Stabilising controller and observer synthesis for uncertain large-scale systems by the Riccati equation approach

Wen-June Wang, MSc, PhD
Chen-Fa Cheng, MSc

Indexing terms: Control theory, Stabilising controllers, Observers, Riccati equations

Abstract: The paper introduces a Riccati equation approach to synthesise of the full state observers and state feedback controllers for uncertain large-scale systems. In this approach, if two given algebraic Riccati equations are solved, their solutions can be applied to synthesise the stabilising state feedback and observer gain matrices. The uncertainties considered in each subsystem may be time-varying and appear in the system matrices (matrix A_i), input connection matrices (matrix B_i), or/and output matrices (matrix C_i). However the values of those uncertainties are constrained to lie within some known admissable bounds. Furthermore, the so-called matching conditions are not needed in the paper.

List of symbols

$\mathscr{R}^n$ = real vector space of dimension n
A^T = transpose of matrix A
$\lambda_i(A)$ = ith eigenvalue of matrix A
$\|A\|_s$ = spectral norm of matrix A, i.e.

$$\|A\|_s = \max_i [\lambda_i(A^TA)]^{1/2}$$

$\|(\cdot)\|$ = Euclidean norm of vector $(\cdot)$ or matrix $(\cdot)$, i.e.

$$\|g\| = \left(\sum_{i=1}^{m} |g_i|^2\right)^{1/2} \quad \|G\| = \left(\sum_{i,j}^{m,n} |G_{ij}|^2\right)^{1/2}$$

where $g = [g_1, \ldots, g_m]$, $G = [G_{ij}]$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$
$\mathscr{R}^{n \times m}$ = real matrix space of dimension $n \times m$
$\lambda_m(A) = \min_i \lambda_i(A)$ = minimal eigenvalue of the matrix A
$\lambda_M(A) = \max_i \lambda_i(A)$ = maximal eigenvalue of the matrix A
In (A) = inertia of a square matrix $A = \{\pi(A), \nu(A), \delta(A)\}$, where $\pi(A)$, $\nu(A)$, $\delta(A)$ denote the number of eigenvalues of A, computed with their algebraic multiplicities, lying the open right half-plane, in the open left-plane, and on the imaginary axis, respectively.

Paper 8298D (C8), first received 14th January and in revised form 3rd June 1991

Dr. Wang is with the Department of Electrical Engineering, National Cestral University, Chung-Li, 32054, Taiwan, Republic of China
Mr. Cheng is with the Institute of Electronics, National Chiao-Tung University, Shin Chu, Taiwan, Republic of China

1 Introduction

In recent years the stabilisation problem of an uncertain system has received great attention. Numerous researchers have used state feedback stabilising control to achieve this task [1–4]. However, in practice, the states of the system may not be available. Luenberger, in 1966, first proposed the concept of the 'observer' and introduced the idea of a 'reduced-order observer' to estimate those states which are inaccessible for direct measurement.

For a system of large dimensions, many researchers have devoted themselves to the investigation of observer design also [5, 6, 15–17]. However, those papers are only concerned with large-scale systems without uncertainties.

In this paper, the Riccati equation approach of Petersen [7–9] is extended to an uncertain large-scale system. If the proposed two algebraic Riccati equations are solved, using their solutions the stabilising feedback controller and observer of each uncertain subsystem can be synthesised simultaneously so that the whole large-scale system is robustly stable.

2 System description and preliminary derivation

Let S be a large-scale system composed of N ($N > 1$) interconnected uncertain subsystems S_i, $i = 1, 2, \ldots, N$. Each S_i is described in the following:

$$S_i: \dot{x}_i(t) = [A_{io} + \Delta A_i(\mathfrak{r}_i(t))]x_i(t) + [B_{io} + \Delta B_i(\mathfrak{s}_i(t))] \times \left(u_i(t) + \sum_{j \neq i}^{N} H_{ij}x_j(t)\right) \tag{1a}$$

$$y_i(t) = ((C_{io} + \Delta C_i(\mathfrak{t}_i(t)))x_i(t)) \quad i = 1, 2, \ldots, N \tag{1b}$$

where $A_{io} \in \mathscr{R}^{n_i \times n_i}$, $B_{io} \in \mathscr{R}^{n_i \times p_i}$, $H_{ij} \in \mathscr{R}^{p_i \times n_j}$ and $C_{io} \in \mathscr{R}^{m_i \times n_i}$ denote the nominal system matrix, input matrix, interconnection matrix, and nominal output matrix, respectively. $\mathfrak{r}_i(\cdot)$, $\mathfrak{s}_i(\cdot)$ and $\mathfrak{t}_i(\cdot)$ are vectors with uncertain elements belonging to the following compact sets $\mathfrak{R}_i$, $\mathfrak{S}_i$ and $\mathfrak{T}_i$, respectively:

$$\mathfrak{R}_i = \{\mathfrak{r}_i : |\mathfrak{r}_{im}| \leqslant \bar{\mathfrak{r}}_i, m = 1, 2, \ldots, k\} \tag{2a}$$

$$\mathfrak{S}_i = \{\mathfrak{s}_i : |\mathfrak{s}_{iq}| \leqslant \bar{\mathfrak{s}}_i, q = 1, 2, \ldots, l\} \tag{2b}$$

$$\mathfrak{T}_i = \{\mathfrak{t}_i : |\mathfrak{t}_{ip}| \leqslant \bar{\mathfrak{t}}_i, p = 1, 2, \ldots, h\} \tag{2c}$$

The uncertainties are assumed to be of the 'rank 1' type [7], i.e.

