e^{-Ts} in a positive unity-feedback configuration. Controller design using such a mode-generating block is also proposed and the resulting controller is called *repetitive controller* [7].

In practice, repetitive controllers may be unnecessary and undesirable for the following reasons:

(i) Repetitive controllers usually result in very narrow closed-loop bandwidth due to the large phase shift in the mode-generating block. This means that the system has sluggish transient response and poor performance in attenuating external disturbances, although it tracks perfectly the periodic input at steady state.

(ii) Most periodic command signals encountered in practice have power concentrated in the first few harmonics, hence a finite number of frequency modes in the loop is usually adequate. For example, if the periodic signal is continuous, its Fourier coefficients converge quadratically to zero, hence the power contained in the high-order harmonics diminishes rapidly.

(iii) The application of repetitive controller requires that the plant be proper rather than strictly proper, which is unrealistic [7].

(iv) Implementation of a repetitive controller is impractical since it contains a perfect time delay.

A practical controller for periodic input tracking should result in large enough closed-loop bandwidth so that the transient response and disturbance attenuation are satisfactory. It should contain enough (yet finite) frequency modes so that the steady-state tracking error is acceptable. Davison and Patel [2] propose a controller design method, based on the parameter optimisation, for MIMO open-loop stable systems with periodic inputs and disturbances which has a finite number of harmonic components. Their design objective is to obtain 'good' asymptotic input tracking and disturbance regulation subject to the controller gain and closed-loop gain margin tolerance requirements.

We propose, in this paper, a controller design method for linear multivariable systems with periodic inputs. The periodic inputs may have different periods in different input channels. The plant is assumed to be minimum phase, but may be unstable. In addition to achieving closed-loop stability and input–output decoupling, the design is to satisfy prespecified bounds on relative steady-state tracking error and sensitivity function. We show that, for minimum phase plants, it is possible to achieve arbitrarily small sensitivity over large bandwidth and arbitrarily small (integral square) tracking error for piecewise continuous periodic inputs in each channel.

1.1 Abbreviations

Throughout this paper, we use the following notations:

$a := b$ means a denotes b
$\mathbb{N}$:= the set of all nonnegative integers
$\mathbb{R}$:= the set of all real numbers
$\boldsymbol{C}$:= the set of all complex numbers
$\boldsymbol{C}_+ := \{s \in \boldsymbol{C} \mid \mathrm{Re}(s) \geq 0\}$
$\boldsymbol{C}_- := \{s \in \boldsymbol{C} \mid \mathrm{Re}(s) < 0\}$
$\mathbb{R}[s](\mathbb{R}(s), \mathbb{R}_p(s), \mathbb{R}_{p,o}(s)$, resp.) := the set of polynomials (rational functions, proper rational functions, strictly proper rational functions, resp.) in s with real coefficients
$\boldsymbol{S} := \{H \in \mathbb{R}_p(s) \mid$ all the poles of H lie in $\boldsymbol{C}_-\}$
$\boldsymbol{S}^{m \times n}(\mathbb{R}_p(s)^{m \times n}, \mathbb{R}_{p,o}(s)^{m \times n}$, resp.) := The $m \times n$ matrix with elements in $\boldsymbol{S}(\mathbb{R}_p(s), \mathbb{R}_{p,o}(s)$, resp.).

For $h \in \mathbb{R}(s)$, the relative degree of h is defined as the degree of its numerator polynomial minus the degree of its denominator polynomial. For $A \in \mathbb{C}^{m \times n}$, $\|A\|$ denotes the largest singular value of A. For $c \in \boldsymbol{C}$, c^* denotes the complex conjugate of c.

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2 Stability and sensitivity bound

Consider the unity-feedback system $S(P, C)$ shown in Fig. 1, where $P(s) \in \mathbb{R}_{p,o}(s)^{n \times n}$ is the plant, $C(s) \in \mathbb{R}_p(s)^{n \times n}$ is the controller. It is assumed that the dynamical system described by $P(s)$ and $C(s)$ contains no unstable hidden modes. The closed-loop transfer matrix $H(s) \in \mathbb{R}_p(s)^{2n \times 2n}$ from $[r^T\ u^T]^T$ to $[e^T\ v^T]^T$ is given by

$$H = \begin{bmatrix} H_{er} & H_{eu} \\ H_{vr} & H_{vu} \end{bmatrix} = \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix} \tag{1}$$

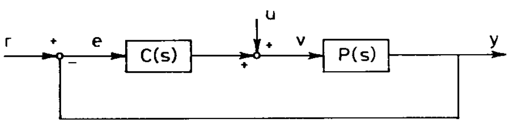

Fig. 1 *Unity-feedback system S(P, C)*

The system $S(P, C)$ is said to be (*interally*) *stable* if and only if $H \in \mathbf{S}^{2n \times 2n}$.

In design, in addition to closed-loop stability, it is of interest to make the sensitivity function $\|H_{er}(j\omega)\|$ small over a certain frequency bandwidth, so that the closed-loop system has good transient response and has good disturbance attenuation. For example, it may be desirable to choose $C(s)$ to make $\|H_{er}(j\omega)\| \leqslant \varepsilon$ for $\omega \in [-\omega_B, \omega_B]$, where $\varepsilon > 0$ is a small number and ω_B is the frequency bandwidth of interest. It is well known [4] that the right-half-plane transmission zeros of the plant limit the achievable lower bound of the sensitivity function. Zames and Bensoussan [11] show that, if the plant is minimum phase, the sensitivity function can be made arbitrarily small over any specified bandwidth while satisfying a prescribed bound at all other frequencies by a controller of the following form:

$$C(s) = \gamma\left(\frac{m}{s + m}\right)\left(\frac{l}{s + l}\right)^k V(s) \tag{2}$$

where $\gamma > 0$, $m > 0$, $l > 0$, $k \in \mathbb{N}$, and $V(s)$ is in the form of a modified plant inverse. In the following, we give criteria for constructing $V(s)$ and selecting γ, m, l, and k.

