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On the Singular Behavior of the Solution of $v''(x) + x \sin v(x) = 0$

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In this paper we study the singular behavior as $a \uparrow \pi$, about the solution $v(x; a)$ of the initial value problem $v''(x) + x \sin v(x) = 0$, $v'(0) = 0$, $v(0) = a$. We also illustrate its application to the large deformation of a heavy cantilever by its own weight.

1. INTRODUCTION

In this paper we are concerned with the singular behavior of the solutions of the following initial value problem:

$$\begin{aligned} v''(x) + x \sin v(x) &= 0, \\ v'(0) &= 0, \\ v(0) &= a, \qquad a \in \mathbb{R}. \end{aligned} \tag{I$_a$}$$

We denote the solution of $(\mathrm{I})_a$ by $v(x; a)$. The qualitative behavior of the solutions $v(x; a)$ is important to the studies of the following mathematical model (1.1) which describes the large deformations of a heavy cantilever by its own weight (see [1] or [2]):

$$\begin{aligned} &v''(x) + x \sin v(x) = 0, \\ &v'(0) = 0, \qquad v(K) = \pi - \alpha, \; 0 \leqslant \alpha \leqslant \pi. \end{aligned} \tag{1.1}$$

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In [2] the authors studied the two-point boundary value problem (1.1) by using the shooting method. From the uniqueness of the solution of the initial value problem $(\mathrm{I})_a$, it follows that

$$\begin{aligned} &v(x;\pi)\equiv\pi, \qquad v(x;-\pi)\equiv-\pi, \qquad v(x;0)\equiv 0;\\ &v(x;2\pi+a)=2\pi+v(x;a), \\ &v(x;2\pi-a)=2\pi-v(x;a), \end{aligned} \tag{1.2}$$

and it suffices to consider the problem $(\mathrm{I})_a$ only for the case $0<a<\pi$. We note that from [2] for all $0<a<\pi$, $v(x;a)$ is oscillatory over $[0,\infty)$ and $-\pi<v(x;a)<\pi$ for all $x\geqslant 0$. We introduce

$$\Delta(x;a)=\frac{dv}{da}(x;a), \qquad \phi(x)=\Delta(x;0).$$

Then differentiating $(\mathrm{I})_a$ with respect to a yields

$$\begin{aligned} &\Delta''(x)+x(\cos v(x;a))\,\Delta(x)=0,\\ &\Delta'(0)=0, \\ &\Delta(0)=1. \end{aligned} \tag{1.3}$$

Setting $a=0$ in (1.3) yields

$$\phi''(x)+x\phi(x)=0, \qquad \phi'(0)=0, \qquad \phi(0)=1. \tag{1.4}$$

Let $y_n(a)$, $z_n(a)$ be the nth zeros of $v(x;a)$ and $v'(x;a)$, respectively, for $n=1,2,\ldots$, with $0=z_1<y_1<z_2<\cdots<y_n<z_{n+1}<y_{n+1}<\cdots$ and λ_n, γ_n be nth zero of $\phi(x)$ and $\phi'(x)$, respectively, for $n=1,2,\ldots$. Then in [2] we have shown the following result.

THEOREM 1.1. *Let $0<a<\pi$; then $\Delta(x;a)$ has an infinite number of isolated zeros $\alpha_n(a)$ and $\Delta'(x;a)$ satisfies the following:*

(i) *If $0<a<\pi/2$, then $\Delta'(x;a)$ has an infinite number of isolated zeros $\beta_n(a)$, $0=\beta_1<\beta_2<\cdots<\beta_n<\cdots$. Furthermore, $\beta_1=z_1=0<y_1<\alpha_1<z_2<\beta_2<y_2<\alpha_2<\cdots<y_n<\alpha_n<z_{n+1}<\beta_{n+1}<y_{n+1}<\cdots$.*

(ii) *If $\pi/2\leqslant a<\pi$ then $\Delta'(x;a)$ has an infinite number of isolated zeros $\beta_n(a)$, $0=\beta_0<\beta_1<\beta_2<\cdots<\beta_n<\cdots$. Furthermore, $\beta_0=z_1=0<\beta_1<y_1<\alpha_1<z_2<\beta_2<y_2<\cdots<y_n<\alpha_n<z_{n+1}<\beta_{n+1}<y_{n+1}<\cdots$.*

(iii) $\lim_{a\to 0^+} y_n(a)=\lambda_n$, $\lim_{a\to 0^+} z_n(a)=\gamma_n$, *and* $\lim_{a\to\pi^-} y_n(a)=\infty$, *for $n=1,2,\ldots$, moreover,*

$$\frac{dy_n}{da}>0, \qquad \frac{dz_n}{da}>0, \qquad \text{for} \quad n=1,2,\ldots.$$

We introduce the following Liapunov function:

$$V(x) = (1 - \cos v(x; a)) + \frac{1}{2} \frac{(v'(x; a))^2}{x}. \tag{1.5}$$

It is easy to verify that

$$V'(x) = -\frac{1}{2} \left[\frac{v'(x; a)}{x} \right]^2 \leqslant 0, \qquad \text{for all} \quad x \geqslant 0. \tag{1.6}$$

Then we have

$$1 - \cos v(0) > 1 - \cos v(z_1) > 1 - \cos v(z_2) > \cdots, \tag{1.7}$$

and it follows that $|v(x; a)| \leqslant a$ for all $x \geqslant 0$. That is, $\{|v(z_n(a); a)|\}$ is a monotone decreasing sequence; moreover from [3] we have

THEOREM 1.2. *Given $a \in (0, \pi)$, we have*

(i) *$v(z_{2n}(a); a)$ monotonically increases to zero as $n \to \infty$;*

(ii) *$v(z_{2n+1}(a); a)$ monotonically decreases to zero as $n \to \infty$.*

Consequently Theorem 1.2 says that for any given a, $0 < a < \pi$, the solution $v(x; a)$ satisfies $\lim_{x \to \infty} v(x; a) = 0$.

2. MAIN RESULTS

In this section we illustrate the singular behavior of the solution $v(x; a)$ as $a \to \pi^-$.

LEMMA 2.1. *Let $h(a) = v^2(z_n(a); a)$. Then $h(a)$ is strictly increasing on $(0, \pi)$.*

Proof. We have

$$\begin{aligned} h'(a) &= 2v(z_n(a); a) \left[v'(z_n(a); a) \frac{dz_n}{da} + \Delta(z_n(a); a) \right] \\ &= 2v(z_n(a); a)\, \Delta(z_n(a); a). \end{aligned}$$

From Theorem 1.1, it is easy to verify $h'(a) > 0$ for any $0 < a < \pi$.

In the following, we state and prove our main result.