$$\Delta A_i(\mathfrak{r}_i(t)) = \sum_{m=1}^{k} A_{im}\mathfrak{r}_{im}(t) \quad (3a)$$

$$\Delta B_i(\mathfrak{s}_i(t)) = \sum_{q=1}^{l} B_{iq}\mathfrak{s}_{iq}(t) \quad (3b)$$

$$\Delta C_i(\mathfrak{t}_i(t)) = \sum_{p=1}^{h} C_{ip}\mathfrak{t}_{ip}(t) \quad (3c)$$

Here suppose constant matrices A_{im}, B_{iq} and C_{ip} can be written as

$$A_{im} = \mathfrak{d}_{im}\mathfrak{e}_{im}^T, \; B_{iq} = \mathfrak{f}_{iq}\mathfrak{g}_{iq}^T \quad \text{and} \quad C_{ip} = \mathfrak{h}_{ip}\mathfrak{m}_{ip}^T \quad (4)$$

where $\mathfrak{d}_{im}, \mathfrak{e}_{im}, \mathfrak{f}_{iq} \in \Re^{n_i}$ and $\mathfrak{g}_{iq} \in \Re^{p_i}$, $\mathfrak{h}_{ip} \in \Re^{m_i}$, $\mathfrak{m}_{ip} \in \Re^{n_i}$. Introduce the notations

$$\begin{aligned} T_i &= \bar{\mathfrak{r}}_i \sum_{m=1}^{k} \mathfrak{d}_{im}\mathfrak{d}_{im}^T & U_i &= \bar{\mathfrak{r}}_i \sum_{m=1}^{k} \mathfrak{e}_{im}\mathfrak{e}_{im}^T \\ L_i &= \bar{\mathfrak{s}}_i \sum_{q=1}^{l} \mathfrak{g}_{iq}\mathfrak{g}_{iq}^T & W_i &= \bar{\mathfrak{s}}_i \sum_{q=1}^{l} \mathfrak{f}_{iq}\mathfrak{f}_{iq}^T \\ Z_i &= \bar{\mathfrak{t}}_i \sum_{p=1}^{h} \mathfrak{h}_{ip}\mathfrak{h}_{ip}^T & Y_i &= \bar{\mathfrak{t}}_i \sum_{p=1}^{h} \mathfrak{m}_{ip}\mathfrak{m}_{ip}^T \end{aligned} \quad (5)$$

By hypothesis, all pairs (A_{io}, B_{io}) and (A_{io}, C_{io}), $i = 1, 2, \ldots, N$, are controllable and observable, respectively. Now suppose the states of the large-scale system (eqn. 1) are all inaccessible to be measured. It is required to construct an observer dynamic equation for each isolated subsystem $(\forall H_{ij} = 0)$ as below [12]:

$$\dot{\hat{x}}_i(t) = A_{io}\hat{x}_i(t) + B_{io}u_i(t) - M_i\tilde{y}_i(t) \quad i = 1, 2, \ldots, N \quad (6)$$

where $\tilde{y}_i(t) \equiv y_i - \hat{y}_i(t)$, $\hat{y}_i(t) \equiv C_{io}\hat{x}_i(t)$ and $\hat{x}_i(t) \in R^{n_i}$ is the observer state. M_i is the unknown observer gain vector and will be found so as to ensure the estimated state $\hat{x}_i(t) \to x_i(t)$ as $t \to \infty$. From eqns. 1a and 6, we have the error equation:

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= A_{io}\tilde{x}_i(t) + \Delta A_i x_i(t) + \Delta B_i u_i(t) \\ &\quad + M_i\tilde{y}_i(t) + (B_{io} + \Delta B_i)\sum_{j \neq i} H_{ij}x_j(t) \end{aligned} \quad (7)$$

where $\tilde{x}_i(t) \equiv x_i(t) - \hat{x}_i(t)$ denotes the error state vector. Applying an observed state feedback

$$u_i(t) = K_i\hat{x}_i(t) \quad i = 1, 2, \ldots, N \quad (8)$$

to eqns. 1 and 7, the closed-loop subsystem S_i^e with the both state and error equations is written as follows:

$$\begin{aligned} S_i^e\colon \dot{x}_i(t) &= [(A_{io} + \Delta A_i) + (B_{io} + \Delta B_i)K_i]x_i(t) \\ &\quad - (B_{io} + \Delta B_i)K_i\tilde{x}_i(t) \\ &\quad + \sum_{j \neq i}^{N} [B_{io} + \Delta B_i]H_{ij}x_j(t) \end{aligned} \quad (9)$$

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= [\Delta A_i + \Delta B_i K_i + M_i\,\Delta C_i]x_i \\ &\quad + [A_{io} - \Delta B_i K_i + M_i C_{io}]\tilde{x} \\ &\quad + (B_{io} + \Delta B_i)\sum_{j \neq i} H_{ij}x_j \quad i = 1, 2, \ldots, N \end{aligned} \quad (10)$$

Let the large-scale system consisting of N $(N > 1)$ closed-loop subsystems be denoted by $\hat{S}$. Our objective is to design a full state observer (eqn. 6) and state feedback (eqn. 8) so that the given uncertain large-scale system is stabilised. For the derivation of the main results we need review some useful lemmas:

Lemma 1: [11] For any matrices or vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with appropriate dimension, we have

$$\boldsymbol{x}^T\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{x} \leqslant \beta\boldsymbol{x}^T\boldsymbol{x} + \frac{1}{\beta}\boldsymbol{y}^T\boldsymbol{y} \quad (11)$$

for any positive constant β.

Lemma 2: [10] Let $\mathscr{A}$ and $\mathscr{B}$ be $n \times n$ Hermitian matrices of the same rank r. If $\mathscr{A} = \mathscr{M}\mathscr{B}\mathscr{M}^*$ for some matrix $\mathscr{M}$, then In $(\mathscr{A})$ = In $(\mathscr{B})$.