Given that $P(s) \in \mathbb{R}_{p,o}(s)^{n \times n}$ is nonsingular and minimum phase, write

$$P(s) = \frac{N(s)}{a(s)} = \frac{N(s)}{a_+(s)a_-(s)} \tag{3}$$

where $N(s) \in \mathbb{R}[s]^{n \times n}$ and $a(s)$ is the monic least common denominator of the entries of $P(s)$, $a_+(s)$ and $a_-(s)$ are monic and only have zeros in $\mathbf{C}_+$ and $\mathbf{C}_-$, respectively. Assume that $p_1, \ldots, p_k$ are the zeros of $a_+(s)$ and let $b \in \mathbb{N}$ be the smallest integer such that

$$b > \max\{|p_1|, \ldots, |p_k|\} \tag{4}$$

and let $V(s)^{-1} := N(s)/(a_+(s + b)a_-(s))$. It is easy to see that $V(s)^{-1}$ has no pole in $\mathbf{C}_+$. Since

$$V(s) = a_+(s + b)a_-(s)N(s)^{-1} \tag{5}$$

we have

$$P(s)V(s) = \frac{a_+(s + b)}{a_+(s)} I := g(s)I \tag{6}$$

Consider the controller $C(s)$ defined by

$$C(s) = V(s)D(s) = V(s)\,\mathrm{diag}\,[d_1(s) \cdots d_n(s)] \tag{7}$$

where $V(s)$ is given by eqn. 5 and

$$d_i = \gamma_i\left(\frac{m_i}{s + m_i}\right)\left(\frac{l_i}{s + l_i}\right)^k \tag{8}$$

$\gamma_i > 0$, $m_i > 0$, $l_i > 0$, $i = 1, 2, \ldots, n$, and k is the largest relative degree of entries of $V(s)$. Note that the controller $C(s)$ defined in eqn. 7 is strictly proper and that the I/O map $H_{yr} = PC(I + PC)^{-1}$ is diagonal.

The following theorem gives conditions on m_i, γ_i, and l_i, $i = 1, 2, \ldots, n$, so that the closed-loop system $S(P, C)$ is stable and satisfies prespecified sensitivity bound in each channel.

Theorem 1

Suppose $P(s) \in \mathbb{R}_{p,o}(s)^{n \times n}$ is nonsingular and minimum phase. Let b satisfy eqn. 4 and let $V(s)$ and $g(s)$ be as defined in eqns. 3, 5, and 6. Suppose, for $i = 1, 2, \ldots, n$, $0 < \varepsilon_i \leqslant 1$, $M_i > 1$, $\omega_{B_i} > 0$, and $0 < \delta_i < (1 - M_i^{-1})$ are given. Under these conditions, if, for $i = 1, 2, \ldots, n$,

(***M1***) $m_i \geqslant \max\{\omega_{B_i}, b\}$ and satisfies

$$\sup_{|\theta| \leqslant \pi/2} |g(m_i e^{j\theta}) - 1| \leqslant 1 - \delta_i$$

and

$$\sup_{|\omega| \geqslant m_i} |g(j\omega) - 1| \leqslant 1 - M_i^{-1} - \delta_i \tag{9}$$

(***M2***) $\gamma_i > 0$ and satisfies

$$\gamma_i > \max\left\{2\xi_{1i}^{-1}\left(1 + \frac{1}{\varepsilon_i}\right), 4\xi_{2i}^{-1}\right\} \tag{10}$$

where

$$\xi_{1i} = \inf_{|\omega| \leqslant m_i} |g(j\omega)|$$

and

$$\xi_{2i} = \inf_{|\theta| \leqslant \pi/2} |g(m_i e^{j\theta})| \tag{11}$$

there exists $l_i > 0$ large enough, $i = 1, 2, \ldots, n$, such that with the controller $C(s)$ defined by eqns. 7 and 8:

(i) the system $S(P, C)$ is stable;
(ii) the sensitivity matrix $(I + PC)^{-1}$ is diagonal; and
(iii) for $i = 1, 2, \ldots, n$,

$$|(I + PC)_{ii}^{-1}(j\omega)| \leqslant \begin{cases} \varepsilon_i & \forall |\omega| \leqslant \omega_{B_i} \\ M_i & \forall |\omega| > \omega_{B_i} \end{cases} \tag{12}$$

Comments:

(i) Since $\lim_{R \to \infty} \sup_{|s| \geqslant R} |g(s) - 1| = 0$, the m_is in (***M1***) exist.

(ii) Since $g(s)$ has no zeros in $\mathbf{C}_+$, $\xi_{1i} > 0$, $\xi_{2i} > 0$, and thus γ_i exists, for $i = 1, 2, \ldots, n$.

(iii) In general, m_i and γ_i increase with decreasing values of M_i and ε_i respectively.

(iv) It follows from the theorem that the design for each channel can be carried out separately.

In design, m_i, γ_i and l_i are tuned sequentially to achieve the prespecified sensitivity bound in each channel.

Proof: See Appendix 10.

3 Decoupling design for periodic inputs

Consider the unity-feedback system $S(P, C)$ shown in Fig. 1, where the external signal $r(t) = [r_1(t), \ldots, r_i(t), \ldots,$

$r_n(t)]^T$ is piecewise continuous and for $i = 1, 2, \ldots, n$, $r_i(t)$ is T_i-periodic with Fourier series expansion:

$$r_i(t) = \sum_{k=-\infty}^{\infty} c_{ik} e^{j2k\pi t/T_i} \tag{13}$$

where $c_{ik} \in \boldsymbol{C}$ and $T_i > 0$. It is well known [9] that is $r_i(t)$ and its first $l-1$ derivatives are continuous, then $|c_{ik}| \to 0$ as $k \to \infty$ at least as rapidly as h/k^{l+1}, where h is a constant independent of k. Suppose that the ith channel input $r_i(t)$ has most its power concentrated in its first q_i harmonics, then, in design, it may be sufficient to track these q_i harmonics at steady-state while keeping the amplification at the frequencies beyond $2\pi q_i/T_i$ within a prespecified bound.