THEOREM 2.2. *For each $n = 1, 2, \ldots$, we have*

$$\lim_{a \to \pi} v(z_n(a); a) = (-1)^{n+1} \pi, \qquad \textit{for all} \quad n = 1, 2, 3, \ldots.$$

Proof. Let $\varepsilon = \pi - a$, denote $v(x;\varepsilon)$ to be the solution of the initial value problem

$$\begin{aligned} &v''(x) + x \sin v = 0 \\ &v'(0) = 0 \\ &v(0) = \pi - \varepsilon \qquad 0 < \varepsilon < \pi, \end{aligned} \tag{2.1}$$

and let $y_n(\varepsilon)$, $z_n(\varepsilon)$ be the nth zero of $v(x;\varepsilon)$ and $v'(x;\varepsilon)$, respectively, for $n = 1, 2, 3, \ldots$. It is equivalent to show $\lim_{\varepsilon \to 0} v(z_n(\varepsilon);\varepsilon) = (-1)^{n+1}\pi$, and we prove it by mathematical induction. We have $v(z_1(a); a) = a$, and Theorem 2.2 holds trivially for $n = 1$.

Step 1. For $n=2$ we prove $\lim_{\varepsilon\to 0} v(z_2(\varepsilon);\varepsilon) = -\pi$ or $\lim_{a\to\pi} v(z_2(a); a) = -\pi$. Multiplying by $v'(x)$ on both sides of (2.1) and integrating the resulting identity from a to b yields

$$\frac{1}{2}(v'(b;\varepsilon))^2 - \frac{1}{2}(v'(a;\varepsilon))^2 = b\cos v(b;\varepsilon) - a\cos v(a;\varepsilon) - \int_a^b \cos v(x;\varepsilon)\,dx. \tag{2.2}$$

For any $\delta > 0$ and sufficiently small $\varepsilon > 0$ with $\delta > \varepsilon > 0$, we define $y(\varepsilon, \delta)$ and $y^*(\varepsilon, \delta)$ to be the first real numbers satisfying $v(y(\varepsilon, \delta);\varepsilon) = \pi - \delta$ and $v(y^*(\varepsilon,\delta);\varepsilon) = \pi - \delta/2$, respectively. Obvious $y^*(\varepsilon,\delta) < y(\varepsilon,\delta)$. Since $v(x;\varepsilon)|_{\varepsilon=0} \equiv \pi$ for all $x \geqslant 0$, and from the continuous dependence on initial data, we have

$$\lim_{\varepsilon\to 0} y^*(\varepsilon;\delta) = +\infty. \tag{2.3}$$

Setting $a = 0$, $b = y(\varepsilon;\delta)$ in (2.2) yields

$$\begin{aligned} \frac{1}{2}[v'(y(\varepsilon;\delta);\varepsilon)]^2 &= y(\varepsilon,\delta)\cos v(y(\varepsilon;\delta);\varepsilon) - \int_0^{y(\varepsilon;\delta)} \cos v(x;\varepsilon)\,dx \\ &= \int_0^{y(\varepsilon;\delta)} [\cos v(y(\varepsilon;\delta);\varepsilon) - \cos v(x;\varepsilon)]\,dx \\ &> \int_0^{y^*(\varepsilon;\delta)} [\cos(\pi-\delta) - \cos v(x;\varepsilon)]\,dx \\ &\geqslant y^*(\varepsilon;\delta)[\cos(\pi-\delta) - \cos(\pi-\delta/2)]. \end{aligned}$$

We have $[\cos(\pi-\delta) - \cos(\pi-\delta/2)] > 0$ and from (2.3)

$$\lim_{\varepsilon\to 0} v'(y(\varepsilon;\delta);\varepsilon) = -\infty. \tag{2.4}$$

Consequently that $v''(x) = -x \sin v < 0$ on $(0, y_1(\varepsilon))$ implies that

$$\lim_{\varepsilon \to 0} v'(y_1(\varepsilon); \varepsilon) = -\infty. \tag{2.5}$$

from the identity

$$\int_{y(\varepsilon;\delta)}^{y_1(\varepsilon)} v'(x;\varepsilon)\,dx = -(\pi - \delta),$$

and the concavity of v on $(y(\varepsilon;\delta), y_1(\varepsilon))$, we have

$$\frac{\delta - \pi}{v'(y_1(\varepsilon);\varepsilon)} < y_1(\varepsilon) - y(\varepsilon;\delta) < \frac{\delta - \pi}{v'(y(\varepsilon;\delta);\varepsilon)}. \tag{2.6}$$

Then from (2.4), (2.5), and (2.6) it follows that

$$\lim_{\varepsilon \to 0} y_1(\varepsilon) - y(\varepsilon;\delta) = 0. \tag{2.7}$$

In the following, we establish that

$$\lim_{\varepsilon \to 0} \frac{[v'(y_1(\varepsilon);\varepsilon)]^2}{y_1(\varepsilon)} = 4. \tag{2.8}$$

Setting $a = 0$, $b = y_1(\varepsilon)$ in (2.2) yields

$$\begin{aligned}\frac{1}{2}(v'(y_1(\varepsilon);\varepsilon))^2 &= y_1(\varepsilon) - \int_0^{y_1(\varepsilon)} \cos v(x;\varepsilon)\,dx \\ &= y_1(\varepsilon) - \int_0^{y(\varepsilon;\delta)} \cos v(x;\varepsilon)\,dx - \int_{y(\varepsilon;\delta)}^{y_1(\varepsilon)} \cos v(x;\varepsilon)\,dx.\end{aligned}$$

It is easy to verify that

$$\begin{aligned}&y_1(\varepsilon) - y(\varepsilon;\delta)\cos(\pi - \delta) - \int_{y(\varepsilon;\delta)}^{y_1(\varepsilon)} \cos v(x;\varepsilon)\,dx \\ &\leqslant \frac{1}{2}(v'(y_1(\varepsilon);\varepsilon))^2 \\ &\leqslant y_1(\varepsilon) - y(\varepsilon;\delta)\cos(\pi - \varepsilon) - \int_{y(\varepsilon;\delta)}^{y_1(\varepsilon)} \cos v(x;\varepsilon)\,dx.\end{aligned}$$

From (2.7) and let $\varepsilon \to 0$ in the above inequality, we have

$$1 - \cos(\pi - \delta) \leqslant \lim_{\varepsilon \to 0} \frac{(v'(y_1(\varepsilon);\varepsilon))^2}{2y_1(\varepsilon)} \leqslant 1 + 1 = 2. \tag{2.9}$$

Since $\delta > 0$ is arbitrary, (2.8) follows directly from (2.9). We note that from (1.5) and (1.6), we have

$$V(0) > V(y(\varepsilon;\delta)) > V(y_1(\varepsilon))$$

or

$$\begin{aligned} 1-\cos(\pi-\varepsilon) &\geqslant [1-\cos(\pi-\delta)] + \frac{1}{2}\frac{(v'(y(\varepsilon;\delta);\varepsilon))^2}{y(\varepsilon;\delta)} \\ &\geqslant \frac{(v'(y_1(\varepsilon);\varepsilon))^2}{2y_1(\varepsilon)}. \end{aligned} \tag{2.10}$$