3 Robustness of a Riccati equation approach design

Let us first consider the stabilisation of S via the stabilising state feedback eqn. 8 and full state observer eqn. 6. The result will be given in terms of the solutions (positive-definite symmetric matrices P_{ic} and P_{io}) of the following two algebraic Riccati equations for $i = 1, \ldots, N$.

$$\begin{aligned} &A_{io}^T P_{ic} + P_{ic}A_{io} - P_{ic} \\ &\times \left\{\frac{2}{\varepsilon_{1i}}(B_{io}R_{1i}^{-1}B_{io}^T - B_{io}R_{1i}^{-1}L_iR_{1i}^{-1}B_{io}^T)\right. \\ &\left. - \left(1 + \frac{1}{\varepsilon_{1i}}\right)B_{io}B_{io}^T - T_i - \eta_i W_i\right\}P_{ic} \\ &+ 2U_i + 2\sum_j H_{ji}^T L_j H_{ji} + H_i \\ &+ \frac{1}{\varepsilon_{2i}}Y_i + \varepsilon_{1i}Q_{1i} = 0 \end{aligned} \quad (12)$$

$$\begin{aligned} &A_{io}^T P_{io} + P_{io}A_{io} - \left(\frac{1}{\varepsilon_{2i}}[C_{io}^T(2R_{2i}^{-1} - R_{2i}^{-1}Z_iR_{2i}^{-1})C_{io}]\right. \\ &\left. - \frac{1}{e_{1i}}P_{ic}B_{io}R_{1i}^{-1}(2L_i + I)R_{1i}^{-1}B_{io}^T P_{ic}\right) \\ &+ P_{io}[T_i + \eta_i W_i + B_{io}B_{io}^T + \varepsilon_{2i}Q_{2i}]P_{io} = 0 \end{aligned} \quad (13)$$

where $\eta_i = (N - 1) + (2/\varepsilon_{1i})$ and $H_i = 2(N - 1)\sum_{j \neq i} \|H_{ji}\|_s^2 I$, both Q_{1i}, Q_{2i} and R_{1i}, R_{2i} are any given positive definite matrices. The ε_{1i} and ε_{2i} are positive constants. The gain matrices K_i and M_i can be synthesised as follows:

$$K_i = -\frac{1}{\varepsilon_{1i}}R_{1i}^{-1}B_{io}^T P_{ic} \quad (14)$$

and

$$M_i = -\frac{1}{\varepsilon_{2i}}P_{io}^{-1}C_{io}^T R_{2i}^{-1} \quad i = 1, 2, \ldots, N \quad (15)$$

These results will be summarised in the following main theorem, for which a proof is given in Appendix 9.

Theorem 1: For the considered uncertain large scale system S (made up of S_i, $i = 1, 2, \ldots, N$, in eqn. 1), if there exist positive constants ε_{1i} and ε_{2i} such that the following two conditions hold:

(a) there exist positive-definite symmetric matrices P_{ic} satisfying eqn. 12

(b) there exist positive-definite symmetric matrices P_{io} satisfying eqn. 13.

Then the resulting closed-loop large-scale system $\hat{S}$ is guaranteed to be asymptotically stabilised by the observer eqn. 6 and state feedback eqn. 8 with the gain matrices (eqns. 15 and 14) respectively. The proof is given in Appendix 9.

4 Discussion

(*a*) The uncertainties discussed in this theorem need not meet the 'matching conditions'.

(*b*) When the system has only one subsystem (i.e. $N = 1$, without the interconnection matrix), the Riccati equations (eqns. 12 and 13) will be reduced to those in References 7 and 8. And the solving procedure will be greatly simplified.

(*c*) Since a system with large dimensions is considered, the effect of interconnection matrices is such that the derived algebraic Riccati equations (eqns. 12 and 13) are much more complicated than those of Reference 7, i.e. there are some extra terms in eqns. 12, 13 compared with eqns. 3.1, 3.2 of Reference 7. We doubt that the algorithm mentioned by Petersen [8, 9] can be used to solve our eqns. 12 and 13. The study of the algorithm to solve the Riccati equation is not the main task of this paper; we mainly propose a sufficient condition (theorem 1) under which the controller and observer can be synthesised.

5 Example

A large-scale system is made up of three uncertain subsystems

$$S_1: \dot{x}_1(t) = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{u}_1(t) + \Delta A_1(\mathfrak{r}_1)x_1(t) + \Delta B_1(\mathfrak{s}_1(\bar{u}_1)(t)$$

$$y_1(t) = [0 \quad 1]x_1(t) + \Delta C_1(\mathfrak{t}_1)x_1(t) \qquad (16a)$$

$$S_2: \dot{x}_2(t) = \begin{bmatrix} 0 & 1 \\ 7 & 1 \end{bmatrix} x_2(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{u}_2(t) + \Delta A_2(\mathfrak{r}_2)x_2(t) + \Delta B_2(\mathfrak{s}_2)\bar{u}_2(t)$$

$$y_2(t) = [1 \quad 1]x_2(t) + \Delta C_2(\mathfrak{t}_2)x_2(t) \qquad (16b)$$

$$S_3: \dot{x}_3(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_3(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{u}_3(t) + \Delta A_3(\mathfrak{r}_3)x_3(t) + \Delta B_3(\mathfrak{s}_3)\bar{u}_3(t)$$

$$y_3(t) = [1 \quad 0]x_3(t) + \Delta C_3(\mathfrak{t}_3)x_3(t) \qquad (16c)$$

and each

$$\bar{u}_i(t) = \sum_{j \neq i} H_{ij} x_j(t) + u_i(t) \qquad (16d)$$

where

$$H_{12} = [0.1 \quad 0.1] \quad H_{13} = [0.1 \quad 0.1]$$

$$H_{21} = [0.2 \quad 0.1] \quad H_{23} = [0.15 \quad 0.2]$$

$$H_{31} = [0.1 \quad 0] \quad \text{and} \quad H_{32} = [0.15 \quad 0.1]$$

The uncertainties satisfy the forms of eqn. 3 for all subsystems:

$$\Delta A_1(\mathfrak{r}_1) = \begin{bmatrix} a'_{11} & 0 \\ a'_{21} & 0 \end{bmatrix} \quad \Delta C_1(\mathfrak{t}_1) = [c'_1 \quad 0] \qquad (17)$$

and $|a'_{11}| \leqslant 0.15$, $|a'_{21}| \leqslant 0.2$, $|c'_1| \leqslant 0.05$.