In order for the unity-feedback system $S(P, C)$ to track T_i-periodic input, $i = 1, 2, \ldots, n$, with small tracking error, let considers the controller

$$C(s) = V(s)D(s)F(s) \tag{14}$$

where $V(s)D(s)$ is defined in eqn. 7, $\omega_i = 2\pi/T_i$, $i = 1, 2, \ldots, n$, and

$$F(s) = \text{diag}\,[f_1(s) \cdots f_n(s)]$$
$$= \text{diag}\left[\frac{s+1}{s}\prod_{k=1}^{q_1}\frac{(s+k\omega_1)^2}{s^2+k^2\omega_1^2} \cdots \frac{s+1}{s}\prod_{k=1}^{q_n}\frac{(s+k\omega_n)^2}{s^2+k^2\omega_n^2}\right] \tag{15}$$

Note that the only difference between the controller defined in eqn. 14 and that defined in eqn. 7 is the diagonal $F(s)$ which is added to provide tracking of the first q_i harmonics of the T_i-periodic input in the ith channel. In design, the number q_i will be determined by the steady-state tracking error requirement. The following corollary, which follows directly from theorem 1, gives conditions on $D(s)$ so that the controller $C(s)$ yields the stable closed-loop system $S(P, C)$ with decoupled sensitivity matrix and achieves prespecified bounds on the sensitivity function.

Corollary 1

Suppose $P(s) \in \mathbb{R}_{p,o}(s)^{n\times n}$ is nonsingular and minimum phase. Let b satisfy eqn. 4 and let $V(s)$ and $g(s)$ be as defined in eqns. 3, 5, and 6. Suppose, for $i = 1, 2, \ldots, n$, $0 < \varepsilon_i \leqslant 1$, $M_i > 1$, $\omega_{B_i} > 0$, and $0 < \delta_i < (1 - M_i^{-1})$ are given. Let $f_i(s)$, $i = 1, 2, \ldots, n$, be as defined in eqn. 15. Under these conditions, if, for $i = 1, 2, \ldots, n$,

($\tilde{M}$1) $m_i \geqslant \max\{\omega_{B_i}, q_i\omega_i, b\}$ and satisfies[1]

$$\sup_{|\theta| \leqslant \pi/2} |gf_i(m_i e^{j\theta}) - 1| \leqslant 1 - \delta_i$$

and

$$\sup_{|\omega| \geqslant m_i} |gf_i(j\omega) - 1| \leqslant 1 - M_i^{-1} - \delta_i \tag{16}$$

($\tilde{M}$2) $\gamma_i > 0$ and satisfies

$$\gamma_i > \max\left\{2\xi_{1i}^{-1}\left(1 + \frac{1}{\varepsilon_i}\right), 4\xi_{2i}^{-1}\right\} \tag{17}$$

where

$$\xi_{1i} = \inf_{|\omega| \leqslant m_i} |gf_i(j\omega)| \quad \text{and} \quad \xi_{2i} = \inf_{|\theta| \leqslant \pi/2} |gf_i(m_i e^{j\theta})| \tag{18}$$

[1] Use $gf_i(s)$ to denote $g(s)f_i(s)$ for simplicity.

there exists $l_i > 0$ large enough, $i = 1, 2, \ldots, n$, such that with the controller $C(s)$ defined by eqns. 14 and 15:

(i) the system $S(P, C)$ is stable

(ii) the sensitivity matrix $(I + PC)^{-1}$ is diagonal and

(iii) for $i = 1, 2, \ldots, n$,

$$(a)\ |(I + PC)_{ii}^{-1}(j\omega)| \leqslant \begin{cases} \varepsilon_i & \forall |\omega| \leqslant \omega_{B_i} \\ M_i & \forall |\omega| > \omega_{B_i} \end{cases} \tag{19}$$

and

$$(b)\ (I + PC)_{ii}^{-1}(jk\omega_i) = 0 \qquad k = 0, \pm 1, \pm 2, \ldots, \pm q_i \tag{20}$$

Comments:

(i) Note that, since $(I + PC)_{ii}^{-1}(jk\omega_i) = 0$, $k = 0, \pm 1, \pm 2, \ldots, \pm q_i$, the system tracks any T_i-periodic input which contains only the first q_i harmonics (in addition to the DC component) in the ith channel.

(ii) The parameters m_i, γ_i, and l_i can be tuned independently for each channel to achieve prespecified sensitivity bound.

(iii) In design, the number q_i is determined by a prescribed relative steady-state tracking error.

4 Time-domain steady-state error analysis

We analyse the time-domain steady-state performance of the system $S(P, C)$ with the controller $C(s)$ prescribed in corollary 1. Assuming that the input $r(t)$ is known, we will derive an upper bound on relative steady-state tracking error in each channel. Since the system $S(P, C)$ is decoupled, it suffices to analyse just one channel. To simplify notations, we assume in this section that the signals $e(t)$, $y(t)$, $u(t)$, $r(t)$ in Fig. 1 are all scalar functions and hence the plant and the controller are SISO. Let $y_{ss}(t)$ be the steady-state output function due to the T-periodic input $r(t)$ with $u(t) = 0$ (see Fig. 1)[2]. Let $e_{ss}(t) := r(t) - y_{ss}(t)$. Note that $e_{ss}(t)$ is the steady-state tracking error. Since $S(P, C)$ is linear time-invariant and stable, $e_{ss}(t)$ is also periodic.

Define the *relative steady-state tracking error* of the system $S(P, C)$

$$\mathfrak{R} = \frac{\mathscr{E}}{\mathscr{P}} \tag{21}$$

where

$$\mathscr{E} = \int_0^T e_{ss}^2(t)\,dt \quad \text{and} \quad \mathscr{P} = \int_0^T r^2(t)\,dt$$

Let $r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2k\pi t/T}$ be the Fourier series representation of $r(t)$. Define

$$r_N(t) = \sum_{k=-N}^{N} c_k e^{j2k\pi t/T} \tag{22}$$

and

$$\mathscr{P}_N = \int_0^T \{r(t) - r_N(t)\}^2\,dt \tag{23}$$

Note that $r_N(t)$ is the sum of the first $N + 1$ harmonics contained in $r(t)$. With these definitions, we are ready to state the following theorem which gives an upper bound on $\mathfrak{R}$.

[2] We note that, although the input u has the interpretation of plant input disturbance, the inclusion of u in Fig. 1 is mainly to allow the determination of closed-loop internal stability [10] through the stability of transfer matrix H in eqn. 1.