From (2.8), (2.10) we obtain

$$\lim_{\varepsilon\to 0} \frac{[v'(y(\varepsilon;\delta);\varepsilon)]^2}{y(\varepsilon;\delta)} = 2(1+\cos(\pi-\delta)). \tag{2.11}$$

Then from (2.7), (2.8), and (2.9) we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{(v'(y_1(\varepsilon);\varepsilon))^2}{(v'(y(\varepsilon;\delta);\varepsilon))^2} &= \lim_{\varepsilon\to 0} \frac{(v'(y_1(\varepsilon);\varepsilon))^2/y_1(\varepsilon)}{(v'(y(\varepsilon;\delta);\varepsilon))^2/y(\varepsilon;\delta)}\,\frac{y_1(\varepsilon)}{y(\varepsilon;\delta)} \\ &= \frac{2}{1-\cos(\pi-\delta)}. \end{aligned}$$

Hence there exists a constant $\hat{C}_1 = \hat{C}_1(\delta) > 0$, such that for $\varepsilon > 0$ sufficiently small

$$|v'(y_1(\varepsilon);\varepsilon)| < \hat{C}_1\,|v'(y_1(\varepsilon;\delta);\varepsilon)|. \tag{2.12}$$

From (2.6) and (2.12), we have

$$0 < y_1(\varepsilon) - y(\varepsilon;\delta) < \frac{\hat{C}_1(\pi-\delta)}{|v'(y_1(\varepsilon);\varepsilon)|} = \frac{C}{|v'(y_1(\varepsilon);\varepsilon)|}, \tag{2.13}$$

where $C = C(\delta) = \hat{C}_1(\pi-\delta) > 0$ independent of ε.

Now we are in a position to show that $\lim_{\varepsilon\to 0} v(z_2(\varepsilon);\varepsilon) = -\pi$. Suppose this does not hold, then there exists $\delta^* > 0$ such that $v(z_2(\varepsilon);\varepsilon) > -\pi + 2\delta^*$ for all $\varepsilon > 0$. Let $u(x;\varepsilon) = v(x + y_1(\varepsilon);\varepsilon)$ and $w(x;\varepsilon) = -v(y_1(\varepsilon) - x;\varepsilon)$, then $u(x;\varepsilon)$ and $w(x;\varepsilon)$ satisfy the following:

$$\begin{aligned} &u''(x) + y_1(\varepsilon)\sin u = -x\sin u, \\ &u(0;\varepsilon) = 0, \qquad u'(0;\varepsilon) = v'(y_1(\varepsilon);\varepsilon), \qquad \text{for all} \quad x \geqslant 0, \end{aligned} \tag{2.14*}$$

and

$$w''(x) + y_1(\varepsilon) \sin w = x \sin w, \qquad w(0; \varepsilon) = 0, \qquad w'(0; \varepsilon) = v'(y_1(\varepsilon); \varepsilon), \qquad \text{for all} \quad x \geqslant 0. \tag{2.15}^*$$

Let $\eta = xA(\varepsilon)$, where $A(\varepsilon) = -v'(y_1(\varepsilon); \varepsilon)$, $u(x) = u(\eta/A(\varepsilon)) = \phi(\eta)$ and $w(x) = w(\eta/A(\varepsilon)) = \psi(\eta)$. Then $\phi(\eta)$ and $\psi(\eta)$ satisfy the following:

$$\phi''(\eta) + y_1(\varepsilon) A(\varepsilon)^{-2} \sin \phi = -\eta A(\varepsilon)^{-3} \sin \phi, \qquad \eta \geqslant 0, \qquad \phi(0; \varepsilon) = 0, \qquad \phi'(0; \varepsilon) = -1. \tag{2.14}$$

and

$$\psi''(\eta) + y_1(\varepsilon) A(\varepsilon)^{-2} \sin \psi = -\eta A(\varepsilon)^{-3} \sin \psi, \qquad \eta \geqslant 0, \qquad \psi(0; \varepsilon) = 0, \qquad \psi'(0; \varepsilon) = -1. \tag{2.15}$$

If we choose $\delta = \delta^* > 0$, then from (2.13) there exists a constant $C = C(\delta^*)$ independent of ε, such that $0 < y_1(\varepsilon) - y(\varepsilon; \delta^*) < CA(\varepsilon)^{-1}$, provided $\varepsilon > 0$ is sufficiently small. Choose $M > C$; from (2.5), (2.8), and the continuous dependence on parameter ε, it follows that for all $\varepsilon > 0$ sufficiently small

$$|\phi(\eta) - \psi(\eta)| \leqslant \delta^*/4, \qquad \text{for} \quad 0 \leqslant \eta \leqslant M.$$

In particular, let $\eta = (y_1(\varepsilon) - y(\varepsilon, \delta^*))\, A(\varepsilon)$; then

$$w(y_1(\varepsilon) - y(\varepsilon; \delta^*)) + \delta^*/4 > u(y_1(\varepsilon) - y(\varepsilon; \delta^*)) > -\pi + 2\delta^*$$

or

$$-(\pi - \delta^*) + \delta^*/4 > -\pi + 2\delta^*.$$

This is a desired contradiction and we complete the proof for the case $n = 2$.

Step 2. We now assume inductively that

$$\lim_{\varepsilon \to 0} v(z_k(\varepsilon); \varepsilon) = (-1)^{k+1}\pi \tag{2.16}$$

and

$$\lim_{\varepsilon \to 0} \frac{(v'(y_{k-1}(\varepsilon); \varepsilon))^2}{y_{k-1}(\varepsilon)} = 4, \tag{2.17}$$

for all $k = 3, 4, 5, ..., n-1$. We show that (2.16), (2.17) hold for $k = n$. For simplicity, we may assume that n is an odd number. We have that (2.16) holds for $k = 2, 3, 4, \ldots, n-1$. Given any $\delta > 0$, there exists $y_1^r(\varepsilon;\delta)$, $y_k^l(\varepsilon;\delta)$ and $y_k^r(\varepsilon,\delta)$, satisfying

$$0 < y_1^r(\varepsilon;\delta) < y_1(\varepsilon), \qquad \text{with} \quad v(y_1(\varepsilon;\delta);\varepsilon) = \pi - \delta;$$

$$y_{k-1}(\varepsilon) < y_k^l(\varepsilon;\delta) < y_k^r(\varepsilon;\delta) < y_k(\varepsilon),$$

$$\text{with} \quad v(y_k^l(\varepsilon;\delta);\varepsilon) = v(y_k^r(\varepsilon;\delta);\varepsilon) = (-1)^{k+1}(\pi-\delta)$$

for $k = 2, 3, \ldots, n-1$, provided ε is sufficiently small, (see Fig. 1).