$$\Delta A_2(\mathfrak{r}_2) = \begin{bmatrix} 0 & 0 \\ a''_{21} & 0 \end{bmatrix}$$

$$\Delta B_2(\mathfrak{s}_2) = \begin{bmatrix} b''_1 \\ 0 \end{bmatrix}$$

$$\Delta C_2(\mathfrak{t}_2) = [0 \quad c''_2] \qquad (18)$$

and $|a''_{21}| \leqslant 0.1$, $|b''_1| \leqslant 0.1$ $|c''_2| \leqslant 0.05$.

$$\Delta A_3(\mathfrak{r}_3) = \begin{bmatrix} a'''_{11} & a'''_{12} \\ 0 & 0 \end{bmatrix} \quad \Delta B_3(\mathfrak{s}_3) = \begin{bmatrix} b'''_1 \\ 0 \end{bmatrix} \qquad (19)$$

and $|a'''_{11}| \leqslant 0.15$, $|a'''_{12}| \leqslant 0.12$, $|b'''_1| \leqslant 0.09$. Putting the problem in the forms of eqns. 2, 3, 4 and 5, let

$$\Delta A_1(\mathfrak{r}_1) = A_{11}\mathfrak{r}_{11} + A_{12}\mathfrak{r}_{12} \quad \Delta C_1(\mathfrak{t}) = C_{11}\mathfrak{t}_{11} \qquad (20)$$

where

$$A_{11} = \mathfrak{d}_{11}\mathfrak{e}_{11}^T = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} [0.3 \quad 0]$$

$$A_{12} = \mathfrak{d}_{12}e_{12}^T = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} [0.4 \quad 0]$$

$$C_{11} = \mathfrak{h}_{11}\mathfrak{m}_{11}^T = 0.25[0.2 \quad 0]$$

$$\Delta A_2(\mathfrak{r}_2) = A_{21}\mathfrak{r}_{21}$$

$$\Delta B_2(\mathfrak{s}_2) = B_{21}\mathfrak{s}_{21}$$

$$\Delta C_2(\mathfrak{t}_2) = C_{21}\mathfrak{t}_{21} + C_{22}\mathfrak{t}_{22} \qquad (21)$$

where

$$A_{21} = \mathfrak{d}_{21}\mathfrak{e}_{21}^T = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} [0.25 \quad 0]$$

$$B_{21} = \mathfrak{f}_{21}\mathfrak{g}_{21}^T = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} 0.5$$

$$C_{21} = \mathfrak{h}_{21}\mathfrak{m}_{21}^T = 0.25[0 \quad 0.2]$$

$$\Delta A_3(\mathfrak{r}_3) = A_{31}\mathfrak{r}_{31} \quad \Delta B_3(\mathfrak{s}_3) = B_{31}\mathfrak{s}_{31} \qquad (22)$$

where

$$A_{31} = \mathfrak{d}_{31}\mathfrak{e}_{31}^T = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} [0.3 \quad 0.24]$$

and

$$B_{31} = \mathfrak{f}_{31}\mathfrak{g}_{31}^T = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} 0.45$$

With $\bar{\mathfrak{r}}_i = 1$, $\bar{\mathfrak{s}}_i = 1$, $\bar{\mathfrak{t}}_i = 1$, select

subsystem 1: $\{\varepsilon_{11} = 0.1, \quad \varepsilon_{21} = 0.01, \quad Q_{11} = Q_{21} = I, R_{11} = 0.1, R_{21} = 1\}$

subsystem 2: $\{\varepsilon_{12} = 0.0125, \quad \varepsilon_{22} = 1, \quad Q_{12} = Q_{22} = I, R_{12} = 1, R_{22} = 0.1\}$

subsystem 3: $\{\varepsilon_{13} = 0.5, \varepsilon_{23} = 1, Q_{13} = Q_{23} = I, R_{13} = 1, R_{23} = 0.1\}$

then the Riccati equations (eqns. 12 and 13) have positive-definite solutions as follows:

$$P_{1c} = \begin{bmatrix} 2.1 & 0.239 \\ 0.239 & 0.0843 \end{bmatrix}$$

$$P_{1o} = \begin{bmatrix} 9.1 & -7.08 \\ -7.08 & 8.084 \end{bmatrix} \qquad (23a)$$

$$P_{2c} = \begin{bmatrix} 0.300 & 0.184 \\ 0.184 & 0.185 \end{bmatrix}$$

$$P_{2o} = \begin{bmatrix} 1.21 & -0.727 \\ -0.727 & 2.298 \end{bmatrix} \qquad (23b)$$

$$P_{3c} = \begin{bmatrix} 1.41 & 0.351 \\ 0.351 & 0.274 \end{bmatrix}$$

$$P_{3o} = \begin{bmatrix} 2.03 & -0.731 \\ -0.731 & 2.902 \end{bmatrix} \qquad (23c)$$

From the above expressions, both conditions of theorem 1 are satisfied. Then, from eqns. 14 and 15, the stabilising control laws are $u_i(t) = K_i \hat{x}_i(t)$, $i = 1, 2, 3$ with

$$K_1 = -[23.85 \quad 8.43]$$
$$K_2 = -[24.00 \quad 14.68]$$
$$K_3 = -[3.522 \quad 1.25] \qquad (24)$$