Theorem 2
Consider the stable system $S(P, C)$ with T-periodic input $r(t)$ and $u(t) = 0$. Assume the sensitivity function satisfies

$$|(1 + PC)^{-1}(j\omega)| \leqslant \begin{cases} \varepsilon & \forall |\omega| \leqslant \omega_B \\ M & \forall |\omega| > \omega_B \end{cases} \tag{24}$$

and

$$(1 + PC)^{-1}(jk\omega_0) = 0 \quad k = 0, \pm 1, \pm 2, \ldots, \pm q \tag{25}$$

where $\omega_0 = 2\pi/T$. Let $Q \in \mathbb{N}$ be such that $Q \leqslant \omega_B/\omega_0 < Q + 1$. Let $\Re$ be as defined in eqn. 21. Under these conditions,

$$\text{(i)}\ \Re \leqslant M^2 \frac{\mathscr{P}_q}{\mathscr{P}} \text{ if } Q \leqslant q \tag{26}$$

$$\text{(ii)}\ \Re \leqslant \varepsilon^2 \frac{\mathscr{P}_q}{\mathscr{P}} + (M^2 - \varepsilon^2) \frac{\mathscr{P}_Q}{\mathscr{P}} \quad \text{if } Q > q \tag{27}$$

The following lemma, which is used in the proof of theorem 2, follows directly from Parseval's theorem [8].

Lemma 1
For $\mathscr{P}_N$ defined in eqn. 23, we have

$$\mathscr{P}_N = 2T \sum_{k=N+1}^{\infty} |c_k|^2 \tag{28}$$

and

$$\mathscr{P}_N = \mathscr{P} - T \sum_{k=-N}^{N} |c_k|^2 \tag{29}$$

Proof of theorem 2
Let $\phi(s) := (1 + PC)^{-1}(s)$. The steady-state tracking error

$$e_{ss}(t) = \sum_{k=-\infty}^{\infty} (1 + PC(jk\omega_0))^{-1} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \phi(jk\omega_0) c_k e^{jk\omega_0 t}$$

Since

$$\phi(jk\omega_0) = 0 \quad k = 0, \pm 1, \pm 2, \ldots, \pm q$$

thus

$$e_{ss}(t) = \sum_{k=q+1}^{\infty} (\phi(jk\omega_0) c_k e^{jk\omega_0 t} + \phi(-jk\omega_0) c_k^* e^{-jk\omega_0 t}) = \sum_{k=q+1}^{\infty} (\phi(jk\omega_0) c_k e^{jk\omega_0 t} + \phi(jk\omega_0)^* c_k^* e^{-jk\omega_0 t})$$

From the orthonormal property,

$$\begin{aligned} \mathscr{E} &= \int_0^T e_{ss}^2(t)\, dt \\ &= \int_0^T \left\{ \sum_{k=q+1}^{\infty} (c_k \phi(jk\omega_0) e^{jk\omega_0 t} + c_k^* \phi(jk\omega_0)^* e^{-jk\omega_0 t}) \right\}^2 dt \\ &= \int_0^T \left\{ \sum_{k=q+1}^{\infty} (2 c_k \phi(jk\omega_0) c_k^* \phi(jk\omega_0)^*) \right\} dt \\ &= 2T \sum_{k=q+1}^{\infty} |\phi(jk\omega_0)|^2 |c_k|^2 \end{aligned} \tag{30}$$

(i) If $Q \leqslant q$, then

$$|\phi(jk\omega_0)| \leqslant M \quad \forall k \geqslant q + 1$$

it follows from eqn. 30 that

$$\mathscr{E} \leqslant 2TM^2 \sum_{k=q+1}^{\infty} |c_k|^2$$

Thus, from eqn. 28, we obtain

$$\mathscr{E} \leqslant M^2 \mathscr{P}_q$$

and eqn. 26 follows.

(ii) If $Q > q$, then

$$|\phi(jk\omega_0)| \leqslant \begin{cases} \varepsilon & \forall q + 1 \leqslant k \leqslant Q \\ M & \forall k \geqslant Q + 1 \end{cases}$$

it follows from eqn. 30 that

$$\begin{aligned} \mathscr{E} &= 2T \sum_{k=q+1}^{Q} |\phi(jk\omega_0)|^2 |c_k|^2 + 2T \sum_{k=Q+1}^{\infty} |\phi(jk\omega_0)|^2 |c_k|^2 \\ &\leqslant 2T\varepsilon^2 \sum_{k=q+1}^{Q} |c_k|^2 + 2TM^2 \sum_{k=Q+1}^{\infty} |c_k|^2 \end{aligned}$$

Thus, from eqn. 28, we obtain

$$\mathscr{E} \leqslant \varepsilon^2 [\mathscr{P}_q - \mathscr{P}_Q] + M^2 \mathscr{P}_Q = \varepsilon^2 \mathscr{P}_q + (M^2 - e^2) \mathscr{P}_Q$$

and eqn. 27 follows.

Based on corollary 1 and theorem 2, we give an algorithm for the design of decoupling controllers for linear multivariable system with periodic inputs to satisfy prespecified bounds on relative steady-state tracking error and sensitivity function.

5 Design algorithm

Consider again the system $S(P, C)$ with $u(t) = 0$, and suppose the T_i-periodic input $r_i(t)$, $i = 1, 2, \ldots, n$, are given. Assume that $P(s) \in \mathbb{R}_{p,o}(s)^{n \times n}$ is nonsingular and minimum phase, and that the numbers $\eta_i > 0$, $0 < \varepsilon_i \leqslant 1$, $M_i > 1$, and $\omega_{B_i} > 0$ are given. Our goal is to find the controller $C(s)$ defined in eqn. 14 such that

(i) $S(P, C)$ is stable
(ii) $(I + PC)^{-1}$ is diagonal
(iii) the sensitivity function satisfies

$$|(I + PC)_{ii}^{-1}(j\omega)| \leqslant \begin{cases} \varepsilon_i & \forall |\omega| \leqslant \omega_{B_i} \\ M_i & \forall |\omega| > \omega_{B_i} \end{cases} \quad i = 1, 2, \ldots, n$$

(iv) the relative steady-state tracking error specification satisfies

$$\Re_i \leqslant \eta_i \quad i = 1, 2, \ldots, n$$

We propose, in the following, a design algorithm to achieve this goal.