We claim that

$$\lim_{\varepsilon \to 0} y_k^l(\varepsilon;\delta) - y_{k-1}(\varepsilon) = 0, \qquad \text{for} \quad k = 2, 3, \ldots, n-1, \tag{2.18}$$

$$\lim_{\varepsilon \to 0} y_k(\varepsilon) - y_k^r(\varepsilon;\delta) = 0, \qquad k = 1, 2, 3, \ldots, n-1. \tag{2.19}$$

Let $V(x) = (1 - \cos v(x;\varepsilon)) + (v'(x;\varepsilon))^2/2x$. Then from the fact that $V'(x) \leqslant 0$, we have $V(y_{k-1}(\varepsilon)) > V(y_k^l(\varepsilon;\delta)) > V(z_k(\varepsilon))$, for $k = 2, 3, \ldots, n-1$. Consequently from (2.16), (2.17), we have $\lim_{\varepsilon \to 0} V(y_k^l(\varepsilon;\delta)) = 2$, or

$$\lim_{\varepsilon \to 0} \frac{(v'(y_k^l(\varepsilon;\delta);\varepsilon))^2}{y_k^l(\varepsilon;\delta)} = 2(1 + \cos(\delta - \pi)). \tag{2.20}$$

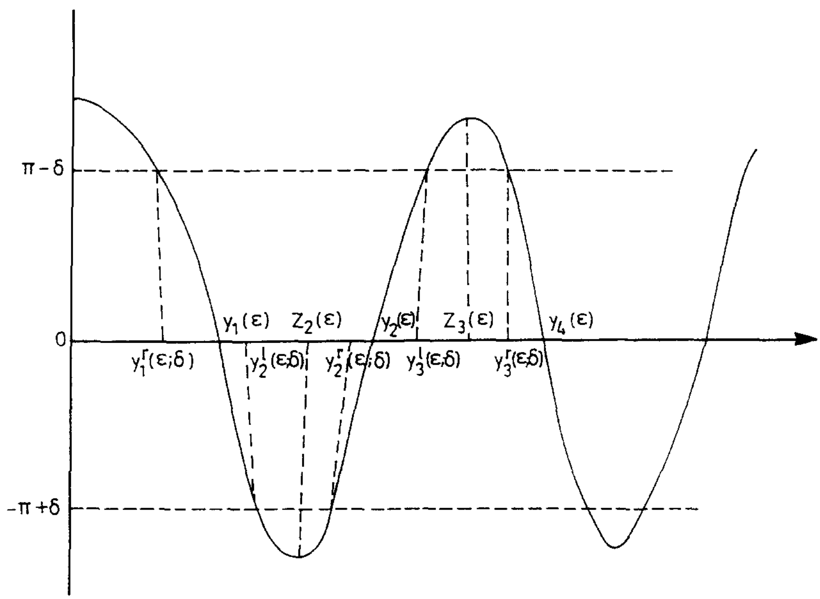

FIG. 1. The graph for the solution $v(x, \varepsilon)$ of (2.1).

From (2.20) we obtain

$$\lim_{\varepsilon \to 0} |v'(y_k^l(\varepsilon;\delta);\varepsilon)| = \infty. \tag{2.21}$$

Consider the following identity:

$$\int_{y_{k-1}(\varepsilon)}^{y_k^l(\varepsilon;\delta)} v'(x;\varepsilon)\,dx = (-1)^{k+1}(\pi-\delta).$$

Then we have

$$v'(y_{k-1}(\varepsilon))(y_k^l(\varepsilon;\delta)-y_{k-1}(\varepsilon)) < \delta-\pi < v'(y_k^l(\varepsilon;\delta))(y_k^l(\varepsilon;\delta)-y_{k-1}(\varepsilon))$$
$$\text{when } k \text{ is even}$$

or

$$v'(y_k^l(\varepsilon;\delta))(y_k^l(\varepsilon;\delta)-y_{k-1}(\varepsilon)) < \pi-\delta < v'(y_{k-1}(\varepsilon))(y_k^l(\varepsilon;\delta)-y_{k-1}(\varepsilon))$$
$$\text{when } k \text{ is odd.}$$

In both cases we have

$$0 < y_k^l(\varepsilon;\delta)-y_{k-1}(\varepsilon) < \frac{\pi-\delta}{|v'(y_k^l(\varepsilon;\delta))|}, \tag{2.22}$$

Hence (2.18) follows directly from (2.21), (2.22). Similarly if we prove

$$\lim_{\varepsilon \to 0} |v'(y_k^r(\varepsilon;\delta);\varepsilon)| = \infty, \tag{2.23}$$

then (2.19) holds.

Setting $a = y_k^l(\varepsilon;\delta)$, $b = y_k^r(\varepsilon;\delta)$ in (2.2) yields

$$\frac{1}{2}(v'(y_k^r(\varepsilon;\delta);\varepsilon))^2 - \frac{1}{2}(v'(y_k^l(\varepsilon;\delta);\varepsilon))^2$$
$$= (y_k^r(\varepsilon;\delta)-y_k^l(\varepsilon;\delta))\cos(\pi-\delta) - \int_{y_k^l(\varepsilon;\delta)}^{y_k^r(\varepsilon;\delta)} \cos v(x;\varepsilon)\,dx \geqslant 0.$$

That is,

$$|v'(y_k^r(\varepsilon;\delta);\varepsilon)| \geqslant |v'(y_k^l(\varepsilon;\delta);\delta)|.$$

Then (2.23) follows directly from (2.21).

We are now in a position to show that (2.17) holds for $k=n$. We set $a=0$, $b=y_{n-1}(\varepsilon)$ in (2.2) to obtain

$$\begin{aligned}\frac{1}{2}(v'(y_{n-1}(\varepsilon);\varepsilon))^2 &= y_{n-1}(\varepsilon)-\int_0^{y_{n-1}(\varepsilon)}\cos v(x;\varepsilon)\,dx\\ &= y_{n-1}(\varepsilon)-\int_0^{y_1^r(\varepsilon;\delta)}\cos v(x;\varepsilon)\,dx\\ &\quad -\sum_{k=2}^{n-1}\int_{y_k^l(\varepsilon;\delta)}^{y_k^r(\varepsilon;\delta)}\cos v(x;\varepsilon)\,dx\\ &\quad -\sum_{k=2}^{n-1}\int_{y_{k-1}^r(\varepsilon;\delta)}^{y_k^l(\varepsilon;\delta)}\cos v(x;\varepsilon)\,dx-\int_{y_{n-1}^r(\varepsilon;\delta)}^{y_{n-1}(\varepsilon)}\cos v(x;\varepsilon)\,dx.\end{aligned}$$