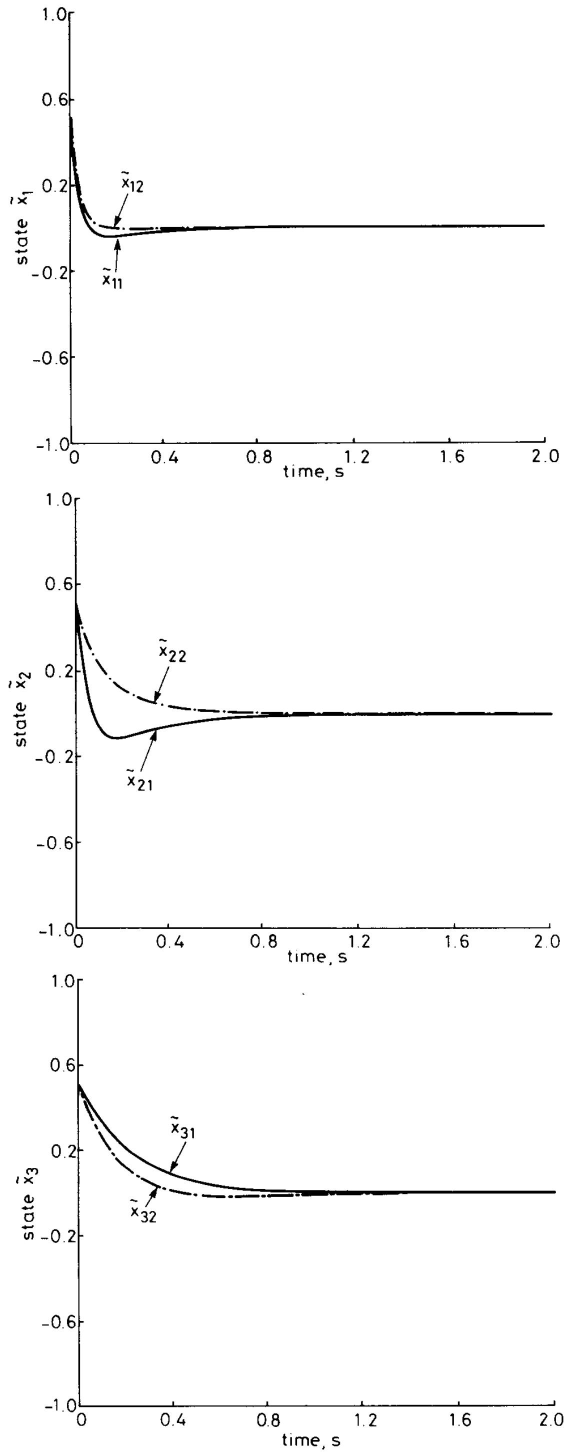

Fig. 1 *Error states*

a Error state $\tilde{x}_1$ of subsystem 1
b Error state $\tilde{x}_2$ of subsystem 2
c Error state $\tilde{x}_3$ of subsystem 3

and the observer gain matrices M_i are

$$M_1 = -[30.21 \quad 38.83]^T$$
$$M_2 = -[13.43 \quad 8.61]^T$$
$$M_3 = -[5.42 \quad 1.37]^T \qquad (25)$$

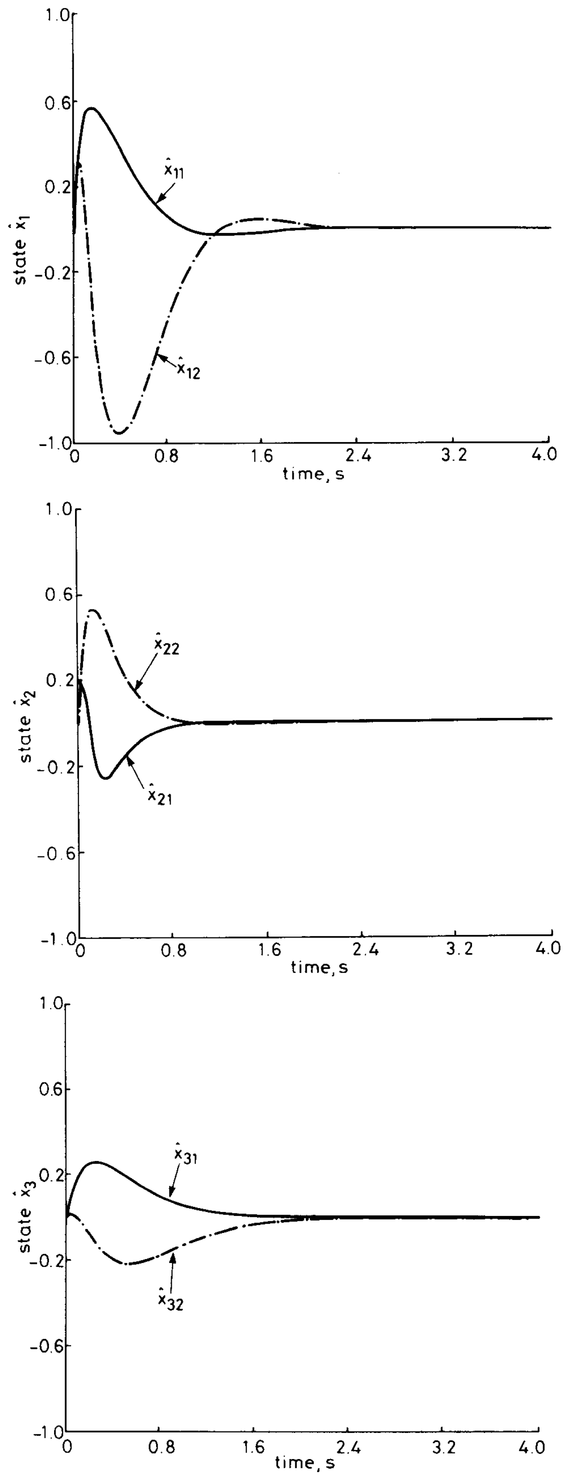

Fig. 2 *Observer states*

a Observer state $\hat{x}_1$ of subsystem 1
b Observer state $\hat{x}_2$ of subsystem 2
c Observer state $\hat{x}_3$ of subsystem 3

Using the above computed control gains and observer gains, we plot the trajectories of the error states and observer states of the resulted closed-loop system in Figs. 1 and 2 with initial conditions $x_i(0) = 0.5$ and $\hat{x}_i(0) = 0$, $i = 1, 2, 3$. The uncertainties in the simulation are given as sinusoidal sin (t) forms, e.g. $a'_{11} = 0.15 \sin(t)$.