Algorithm 1
Data: $0 < \varepsilon_i \leqslant 1$, $M_i > 1$, $\omega_{B_i} > 0$, $\eta_i > 0$, $\omega_i = 2\pi/T_i$, and $0 < \delta_i < (1 - M_i^{-1})$, for $i = 1, 2, \ldots, n$.

Step 0: Set $i = 1$.

Step 1: Determine the harmonic number q_i such that $\Re_i \leqslant \eta_i$.

1 Find $Q_i \in \mathbb{N}$ such that $Q_i \leqslant \omega_{B_i}/\omega_i < Q_i + 1$.
2 Compute the Fourier coefficients c_{ik} of $r_i(t)$.
3 Compute $\mathscr{P}_{Q_i}$ from eqn. 29 and use eqn. 26 or 27 to determine q_i such that $\Re_i \leqslant \eta_i$.

Step 2: Determine $V(s)$, $f_i(s)$, $g(s)$, and k.

1 From eqns. 5 and 15, determine $V(s)$ and $f_i(s)$ respectively.

2 Obtain k and $g(s)$ from $V(s)$ snd $P(s)$.

Step 3: Determine m_i.

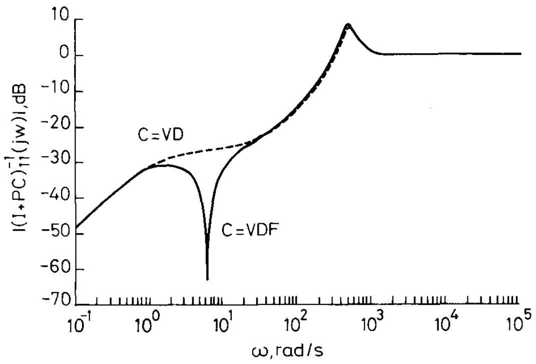

Fig. 2 *Plot of sensitivity function* $|(I + PC)^{-1}_{11}(j\omega)|$

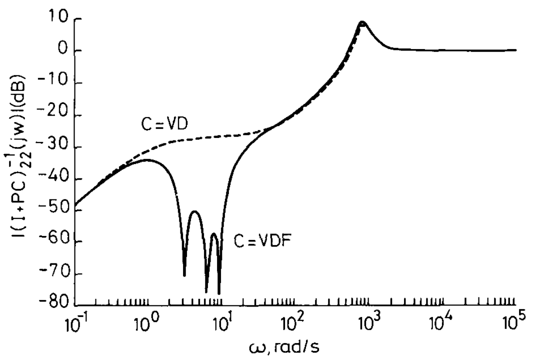

Fig. 3 *Plot of sensitivity function* $|(I + PC)^{-1}_{22}(j\omega)|$

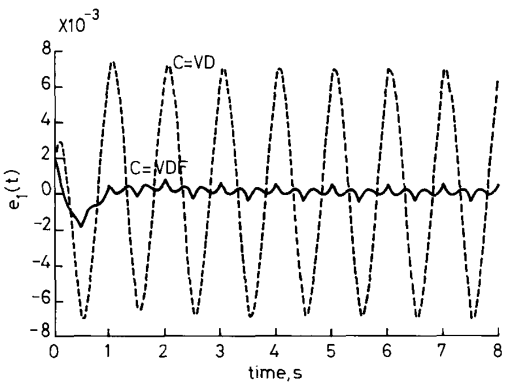

Fig. 4 *Plot of tracking error* $e_1(t)$

1 Let $m_{1i} = \max\{\omega_{B_i}, q_i\omega_i, b\}$.

2 Choose m_{2i} so that $|gf_i(m_{2i}e^{j\theta}) - 1| \leqslant 1 - \delta_i$ for $\theta \in [0, \pi/2]$.

3 Choose m_{3i} so that $|gf_i(j\omega) - 1| \leqslant 1 - M_i^{-1} - \delta_i$ for $\omega \in [m_{3i}, \infty)$.

4 Let $m_i = \max\{m_{1i}, m_{2i}, m_{3i}\}$.

Step 4: Determine γ_i. Compute

$$\xi_{1i} = \inf_{\omega \in [0, m_i]} |gf_i(j\omega)|$$

and

$$\xi_{2i} = \inf_{\theta \in [0, \pi/2]} |gf_i(m_i e^{j\theta})|$$

and let

$$\gamma_i = \max\left\{2\xi_{1i}^{-1}\left(1 + \frac{1}{\varepsilon_i}\right),\ \ 4\xi_{2i}^{-1}\right\} + 1$$

Step 5: Determine l_i. Choose $l_i > m_i/(2^{1/k} - 1)$ such that the poles of $(1 + gd_i f_i)^{-1}(s) \in \boldsymbol{C}_-$, which then guarantee that $(1 + gd_i f_i)^{-1} \in \boldsymbol{S}$ and

$$|(1 + gd_i f_i)^{-1}(j\omega)| \leqslant \varepsilon_i \quad \forall |\omega| \leqslant \omega_{B_i} \tag{31}$$

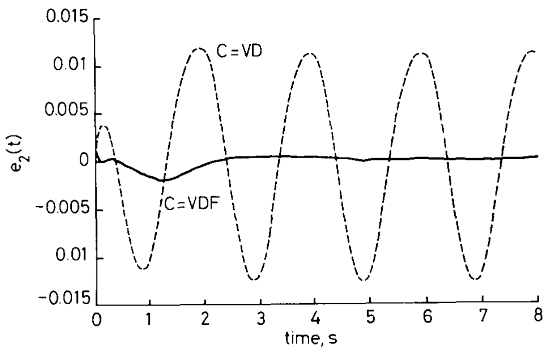

Fig. 5 *Plot of tracking error* $e_2(t)$

If

$$|(1 + gd_i f_i)^{-1}(j\omega)| \leqslant M_i \quad \forall |\omega| > \omega_{B_i} \tag{32}$$

then go to step 6; else increase l_i until eqn. 32 holds.

Step 6: If $i = n$, then stop; else $i = i + 1$, go to step 1.

Comments:

(i) The controller is given by $C(s) = V(s)D(s)F(s)$.