It is easy to verify the following inequality

$$\begin{aligned}&1-\frac{y_1^r(\varepsilon;\delta)}{y_{n-1}(\varepsilon)}\cos(\pi-\delta)-\frac{\cos(\pi-\delta)}{y_{n-1}(\varepsilon)}\sum_{k=2}^{n-1}(y_k^r(\varepsilon;\delta)-y_k^l(\varepsilon;\delta))\\ &\quad-\frac{1}{y_{n-1}(\varepsilon)}\sum_{k=2}^{n-1}(y_k^l(\varepsilon;\delta)-y_{k-1}^r(\varepsilon;\delta))-\frac{1}{y_{n-1}(\varepsilon)}(y_{n-1}(\varepsilon)-y_{n-1}^r(\varepsilon;\delta))\\ &\leqslant\frac{1}{2}\frac{(v'(y_{n-1}(\varepsilon);\varepsilon))^2}{y_{n-1}(\varepsilon)}\\ &\leqslant 1+\frac{y_1^r(\varepsilon;\delta)}{y_{n-1}(\varepsilon)}+\frac{1}{y_{n-1}(\varepsilon)}\sum_{k=2}^{n-1}(y_k^r(\varepsilon;\delta)-y_k^l(\varepsilon;\delta))\\ &\quad+\frac{1}{y_{n-1}(\varepsilon)}\sum_{k=2}^{n-1}(y_k^l(\varepsilon;\delta)-y_{k-1}^r(\varepsilon;\delta))+\frac{1}{y_{n-1}(\varepsilon)}(y_{n-1}(\varepsilon)-y_{n-1}^r(\varepsilon;\delta)).\end{aligned}$$

Since $\lim_{\varepsilon\to 0} y_{n-1}(\varepsilon)=\infty$ and $\delta>0$ is arbitrary, (2.18) and (2.19) imply

$$\lim_{\varepsilon\to 0}\frac{(v'(y_{n-1}(\varepsilon)))^2}{y_{n-1}(\varepsilon)}=4. \tag{2.24}$$

Hence we establish (2.17) for $k=n$.

Using the same argument as we did in Step 1 yields

$$0<y_{n-1}(\varepsilon)-y_{n-1}(\varepsilon;\delta)<\frac{C}{|v'(y_{n-1}(\varepsilon))|} \tag{2.25}$$

for some $C=C(\delta)>0$ and for all $\varepsilon>0$ sufficiently small. Since n is odd, we show that

$$\lim_{\varepsilon\to 0} v(z_n(\varepsilon);\varepsilon)=\pi. \tag{2.26}$$

Suppose (2.26) does not hold. Then there exists a $\delta^* > 0$, such that

$$v(z_n(\varepsilon); \varepsilon) < \pi - 2\delta^* \qquad \text{for all} \quad \varepsilon > 0. \tag{2.27}$$

Let $u(x; \varepsilon) = v(x + y_{n-1}(\varepsilon); \varepsilon)$, $w(x; \varepsilon) = -v(y_{n-1}(\varepsilon) - x; \varepsilon)$. From (2.24), (2.17) and the arguments for the case $n = 2$, we obtain

$$\begin{aligned}\pi - 2\delta^* &> u(y_{n-1}(\varepsilon) - y^{\mathrm{r}}_{n-1}(\varepsilon; \delta^*); \varepsilon) \\ &> w(y_{n-1}(\varepsilon) - y^{\mathrm{r}}_{n-1}(\varepsilon; \delta^*); \varepsilon) - \delta^*/4 \\ &= -(-\pi + \delta^*) - \delta^*/4 \\ &= \pi - 5\delta^*/4.\end{aligned}$$

This is a desired contradiction. Thus we complete the proof of Theorem 2.2.

3. The Application

In [1, 2] the authors discussed a mathematical model describing the deformation of a cantilever by its own weight. It is assumed that a cantilever of uniform cross-section, uniform density ρ, and total length L is held fixed at an angle α at one end, say the origin, and is free at the other end. Let s' be the arc length from the origin, and $\theta = \theta(s')$ be the local angle of inclination. Then we have the governing equation

$$\begin{gathered}\mathrm{EI}\, \frac{d^2\theta}{ds'^2} = \rho(L - s') \sin \theta, \\ \theta(0) = \alpha, \qquad \frac{d\theta}{ds'}(L) = 0,\end{gathered} \tag{3.1}$$

where EI is the flexural rigidity of the material. Let $s = s'/L$, then the governing equation becomes

$$\begin{gathered}\frac{d^2\theta}{ds^2} = K^3(1 - s) \sin \psi, \qquad 0 \leqslant s \leqslant 1,\ K > 0, \\ \theta'(1) = 0, \qquad \theta(0) = \alpha, \qquad -\pi \leqslant \alpha \leqslant \pi,\end{gathered} \tag{P$_\alpha$}$$

where $K = (\rho L^3/\mathrm{EI})^{1/3}$ represents the importance of density and length relative to that of flexural rigidity. Let $s = x$, $v(x) = \theta(1 - x/K) - \pi$; then we reformulate our equation as the following:

$$\begin{gathered}v''(x) + x \sin v(x) = 0, \qquad ' = d/dx, \\ v'(0) = 0, \qquad v(K) = \alpha - \pi.\end{gathered} \tag{3.2}$$

The vertical case $\alpha = \pi$ was completely analyzed in [2].

We note that from (1.2) we have $v(K; a) = \pi - \alpha$ if and only if $v(K; -a) = \alpha - \pi$. For simplicity, instead of (3.2), we study the multiplicities of the solutions of the following boundary value problem:

$$v''(x) + x \sin v = 0$$
$$v'(0) = 0, \qquad v(K) = \pi - \alpha, \qquad \text{for} \quad 0 < \alpha < \pi. \tag{3.3}$$

To solve (3.3) by the shooting method, we consider the following initial value problem:

$$v''(x) + x \sin v = 0,$$
$$v'(0) = 0, \qquad v(0) = a, \qquad \text{for} \quad -\pi < a < \pi. \tag{3.4}$$

THEOREM 3.1. *Given* $\alpha \in (0, \pi)$,

(i) *For each* $n = 0, 1, 2, \ldots$, *there exists a unique* $a_{2n+1} = a_{2n+1}(\alpha)$, $a_{2n+1} \in (\pi - \alpha, \pi)$, *satisfying* $v(z_{2n+1}(a_{2n+1}); a_{2n+1}) = \pi - \alpha$; *moreover*, $a_1 = \pi - \alpha < a_3 < a_5 < \cdots < a_{2n+1} < \cdots < \pi$, *and* $\lim_{n \to \infty} a_{2n+1} = o$.

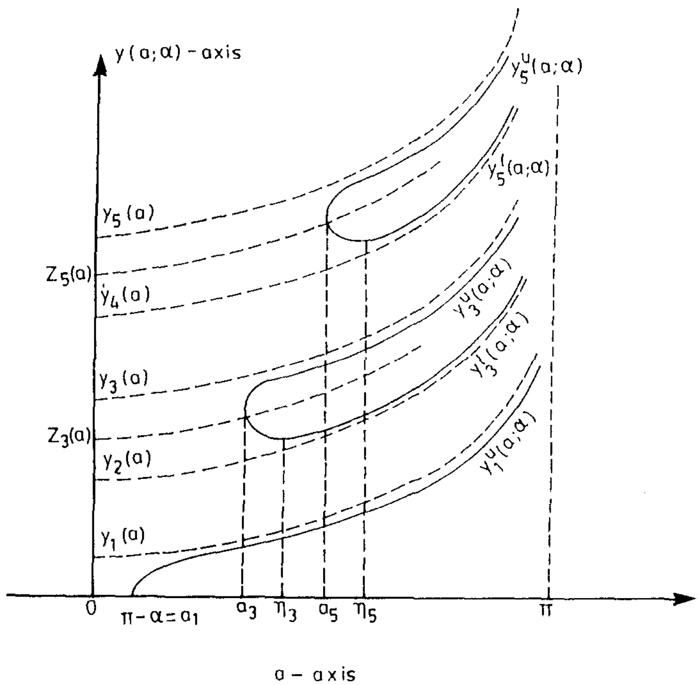

FIG. 2. The graph of the functions $Y^u_{2n+1}(a)$ and $Y^l_{2n+1}(a)$ for fixed $0 < a < \pi$, $n = 0, 1, 2, \ldots$.