6 Conclusions

In this paper, we have extended the technique of Reference 7 to the uncertain large-scale system. Similarly, two algebraic Riccati equations are required to be solved in our main results. If these Riccati equations possess positive-definite solutions, the observer and the feedback control gain can be synthesised by their solutions so that the whole uncertain large scale system is stabilised. Whether the algebraic Riccati equations are solvable possibly depends on the magnitudes of uncertainties and norms of interconnection matrices and the number of subsystems. Hence determining the conditions of the solvability of the Riccati equations (eqns. 12, 13) and seeking an algorithm to solve them are two open problems for the future.

7 Acknowledgement

This research was supported by the National Science Council, Taiwan, under contract NSC 77-0404-E008-04.

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9 Appendix

Proof of theorem 1: By the gain matrices eqns. 14 and 15, the state and error equations for the closed-loop uncertain large-scale system are as follows:

$$\dot{x}_i = \left[A_{io} + \sum_m A_{im}\mathfrak{r}_{im}(t) - \frac{1}{\varepsilon_{\varepsilon 1i}}\left(B_{io} + \sum_q B_{iq}\mathfrak{s}_{iq}\right)R_{1i}^{-1}B_{io}^T P_{ic}\right]x_i + \frac{1}{\varepsilon_{1i}}\left[B_{io} + \sum_q B_{iq}\mathfrak{s}_{iq}\right]R_{1i}^{-1}B_{io}^T P_{ic}\tilde{x}_i + \sum_{j\neq i}\left(B_{io} + \sum_q B_{iq}\mathfrak{s}_{iq}\right)H_{ij}x_j \tag{26}$$

$$\dot{\tilde{x}}_i = \left[\sum_m A_{im}\mathfrak{r}_{im} - \frac{1}{\varepsilon_{1i}}\sum_q B_{iq}\mathfrak{s}_{iq}R_{1i}^{-1}B_{io}^T P_{ic} - \frac{1}{\varepsilon_{2i}}P_{io}^{-1}C_{io}^T R_{2i}^{-1}\sum_p C_{ip}\mathfrak{t}_{ip}\right]x_i + \left[A_{io} + \frac{1}{\varepsilon_{1i}}\sum_q B_{iq}\mathfrak{s}_{iq}R_{1i}^{-1}B_{io}^T P_{ic} - \frac{1}{\varepsilon_{2i}}P_{io}^{-1}C_{io}^T R_{2i}^{-1}C_{io}\right]\tilde{x}_i + \left(B_{io} + \sum_q B_{iq}\mathfrak{s}_{iq}\right)\sum_j H_{ij}x_j \tag{27}$$

Define a Lyapunov function $V = \sum_i v_i$ to be

$$V = \sum_i (x_i^T P_{ic} x_i + \tilde{x}_i^T P_{io}\tilde{x}_i) \tag{28}$$

where both P_{ic} and P_{io} are positive-definite matrices satisfying Riccati eqns. 12 and 13, respectively. Taking the derivative $\dot{V}$ and substituting eqns. 25 and 26 into $\dot{V}$ yields

$$\dot{V} = \sum_i [(\dot{x}_i^T P_{ic} x_i + x_i^T P_{ic}\dot{x}_i) + (\dot{\tilde{x}}_i^T P_{io}\tilde{x}_i + \tilde{x}_i^T P_{io}\dot{\tilde{x}}_i)] \tag{29}$$

$$= \sum_i \{x_i^T(P_{ic}A_{io} + A_{io}^T P_{ic})x_i + 2x_i^T P_{ic}\left(\sum_{m=1}^k A_{im}\mathfrak{r}_{im}(t)\right)x_i(t) - \frac{2}{\varepsilon_{1i}}x_i^T P_{ic}\left(\sum_{q=1}^l B_{iq}\mathfrak{s}_{iq}(t)\right)R_{1i}^{-1}B_{io}^T P_{ic}x_i + 2\sum_{j\neq i}(B_{io}H_{ij}x_j)^T P_{ic}x_i + 2\sum_{j\neq i}\left(\left(\sum_{q=1}^l B_{iq}\mathfrak{s}_{iq}(t)\right)H_{ij}x_j\right)^T P_{ic}x_i - \frac{2}{\varepsilon_{1i}}x_i^T P_{ic}B_{io}R_{1i}^{-1}B_{io}^T P_{ic}x_i + \frac{2}{\varepsilon_{1i}}x_i^T P_{ic}\left[B_{io} + \left(\sum_{q=1}^l B_{iq}\mathfrak{s}_{iq}(t)\right)\right]R_{1i}^{-1}B_{io}^T P_{ic}\tilde{x}_i + \tilde{x}_i^T[A_{io}^T P_{io} + P_{io}A_{io}]\tilde{x}_i + \frac{2}{\varepsilon_{1i}}\tilde{x}_i^T P_{io}\left(\sum_q B_{iq}\mathfrak{s}_{iq}(t)\right)R_{1i}^{-1}B_{io}^T P_{ic}\tilde{x}_i - \frac{2}{\varepsilon_{2i}}\tilde{x}_i^T C_{io}^T R_{2i}^{-1}C_{io}\tilde{x}_i + 2\sum_i \tilde{x}_i^T P_{io}\left(\sum_q B_{iq}\mathfrak{s}_{iq}(t)\right)\sum_{j\neq i}H_{ij}x_j$$