(ii) In general, m_i and γ_i increase with decreasing values M_i and ε_i, respectively.

(iii) l_i determination plays an important role in this algorithm to satisfy eqn. 31 and 32. Also, from the proof of theorem 1 (see Appendix 10), l_i must be greater than $m_i/(2^{1/k} - 1)$ at least.

(iv) In practical design, m_{2i}, m_{3i}, ξ_{1i}, and ξ_{2i} can be determined by a few magnitude plot of the respective functions. For example, to determine ξ_{1i} and ξ_{2i}, we only have to plot the magnitude $|gf_i(j\omega)|$ for $0 \leqslant \omega \leqslant m_i$ and the magnitude $|gf_i(m_i e^{j\theta})|$ for $0 \leqslant \theta \leqslant \pi/2$ respectively.

6 Illustrative example

The plant is

$$P(s) = \frac{1}{s(s+4)(s-3)}\begin{bmatrix} s-1 & -3 \\ 4 & s+6 \end{bmatrix}$$

The periodic inputs are

$$r_1(t) = \begin{cases} t - 2t^2 & 0 \leqslant t \leqslant 1/2 \\ 2t^2 - 5t + 5 & 1/2 \leqslant t \leqslant 1 \end{cases}$$

and

$$r_2(t) = \begin{cases} \dfrac{1}{\pi}\sin \pi t & 0 \leqslant t \leqslant 1 \\ t^2 - 3t + 2 & 1 \leqslant t \leqslant 2 \end{cases}$$

In addition to achieving closed-loop stability, the design is required to satisfy the following specifications[3]

(i) $(I + PC)^{-1}$ is diagonal

$$\text{(ii) } \mathfrak{R}_1 \leqslant 0.05\% \quad \text{and} \quad \mathfrak{R}_2 \leqslant 0.0001\% \tag{33}$$

$$\text{(iii) } 20 \log_{10} |(I + PC)^{-1}_{11}(j\omega)| \leqslant \begin{cases} -20 \text{ dB} & \forall |\omega| \leqslant 25 \\ 8 \text{ dB} & \forall |\omega| > 25 \end{cases} \tag{34}$$

and

$$\text{(iv) } 20 \log_{10} |(I + PC)^{-1}_{22}(j\omega)| \leqslant \begin{cases} -20 \text{ dB} & \forall |\omega| \leqslant 45 \\ 10 \text{ dB} & \forall |\omega| > 45 \end{cases} \tag{35}$$

The following values are given:

(*a*) $\omega_{B_1} = 25$, $\omega_1 = 2\pi$, $\eta_1 = 5 \times 10^{-4}$, $\varepsilon_1 = 0.1$, $M_1 = 2.5$, and $\delta_1 = \frac{1}{100}(1 - M_1^{-1}) = 0.006$.

(*b*) $\omega_{B_2} = 45$, $\omega_2 = \pi$, $\eta_2 = 10^{-6}$, $\varepsilon_2 = 0.1$, $M_2 = 3.16$, and $\delta_2 = \frac{1}{100}(1 - M_2^{-1}) = 0.0067$.

By computations, $q_1 = 1$, $q_2 = 3$, $Q_1 = 3$, and $Q_2 = 14$ satisfy eqn. 33. Let $b = 4$,

$$V(s) = \frac{(s+1)(s+4)^2}{(s+2)(s+3)} \begin{bmatrix} s+6 & 3 \\ -4 & s-1 \end{bmatrix} \quad \text{and} \quad k = 2$$

Since the plant already has a pole at $s = 0$, choose

$$F(s) = \text{diag}\,[f_1(s), f_2(s)]$$

where

$$f_1(s) = \frac{(s + 2\pi)^2}{s^2 + 4\pi^2}$$

and

$$f_2(s) = \frac{(s + \pi)^2}{s^2 + \pi^2} \frac{(s + 2\pi)^2}{s^2 + 4\pi^2} \frac{(s + 3\pi)^2}{s^2 + 9\pi^2}$$

By steps 3 and 4, it is determined that $m_1 = 28$, $m_2 = 50$, $\xi_{11} = 1.2$, $\xi_{21} = 1.15$, $\gamma_1 = 20$, $\xi_{12} = 1.1$, $\xi_{22} = 1$, and $\gamma_2 = 20$. By step 5, $l_1 = 650$ and $l_2 = 1100$ satisfy eqns. 34 and 35, respectively. The controller is given by $C(s) = V(s)D(s)F(s)$, where

$$D(s) = \text{diag}\left[20 \frac{28}{s + 28} \frac{650^2}{(s + 650)^2} \quad 20 \frac{50}{s + 50} \frac{1100^2}{(s + 1100)^2}\right]$$

It is easy to check that $(I + PC)^{-1}$ is diagonal. The plots of the sensitivity function $|(I + PC)^{-1}_{ii}(j\omega)|$ and the tracking error $e_i(t)$, $i = 1, 2$, are given from Figs. 2 to 5. For comparison, the sensitivity function and tracking error corresponding to the controller $C(s) = V(s)D(s)$ without frequency modes are also plotted. By computation, $\mathfrak{R}_1 = 4.8 \times 10^{-4}$ and $\mathfrak{R}_2 = 9.3 \times 10^{-7}$. Thus eqn. 33 is satisfied. It can be seen from Figs. 2 and 3 that sensitivity functions satisfy eqns. 34 and 35, respectively. Note that the upper bounds are almost reached at $\omega \approx 500$ rad/s and $\omega \approx 800$ rad/s, respectively. Figs. 4 and 5 show that the tracking errors decrease considerably by the introduction of frequency modes $F(s)$.

7 Conclusion

We propose an algorithm for the design of controller for linear multivariable minimum phase plants with periodic inputs. The periodic inputs may have different periods in different input channels. The controller designed yields stable closed-loop system and decoupled sensitivity transfer matrix. Other design specifications include an upper bound on the relative steady-state tracking error and an prescribed bound on the sensitivity function in each channel. The design method is a practical alternative to the so-called repetitive controller. Interesting topics for further study include the extension of this result to non-minimum phase plants and the effect of robustness requirement on the achievable time-domain and frequency-domain specifications.

[3] Specifications on disturbance attenuation and transient response are reflected in the bounds of the sensitivity functions.