(ii) *Given any* $n \geqslant 1$, *for each* $a \in (a_{2n-1}, a_{2n+1})$, *the equation* $v(x;a) = \pi - \alpha$, *has exactly* $2n-1$ *isolated zeros* $\{y_1^u, y_{2m+1}^l, y_{2m+1}^u\}_{m=1}^{n-1}$, *where* $y_1^u = y_1^u(a;\alpha)$, $y_{2m+1}^l = y_{2m+1}^l(a;\alpha)$, $y_{2m+1}^u = y_{2m+1}^u(a;\alpha)$, *satisfying* $0 = z_1(a) < y_1^u < y_3^l < z_3(a) < y_3^u < \cdots < y_{2n-1}^l < z_{2n-1}(a) < y_{2n-1}^u$; *moreover, for* $a = a_{2n-1}$, *we have* $y_{2n-1}^l(a_{2n-1}, \alpha) = y_{2n-1}^u(a_{2n-1}, \alpha) = z_{2n-1}(a_{2n-1})$.

(iii) *For each* $n = 1, 2, ...,$ *as a function of* a, $y_{2n+1}^l(a)$ *attains global minimum at a point* $\eta_{2n+1} \in (\pi - \alpha, \pi)$, $a_{2n+1} < \eta_{2n+1}$, *satisfying* $y_{2n+1}^l(\eta_{2n+1}) = \alpha_{2n}(\eta_{2n+1})$, *where* $\alpha_{2n}(a)$ *is the* $2n$*th zero of* $\Delta(x;a)$, *and* $\lim_{a \to \pi^-} y_{2n+1}^l(a) = +\infty$. *On the other hand,* $y_{2n+1}^u(a)$ *is strictly increasing on* $[a_{2n+1}, \pi)$ *and* $\lim_{a \to \pi^-} y_{2n+1}^u(a) = +\infty$. *(See Fig. 2.)*

For analogous results, we have

(i)* *For each* $n = 1, 2, ...,$ *there exists a unique* $a_{2n} = a_{2n}(\alpha)$, $a_{2n} \in (-\pi, -\pi + \alpha)$, *satisfying* $v(z_{2n}(a_{2n}); a_{2n}) = \pi - \alpha$. *Moreover,* $0 > a_2 > a_4 > \cdots > a_{2n} > \cdots > -\pi$, *and* $\lim_{n \to \infty} a_{2n} = -\pi$.

(ii)* *Given any* $n \geqslant 1$, *for each* $a \in (a_{2n}, a_{2n+2})$, *the equation* $v(x;a) = \pi - \alpha$ *has exactly* $2n$ *isolated zeros* $\{y_{2m}^l, y_{2m}^u\}_{m=1}^{n}$, *where* $y_{2m}^l = y_{2m}^l(a;\alpha)$,

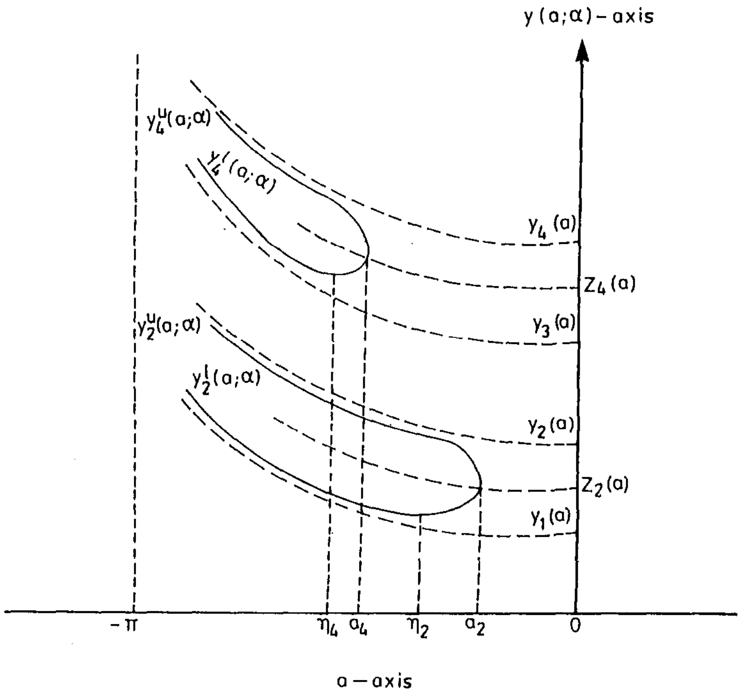

FIG. 3. The graph of the functions $Y_{2n}^u(a)$ and $Y_{2n}^l(a)$ for fixed $0 < \alpha < \pi$, $n = 1, 2, ...$.

$y^u_{2m} = y^u_{2m}(a;\alpha)$, *satisfying* $y^l_2 < z_2(a) < y^u_2 < \cdots < y^l_{2n} < z_{2n}(a) < y^u_{2n}$, *moreover, for* $a = a_{2n}$, *we have* $y^l_{2n}(a_{2n}, \alpha) = y^u_{2n}(a_{2n}, \alpha) = z_{2n}(a_{2n})$.

(iii)* *For each* $n = 1, 2, \ldots, y^l_{2n}(a)$, *defined on* $(-\pi, a_{2n}]$, *attains a global minimum at a point* $\eta_{2n} \in (-\pi, \alpha - \pi)$ *with* $a_{2n} > \eta_{2n}$, *satisfying* $y^l_{2n}(\eta_{2n}) = \alpha_{2n-1}(\eta_{2n})$, *where* $\alpha_{2n-1}(a)$ *is the* $(2n-1)$*th zero of* $\Delta(x;a)$ *and* $\lim_{a \to -\pi^+} y^l_{2n}(a) = +\infty$. *On the other hand,* $y^u_{2n}(a)$ *is strictly increasing on* $(-\pi, a_{2n}]$ *and* $\lim_{a \to -\pi^+} y^u_{2n}(a) = +\infty$. *(See Fig.* 3.)