$$+ 2\tilde{x}_i^T P_{io} B_{io} \sum_{j \neq i} H_{ij} x_j$$

$$- \frac{2}{\varepsilon_{1i}} \tilde{x}_i^T P_{io} \left(\sum_q B_{iq} \mathfrak{s}_{iq}(t) \right) R_{1i}^{-1} B_{io}^T P_{ic} x_i$$

$$+ 2x_i^T P_{io} \left(\sum_m A_{im} \mathfrak{r}_{im}(t) \right) x_i$$

$$- \frac{2}{\varepsilon_{2i}} \tilde{x}_i^T C_{io}^T R_{2i}^{-1} \left(\sum_p C_{ip} \mathfrak{t}_{ip}(t) \right) x_i \Bigg\} \tag{30}$$

By the inequality $2|ab| \leqslant a^2 + b^2$ and eqns. 4 and 5, we have

$$2x_i^T P_{ic} \left(\sum_{m=1}^{k} A_{im} \mathfrak{r}_{im}(t) \right) x_i$$

$$\leqslant \bar{\mathfrak{r}}_i \sum_{m=1}^{k} (x_i^T P_{ic} \mathfrak{d}_{im})^2 + \bar{\mathfrak{r}}_i \sum_{m=1}^{k} (\mathfrak{e}_{im}^T x_i)^2$$

$$= x_i^T P_{ic} T_i P_{ic} x_i + x_i^T U_i x_i \tag{31}$$

Further, by lemma 1, the following inequalities hold:

$$2 \sum_i \sum_{j \neq i} (B_{io} H_{ij} x_j)^T P_{ic} x_i$$

$$\leqslant \sum_i \sum_{j \neq i} \left(\frac{1}{N-1} x_i^T P_{ic} B_{io} B_{io}^T P_{ic} x_i + (N-1) \|H_{ij}\|_s^2 x_j^T x_j \right) \quad \text{by Lemma 1}$$

$$\leqslant \sum_i x_i^T P_{ic} B_{io} B_{io}^T P_{ic} x_i + (N-1) \sum_i \sum_{j \neq i} \|H_{ji}\|_s^2 x_i^T x_i \tag{32}$$

Similarly, we derive upper bounds of the other terms as follows:

$$- \frac{2}{\varepsilon_{1i}} x_i^T P_{ic} \left(\sum_{q=1}^{l} B_{iq} \mathfrak{s}_{iq}(t) \right) R_{1i}^{-1} B_{io}^T P_{ic} x_i$$

$$\leqslant \frac{1}{\varepsilon_{1i}} x_i^T P_{ic} W_i P_{ic} x_i + \frac{1}{\varepsilon_{1i}} x_i^T P_{ic} B_{io} R_{1i}^{-1} L_i R_{1i}^{-1} B_{io}^T P_{ic} x_i$$

$$2 \sum_i \sum_{j \neq i} \left(\left(\sum_{q=1}^{l} B_{iq} \mathfrak{s}_{iq}(t) \right) H_{ij} x_j \right)^T P_{ic} x_i$$

$$\leqslant (N-1) \sum_l x_i^T P_{ic} W_i P_{ic} x_i + \sum_i \sum_j x_i^T H_{ji}^T L_j H_{ji} x_i$$

$$\frac{2}{\varepsilon_{1i}} x_i^T P_{ic} B_{io} R_{1i}^{-1} B_{io}^T P_{ic} \tilde{x}_i$$

$$\leqslant \frac{1}{\varepsilon_{1i}} x_i^T P_{ic} B_{io} B_{io}^T P_{ic} x_i + \frac{1}{\varepsilon_{1i}} \tilde{x}_i^T P_{ic} B_{io} R_{1i}^{-2} B_{io}^T P_{ic} \tilde{x}_i$$

$$\frac{2}{\varepsilon_{1i}} x_i^T P_{ic} \left(\sum_{q=1}^{l} B_{iq} \mathfrak{s}_{iq}(t) \right) R_{1i}^{-1} B_{io}^T P_{ic} \tilde{x}_i$$

$$\leqslant \frac{1}{\varepsilon_{1i}} x_i^T P_{ic} W_i P_{ic} x_i + \frac{1}{\varepsilon_{1i}} \tilde{x}_i^T P_{ic} B_{io} R_{1i}^{-1} L_i R_{1i}^{-1} B_{io}^T P_{ic} \tilde{x}_i$$

$$\frac{2}{\varepsilon_{1i}} \tilde{x}_i^T P_{io} \left(\sum_q B_{iq} \mathfrak{s}_{iq}(t) \right) R_{1i}^{-1} B_{io}^T P_{ic} \tilde{x}_i$$

$$\leqslant \frac{1}{\varepsilon_{1i}} \tilde{x}_i^T P_{io} W_i P_{io} \tilde{x}_i + \frac{1}{\varepsilon_{1i}} \tilde{x}_i^T P_{ic} B_{io} R_{1i}^{-1} L_i R_{1i}^{-1} B_{io}^T P_{ic} \tilde{x}_i$$

$$2 \sum_i \tilde{x}_i^T P_{io} \left(\sum_q B_{iq} \mathfrak{s}_{iq}(t) \right) \sum_{j \neq i} H_{ij} x_j$$

$$\leqslant (N-1) \sum_i \tilde{x}_i^T P_{io} W_i P_{io} \tilde{x}_i + \sum_i \sum_j x_i^T H_{ji}^T L_j H_{ji} x_i$$