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10 Appendix

Proof of theorem 1

We define $\Omega[p, r) := \{s \in C \mid \text{Re}(s) > 0,\ p \leqslant |s| < r\}$; $\Theta[p, r] :=$ The boundary of $\Omega(p, r)$. Let

$$\psi_i = (1 + gd_i)^{-1} \quad i = 1, 2, \ldots, n$$

we have

$$(I + PC)^{-1} = \text{diag}\,[(1 + gd_1)^{-1} \cdots (1 + gd_n)^{-1}] = \text{diag}\,[\psi_1 \cdots \psi_n]$$

We shall prove

(*a*) for $i = 1, 2, \ldots, n$, $\psi_i(s)$ is bounded in $\Omega[0, \omega_{B_i}]$ and $|\psi_i(j\omega)| \leqslant \varepsilon_i$, $\forall |\omega| \leqslant \omega_{B_i}$;

(*b*) for $i = 1, 2, \ldots, n$, $\psi_i(s)$ is bounded in $\Omega(\omega_{B_i}, \infty)$ and $|\psi_i(j\omega)| \leqslant M_i$, $\forall |\omega| > \omega_{B_i}$; and

(*c*) $H(s) \in S^{2n \times 2n}$.

To prove (*a*), note that if

$$\inf_{s \in \Omega[0,\, \omega_{Bi}]} |g(s)d_i(s)| = \alpha_i > 1 \tag{36}$$

then

$$|\psi_i(s)| \leqslant (|g(s)d_i(s)| - 1)^{-1} \leqslant (\alpha_i - 1)^{-1} \quad \forall s \in \Omega[0, \omega_{B_i}]$$

Similarly, if

$$\inf_{|\omega| \leqslant \omega_{Bi}} |g(j\omega)d_i(j\omega)| \geqslant 1 + \frac{1}{\varepsilon_i} \tag{37}$$

then

$$|\psi_i(j\omega)| \leqslant \varepsilon_i \quad \forall |\omega| \leqslant \omega_{B_i}$$

Thus (*a*) holds if eqns. 36 and 37 are true. We now show that eqns. 36 and 37 are true, provided that l_i is large enough. Since $|m_i/(m_i + j\omega)| \geqslant 1/\sqrt{(2)}$, $\forall |\omega| \leqslant m_i$ and $|l_i/(l_i + j\omega)|^k \geqslant 1/\sqrt{(2)}$, $\forall |\omega| \leqslant l_i\sqrt{(2^{1/k} - 1)}$ if $l_i \geqslant m_i/\sqrt{(2^{1/k} - 1)}$, then

$$\inf_{|\omega| \leqslant m_i} |d_i(j\omega)| \geqslant \frac{\gamma_i}{2} \tag{38}$$

Similarly, since $|m_i/(s + m_i)| \geqslant m_i/(|s| + m_i) \geqslant 1/2$, $\forall s \in \Omega[0, m_i]$ and $|l_i/(s + l_i)|^k \geqslant l_i^k/(|s| + l_i)^k \geqslant 1/2$, $\forall s \in \Omega[0, l_i(2^{1/k} - 1)]$ if $l_i \geqslant m_i/(2^{1/k} - 1)$, then

$$\inf_{s \in \Omega[0, m_i]} |d_i(s)| \geqslant \frac{\gamma_i}{4} \tag{39}$$

Thus, if $l_i \geqslant m_i/(2^{1/k} - 1)$, then eqns. 38 and 39 hold.

Since $g(s)^{-1}$ is analytic in C_+, thus by the maximum modulus principle [1] $\forall s \in \Omega[0, m_i]$,

$$|g(s)|^{-1} = |g(s)^{-1}| \leqslant \sup_{s \in \Theta[0, m_i]} |g(s)^{-1}|$$

$$= \frac{1}{\inf_{s \in \Theta[0, m_i]} |g(s)|} \tag{40}$$

It follows from eqn. 40 that $\forall s \in \Omega[0, m_i]$,

$$|g(s)| \geqslant \inf_{s \in \Theta[0, m_i]} |g(s)| = \min\{\xi_{1i}, \xi_{2i}\} \tag{41}$$

where ξ_{1i} and ξ_{2i} are defined in eqn. 11.

Note that $\xi_{1i} > 0$ and $\xi_{2i} > 0$, since all the zeros of $g(s) \in C_-$. Since $2\xi_{1i}^{-1}(1 + (1/\varepsilon_i)) \geqslant 4\xi_{1i}^{-1}$ and γ_i satisfies eqn. 10, thus

$$\gamma_i > \max\left\{2\xi_{1i}^{-1}\left(1 + \frac{1}{\varepsilon_i}\right), \frac{4}{\min\{\xi_{1i}, \xi_{2i}\}}\right\} \tag{42}$$

From eqns. 38, 39, 41, and 42, if $l_i \geqslant m_i/(2^{1/k} - 1)$, we have that $\forall s \in \Omega[0, m_i]$, then

$$\begin{aligned}
&|g(s)d_i(s)| \\
&\quad = |d_i(s)|\,|g(s)| \geqslant \inf_{s \in \Omega[0, m_i]} |d_i(s)| \inf_{s \in \Theta[0, m_i]} |g(s)| \\
&\quad \geqslant \tfrac{1}{4}\gamma_i \min\{\xi_{1i}, \xi_{2i}\} \\
&\quad > \max\left\{\tfrac{1}{2}\xi_{1i}^{-1}\left(1 + \frac{1}{\varepsilon_i}\right), \frac{1}{\min\{\xi_{1i}, \xi_{2i}\}}\right\} \\
&\quad \times \min\{\xi_{1i}, \xi_{2i}\} \\
&\quad \geqslant 1
\end{aligned}$$

and $\forall \omega \in [-m_i, m_i]$,

$$\begin{aligned}
&|g(j\omega)d_i(j\omega)| \\
&\quad = |d_i(j\omega)|\,|g(j\omega)| \geqslant \inf_{|\omega| \leqslant m_i} |d_i(j\omega)| \inf_{|\omega| \leqslant m_i} |g(j\omega)| \\
&\quad \geqslant \tfrac{1}{2}\gamma_i \xi_{1i} \\
&\quad > \max\left\{\xi_{1i}^{-1}\left(1 + \frac{1}{\varepsilon_i}\right), \frac{2}{\min\{\xi_{1i}, \xi_{2i}\}}\right\}\xi_{1i} \\
&\quad \geqslant 1 + \frac{1}{\varepsilon_i}
\end{aligned}$$