Proof. From Lemma 2.1 and Theorem 2.2, it is easy to show that there exists a unique a_{2n+1}, depending on α, $a_{2n+1} \in (\pi - \alpha, \pi)$, satisfying $v(z_{2n+1}(a_{2n+1}); a_{2n+1}) = \pi - \alpha$, and $a_1 = \pi - \alpha < a_3 < a_5 < \cdots < a_{2n+1} < \pi$. We claim $\lim_{n \to \infty} a_{2n+1} = \pi$. If not, then $\lim_{n \to \infty} a_{2n+1} = \pi - \delta_0$ for some $\delta_0 > 0$. Choose $a = \pi - \delta_0/2$. Then from Lemma 3.2 for any $n = 1, 2, \ldots$, $v(z_{2n+1}(a); a) > v(z_{2n+1}(a_{2n+1}); a_{2n+1}) = \pi - \alpha$. This is a desired contradiction to Lemma 3.1. Thus we complete the proof for part (i). Part (ii) follows directly from Lemma 3.2 and the oscillatory behavior of the solution $v(x;a)$ and (3.3). We have the relation

$$v(y^l_{2n+1}(a;\alpha); a) = \pi - \alpha. \tag{3.5}$$

Differentiating (3.5) with respect to a yields

$$\frac{dy^l_{2n+1}(a;\alpha)}{da} = \frac{-\Delta(y^l_{2n+1}(a;\alpha); a)}{v'(y^l_{2n+1}(a;\alpha); a)}. \tag{3.6}$$

Since $y^l_{2n+1}(a_{2n+1}, \alpha) = z_{2n+1}(a_{2n+1})$, we have

$$\left.\frac{dy^l_{2n+1}(a;\alpha)}{da}\right|_{a = a_{2n+1}} = -\infty.$$

However, $y_{2n}(a) \leqslant y^l_{2n+1}(a, \alpha)$ and $\lim_{a \to \pi} y_{2n}(a) = +\infty$; this shows $\lim_{a \to \pi} y^l_{2n+1}(a) = +\infty$ and the existence of a global minimum η_{2n+1} of $y^l_{2n+1}(a, \alpha)$. From (3.6) we have

$$0 = \frac{dy^l_{2n+1}(\eta_{2n+1};\alpha)}{da} = -\frac{\Delta(y^l_{2n+1}(\eta_{2n+1};\alpha); \eta_{2n+1})}{v'(y^l_{2n+1}(\eta_{2n+1};\alpha); \eta_{2n+1})}, \tag{3.7}$$

and $y^l_{2n+1}(\eta_{2n+1};\alpha) = \alpha_{2n}(\eta_{2n+1})$ follows directly from (3.7). On the other hand, we have

$$\frac{dy^u_{2n+1}(a;\alpha)}{da} = \frac{-\Delta(y^u_{2n+1}(a;\alpha); a)}{v'(y^u_{2n+1}(a;\alpha); a)}.$$

From the relation $\alpha_{2n}(a) < z_{2n+1}(a) < y^u_{2n+1}(a) < y_{2n+1}(a)$, $n = 1, 2, \ldots$, it follows that $dy^{u}_{2n+1}/da > 0$ for all $a \in (\alpha_{2n+1}, \pi)$. The analogous results for (i)*, (ii)*, and (iii)* can be proved similarly.

Remark 1. For each $n = 2, 3, \ldots$, if every extremum of the function $y^{1}_{n}(a; \alpha)$ is a local minimum, then η_n is the unique local minimum and $y^{1}_{n}(a; \alpha)$ is strictly increasing (decreasing) for $a \geqslant \eta_n$ ($a \leqslant \eta_n$) provide n is odd (even). Differentiating the identity

$$v(y^{1}_{n}(a; \alpha); a) = \pi - \alpha,$$

twice with respect to a and setting $dy^{1}_{n}/da = 0$ yields

$$\frac{d^2 y^{1}_{n}(a, \alpha)}{da^2} = \frac{-(d\Delta/da)(y^{1}_{n}(a, \alpha); a)}{v'(y^{1}(a, \alpha); a)}, \tag{3.8}$$

From Theorem 3.1 (iii) and (iii)*, if $dy^{1}_{n}/da = 0$ then $y^{1}_{n}(a, \alpha) = \alpha_{n-1}(a)$, and (3.8) becomes

$$\frac{d^2 y^{1}_{n}(a, \alpha)}{da^2} = \frac{-(d\Delta/da)(\alpha_{n-1}(a), a)}{v'(\alpha_{n-1}(a), a)}. \tag{3.9}$$

Since $\Delta(\alpha_{n-1}(a), a) = 0$ for all $a \in (0, \pi)$, it follows that

$$\frac{d\alpha_{n-1}}{da} = \frac{-(d\Delta/da)(\alpha_{n-1}(a), a)}{\Delta'(\alpha_{n-1}(a), a)}. \tag{3.10}$$

From (3.9), (3.10), and Theorem 2.1(i), (ii), $d^2 y^{1}_{n}(a, \alpha)/da^2 > 0$ provide $d\alpha_{n-1}/da > 0$, for all $a \in (0, \pi)$, $n = 2, 3, \ldots$. Let $w(x; a) = (d\Delta/da)(x; a)$, then $w(x; a)$ satisfies

$$w''(x) + xw \cos v = x\Delta^2 \sin v, \qquad w(0) = 0, \qquad w'(0) = 0. \tag{3.11}$$

We recall that

$$v''(x) + x \sin v = 0, \qquad v(0) = a, \qquad v'(0) = 0. \tag{3.12}$$

$$\Delta''(x) + x\,\Delta \cos v = 0, \qquad \Delta(0) = 1, \qquad \Delta'(0) = 0. \tag{3.13}$$

We conjecture that the following hold:

(i) For $0 < a < \pi$, $w(x; a)$ and $w'(x; a)$ are oscillatory over $[0, \pi)$ with zeros $p_n = p_n(a)$, $q_n = q_n(a)$, respectively, for $n = 1, 2, \ldots$, where $p_1 = q_1 = 0$.

(ii) For $0 < a < \pi/2$, we have

$$\begin{aligned} 0 = p_1 = q_1 = z_1 = \beta_1 < y_1 < q_2 < \alpha_1 < z_2 < p_2 < \beta_2 < y_2 < q_3 < \cdots \\ < \cdots < y_n < q_{n+1} < \alpha_n < z_{n+1} < p_{n+1} < \beta_{n+1} < y_{n+1} < \cdots. \end{aligned} \tag{3.14}$$

For $\pi/2 \leqslant a < \pi$, we have

$$0 = p_1 = q_1 = z_1 = \beta_0 < \beta_1 < y_1 < q_2 < \alpha_1 < z_2 < p_2 < \beta_2 < y_2 < q_3 < \cdots$$
$$< \cdots < y_n < q_{n+1} < \alpha_n < z_{n+1} < p_{n+1} < \beta_{n+1} < y_{n+1} < \cdots. \tag{3.15}$$

From (3.10), (3.14), and (3.15) it is easy to verify that $\alpha_n(a)$ is strictly increasing on $(0, \pi)$. In Fig. 4 we plot a graph for the functions $K = y_n^{\mathrm{u}}(a; \alpha)$ and $K = y_n^{\mathrm{l}}(a; \alpha)$ for $0 < \alpha < \pi$. From the figure there follow the bifurcation phenomena of problem $(\mathrm{P})_\alpha$, $0 < \alpha < \pi$, or (4.2) as the parameter K varies. It is interesting to note that when $\alpha = 0$ the problem $(\mathrm{P})_0$ has a unique solution [2] for any K while our results show that given any α, $0 < \alpha < \pi$, and any positive integer n, there exists K such that $(\mathrm{P})_\alpha$ has n distinct solutions.