$$2 \sum_i \tilde{x}_i^T P_{io} B_{io} \sum_{j \neq i} H_{ij} x_j$$

$$\leqslant \sum_i \tilde{x}_i^T P_{io} B_{io} B_{io}^T P_{io} \tilde{x}_i + \sum_i \sum_j (N-1) x_i^T \|H_{ji}\|_s^2 x_i$$

$$- \frac{2}{\varepsilon_{1i}} \tilde{x}_i^T P_{io} \left(\sum_q B_{iq} \mathfrak{s}_{iq}(t) \right) R_{1i}^{-1} B_{io}^T P_{ic} x_i$$

$$\leqslant \frac{1}{\varepsilon_{1i}} \tilde{x}_i^T P_{io} W_i P_{io} \tilde{x}_i + \frac{1}{\varepsilon_{1i}} x_i^T P_{ic} B_{io} R_{1i}^{-1} L_i R_{1i}^{-1} B_{io}^T P_{ic} x_i$$

$$2\tilde{x}_i^T P_{io} \left(\sum_m A_{im} \mathfrak{r}_{im}(t) \right) x_i \leqslant \tilde{x}_i^T P_{io} T_i P_{io} \tilde{x}_i + x_i^T U_i x_i$$

$$- \frac{2}{\varepsilon_{2i}} \tilde{x}_i^T C_{io}^T R_{2i}^{-1} \left(\sum_p C_{ip} \mathfrak{t}_{ip}(t) \right) x_i$$

$$\leqslant \frac{1}{\varepsilon_{2i}} \tilde{x}_i^T C_{io}^T R_{2i}^{-1} Z_i R_{2i}^{-1} C_{io} \tilde{x}_i + \frac{1}{\varepsilon_{2i}} x_i^T Y_i x_i$$

Hence

$$\begin{aligned}
\dot{V} = \leqslant \sum_i \Bigg\{ x_i^T \Bigg[& A_{io}^T P_{ic} + P_{ic} A_{io} + P_{ic} T_i P_{ic} + 2U_i \\
& - \frac{2}{\varepsilon_{1i}} P_{ic} B_{io} R_{1i}^{-1} B_{io}^T P_{ic} \\
& + \frac{2}{\varepsilon_{1i}} P_{ic} W_i P_{ic} + \frac{2}{\varepsilon_{1i}} P_{ic} B_{io} R_{1i}^{-1} L_i \\
& \times R_{1i}^{-1} B_{io}^T P_{ic} + (N-1) P_{ic} W_i P_{ic} \\
& + P_{ic} B_{io} B_{io}^T P_{ic} + 2(N-1) \sum_{j \neq i} \|H_{ji}\|_s^2 I \\
& + 2 \sum_{j \neq i} H_{ji}^T L_j H_{ji} + \frac{1}{\varepsilon_{2i}} Y_i \Bigg] x_i \\
& + \frac{1}{\varepsilon_{1i}} x_i^T P_{ic} B_{io} B_{io}^T P_{ic} x_i \\
& + \frac{1}{\varepsilon_{1i}} \tilde{x}_i^T P_{ic} B_{io} R_{1i}^{-1} R_{1i}^{-1} B_{io}^T P_{ic} \tilde{x}_i \\
& + \tilde{x}_i^T \Bigg[A_{io}^T P_{io} + P_{io} A_{io} + \frac{2}{\varepsilon_{1i}} P_{io} W_i P_{io} \\
& + \frac{2}{\varepsilon_{1i}} P_{ic} B_{io} R_{1i}^{-1} L_i R_{1i}^{-1} B_{io}^T P_{ic} \\
& - \frac{2}{\varepsilon_{2i}} C_{io}^T R_{2i}^{-1} C_{io} + (N-1) P_{io} W_i P_{io} + P_{io} T_i P_{io} \\
& + \frac{1}{\varepsilon_{2i}} C_{io}^T R_{2i}^{-1} Z_i R_{2i}^{-1} C_{io} + P_{io} B_{io} B_{io}^T P_{io} \Bigg] \tilde{x}_i \Bigg\}
\end{aligned} \tag{33}$$

Using the Riccatic equations described in eqns. 12 and 13, eqn. 33 yields

$$\dot{V} = \sum_i [-\varepsilon_{1i} x_i^T Q_{1i} x_i - \varepsilon_{2i} \tilde{x}_i^T P_{io} Q_{2i} P_{io} \tilde{x}_i]$$

$$= -\sum_l [x_i^T \quad \tilde{x}_i^T] \Omega_i \begin{bmatrix} x_i \\ \tilde{x}_i \end{bmatrix} \tag{34}$$

where

$$\Omega_i = \begin{bmatrix} \varepsilon_{1i} Q_{1i} & 0 \\ 0 & \varepsilon_{2i} P_{io} Q_{2i} P_{io} \end{bmatrix} \tag{35}$$

By lemma 2, $P_{io} Q_{2i} P_{io}$ is positive-definite owing to the given positive-definite matrices Q_{2i} and P_{io}. In addition, Q_{1i} is positive-definite matrix and ε_{1i} and ε_{2i} are all positive constants, then Ω_i is positive-definite also. Define $\alpha_i = \lambda_{min}(\Omega_i)$, then for any given admissible uncertainties

$\mathfrak{r}(\cdot)$, $\mathfrak{s}(\cdot)$ and $\mathfrak{t}(\cdot)$:

$$\dot{V} \leqslant -\sum_i \alpha_i(\|x_i\|^2 + \|\tilde{x}_i\|^2) \tag{36}$$

We conclude that the choice of the gain matrices given by eqns. 14 and 15 guarantees the uncertain large-scale system $\hat{S}$ to be stabilised if assumptions (*a*) and (*b*) hold.