Since $m_i \geqslant \omega_{B_i}$ thus eqns. 36 and 37 are true, provided that $l_i \geqslant m_i/(2^{1/k} - 1)$. To prove (*b*), let $J_{n_i}(s) := n_i/(s + n_i)$, then

$$d_i(s) = \gamma_i \frac{m_i}{s + m_i}\left(\frac{l_i}{s + l_i}\right)^k = \gamma_i J_{m_i}(s) J_{l_i}^k(s) \tag{43}$$

Using eqn. 43 we have:

$$\begin{aligned}
|\psi_i(s)|^{-1} &= |1 + g(s)d_i(s)| \\
&= |1 + \gamma_i J_{m_i}(s) J_{l_i}^k(s) g(s)| \\
&= |1 + \gamma_i J_{m_i}(s) + \gamma_i J_{m_i}(s)(g(s) - 1) \\
&\quad + \gamma_i J_{m_i}(s)(J_{l_i}^k(s) - 1)g(s)| \\
&= \left|1 + \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(g(s) - 1)\right. \\
&\quad \left. + \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(J_{l_i}^k(s) - 1)g(s)\right| |1 + \gamma_i J_{m_i}(s)| \\
&\geqslant \left|1 + \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(g(s) - 1)\right. \\
&\quad \left. + \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(J_{l_i}^k(s) - 1)g(s)\right| \\
&:= |1 + \Lambda_i(s)| \geqslant 1 - |\Lambda_i(s)|
\end{aligned} \tag{44}$$

where

$$\Lambda_i(s) = \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(g(s) - 1) + \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(J_{l_i}^k(s) - 1)g(s)$$

From eqn. 44, if

$$\sup_{s \in \Omega(m_i, \infty)} |\Lambda_i(s)| = \beta_i < 1 \tag{45}$$

then

$$|\psi_i(s)| \leqslant |1 + \Lambda_i(s)|^{-1} \leqslant (1 - \beta_i)^{-1} \quad \forall s \in \Omega(m_i, \infty)$$

Similarly, if

$$\sup_{|\omega| > m_i} |\Lambda_i(j\omega)| \leqslant 1 - \frac{1}{M_i} \tag{46}$$

then

$$|\psi_i(j\omega)| \leqslant |1 + \Lambda_i(j\omega)|^{-1} \leqslant M_i \quad \forall |\omega| > m_i$$

Thus (*b*) holds if eqns. 45 and 46 are true.

We show that eqns. 45 and 46 are true, provided l_i is large enough. Now,

$$\begin{aligned}
|\Lambda_i(s)| &= \left|\frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(g(s) - 1)\right. \\
&\quad \left. + \frac{\gamma_i J_{m_i}(s)}{1 + \gamma_i J_{m_i}(s)}(J_{l_i}^k(s) - 1)g(s)\right| \\
&= \left|\frac{\gamma_i}{\gamma_i + 1} J_{m_i(\gamma_i + 1)}(s)(g(s) - 1)\right. \\
&\quad \left. + \frac{\gamma_i}{\gamma_i + 1} J_{m_i(\gamma_i + 1)}(s)(J_{l_i}^k(s) - 1)g(s)\right| \\
&\leqslant \left|\frac{\gamma_i}{\gamma_i + 1} J_{m_i(\gamma_i + 1)}(s)\right| |g(s) - 1| \\
&\quad + \left|\frac{\gamma_i}{\gamma_i + 1} J_{m_i(\gamma_i + 1)}(s)\right| |J_{l_i}^k(s) - 1|\,|g(s)| \\
&< |g(s) - 1| + \frac{\gamma_i}{\gamma_i + 1}|J_{m_i(\gamma_i + 1)}(s)| \\
&\quad \times |J_{l_i}^k(s) - 1|\,|g(s)|
\end{aligned} \tag{47}$$

Since $g(s)$ is proper and $m_i \geqslant b$, $|g(s)|$ is bounded in $\Omega(m_i, \infty)$. Also,

$$|J_{m_i(\gamma_i+1)}(s)|\,|J^k_{l_{0i}}(s) - 1| \to 0 \text{ uniformly in } \Omega(m_i, \infty) \quad \text{as } l_{0i} \to \infty \tag{48}$$

where l_{0i} belongs to positive integers; thus, given any $\varepsilon_{0i} > 0$, there exists $l_i \geqslant l_{0i} > 0$ such that

$$|\Lambda_i(s)| < |g(s) - 1| + \varepsilon_{0i} \quad \forall s \in \Omega(m_i, \infty) \tag{49}$$

Since m_i satisfies eqn. 9, we get

$$|g(s) - 1| \leqslant 1 - \delta_i \quad \forall s \in \Omega(m_i, \infty)$$

and

$$|g(j\omega) - 1| \leqslant 1 - M_i^{-1} - \delta_i \quad \forall |\omega| > m_i$$

Therefore, if we choose $\varepsilon_{0i} = \delta_i$, then there exists $l_i \geqslant l_{0i} \geqslant m_i/(2^{1/k} - 1)$ large enough such that eqns. 45 and 46 are true.

Finally, we show that $H(s)$ belong to $S^{2n \times 2n}$.

Since $P(s)$ and $C(s)$ are strictly proper, thus $H(s)$ belongs to $\mathbb{R}_p(s)^{2n \times 2n}$. We have shown that $(I + PC)^{-1} \in S^{n \times n}$, By assumption, $P(s)^{-1}$ is analytic in C_+; By construction, $C(s)^{-1}$ is analytic in C_+. Thus

$$P(I + CP)^{-1} = (I - (I + PC)^{-1})C^{-1} \in S^{n \times n}$$

$$(I + CP)^{-1} = P^{-1}P(I + CP)^{-1} \in S^{n \times n}$$

and

$$C(I + PC)^{-1} = (I - (I + CP)^{-1})P^{-1} \in S^{n \times n}$$