Remark 2. We note that for any $n = 1, 2, \ldots,$

$$\lim_{\alpha \to 0} a_n(\alpha) = (-1)^{n+1}\pi. \tag{3.16}$$

It follows directly from $\pi - \alpha = a_1(\alpha) < a_3(\alpha) < \cdots < a_{2n+1}(\alpha) < \cdots < \pi$, and $-\pi + \alpha > a_2(\alpha) > a_4(\alpha) > \cdots > a_{2n}(\alpha) > \cdots > -\pi$. (3.16) indicates that the bifurcation phenomena will disappear as $\alpha = 0$.

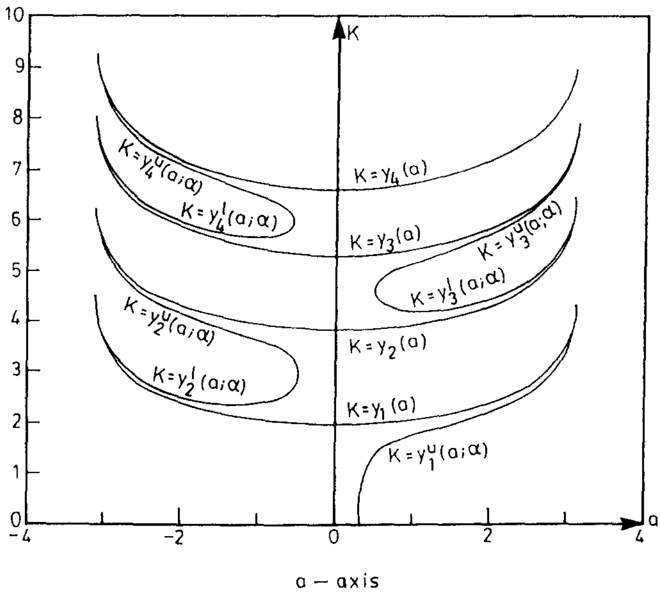

FIG. 4. The graph of the functions $K = Y_n^{\mathrm{u}}(a; \alpha)$ and $K = Y_n^{\mathrm{l}}(a; \alpha)$ for $0 < \alpha < \pi$, which are the bifurcation pictures for the boundary value problem (3.2).

Remark 3. As $\alpha \to \pi$ we have

(i) $\lim_{\alpha \to \pi} a_n(\alpha) = 0$, for all $n = 1, 2, \ldots$.

(ii) $\lim_{\alpha \to \pi} y_n^{\mathrm{l}}(a_n(\alpha); \alpha) = \lim_{\alpha \to \pi} y_n^{\mathrm{u}}(a_n(\alpha); \alpha) = \gamma_n$.

(iii) For arbitrary $\mu > 0$, we have

$$\lim_{\alpha \to \pi} y_n^{\mathrm{l}}(a; \alpha) = y_{n-1}(a) \text{ uniformly for all } a \in [\mu, \pi) \text{ if } n \text{ is odd.}$$

$$\lim_{\alpha \to \pi} y_n^{\mathrm{u}}(a; \alpha) = y_n(a) \text{ uniformly for all } a \in [\mu, \pi) \text{ if } n \text{ is odd.}$$

$$\lim_{\alpha \to \pi} y_n^{\mathrm{l}}(a; \alpha) = y_{n-1}(a) \text{ uniformly for all } a \in (-\pi, -\mu] \text{ if } n \text{ is even.}$$

$$\lim_{\alpha \to \pi} y_n^{\mathrm{u}}(a; \alpha) = y_n(a) \text{ uniformly for all } a \in (-\pi, -\mu] \text{ if } n \text{ is even.}$$

To prove (i) we shall only consider the case n is odd; the argument is similar for the case n is even. We have the relation

$$v(z_n(a_n(\alpha)); a_n(\alpha)) = \pi - \alpha. \tag{3.17}$$

Differentiating (3.17) with respect to α yields

$$v'(z_n; a_n(\alpha)) \frac{dz_n}{da} \frac{da_n(\alpha)}{d\alpha} + \Delta(z_n; a_n(\alpha)) \frac{da_n(\alpha)}{d\alpha} = -1.$$

From Theorem 1.1 $\Delta(z_n; a_n(\alpha))$ is positive for odd n and $v'(z_n; a_n(\alpha)) = 0$, then we have $da_n(\alpha)/d\alpha < 0$. If $\lim_{\alpha \to \pi} a_n(\alpha) \neq 0$, say $A = \lim_{\alpha \to \pi} a_n(\alpha) > 0$, by Lemma 2.1, we have

$$0 < v^2(z_n(A); A) < v^2(z_n(a_n(\alpha)); a_n(\alpha)) = (\pi - \alpha)^2. \tag{3.18}$$

Let $\alpha \to \pi$ in (3.18), then this leads to a contradiction, $v(z_n(A); A) = 0$. Part (ii) follows directly from Theorem 1.1 (iii) and Theorem 3.1 (ii), (ii)*. For part (iii), we consider only the first case, n an odd number. Given $\mu > 0$, from (i) there exists $\delta_1 = \delta_1(\mu) > 0$, such that $a_n(\alpha) < \mu < \pi$, provided $|\pi - \alpha| < \delta_1$. Hence $y_n^{\mathrm{l}}(a; \alpha)$ is well-defined for $a \in [\mu, \pi)$ and $|\pi - \alpha| < \delta_1$. Consider the identity

$$\int_{y_{n-1}(a)}^{y_n^{\mathrm{l}}(a;\alpha)} v'(x; a)\, dx = \pi - \alpha.$$

We have

$$v'(y_n^{\mathrm{l}}(a; \alpha), a)(y_n^{\mathrm{l}}(a; \alpha) - y_{n-1}(a)) < \pi - \alpha$$

or

$$0 < y_n^1(a;\alpha) - y_{n-1}(a) < \frac{\pi - \alpha}{|v'(y_n^1(a;\alpha), a)|}. \tag{3.19}$$

From (2.21) we have

$$\lim_{a \to \pi} v'(y_n^1(a;\alpha), a) = +\infty.$$

Then

$$M = \max_{a \in [\mu, \pi)} \frac{1}{|v'(y_n^1(a;\alpha), a)|} < \infty.$$

Hence given $\mu > 0$ for any $\varepsilon > 0$, choose $\delta = \min\{\varepsilon/M, \delta_1\}$; then

$$|y_n^1(a;\alpha) - y_{n-1}(a)| < \varepsilon, \qquad \text{for all} \quad a \in [\mu, \pi),$$

provided $|\pi - \alpha| < \delta$. Hence the first case of part (iii) holds. By similar arguments it is easy to verify the other cases of part (iii). (See Fig. 4.)

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