Generalized Friedmann-Robertson-Walker metric and redundancy in the generalized Einstein equations

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A nontrivial redundancy relation, due to the differential structure of the gravitational Bianchi identity as well as the symmetry of the Friedmann-Robertson-Walker metric, in the gravitational field equation is clarified. A generalized Friedmann-Robertson-Walker metric is introduced in order to properly define a one-dimensional reduced problem which offers an alternative approach to obtain the gravitational field equations on Friedmann-Robertson-Walker spaces.

I. INTRODUCTION

In this Brief Report we will study some gravitational field equations which appear when considering physics at the cosmological scale.

This Brief Report is organized as follows. In Sec. II we solve a simple gravitational theory with nonvanishing cosmological constant on the Friedmann-Robertson-Walker (FRW) spaces [1–6]. We also describe how redundancy arises among the Einstein equations as a result of the Bianchi identity (BI) [2]. It is also remarked that the exclusive Friedmann equation cannot be considered as a redundant equation because of the differential structure of the gravitational BI [2]. In Sec. III the reduced problem is defined by directly substituting the FRW metric into the gravitational action introduced in Sec. II. A generalized Friedmann-Robertson-Walker (GFRW) metric is thus introduced in order to make the reduced problem a complete theoretical approach. In Sec. IV, the advantage of the independent reduced problem is demonstrated by analyzing a general covariant theory with scalar-metric coupling [7–10]. Finally, in Sec. V, we make some concluding remarks.

The main purpose of this Brief Report is to reduce the labor in dealing with the complicated gravitational system by defining an appropriate reduced problem which is a simple one-dimensional Lagrangian system. We show that the FRW metric needs to be generalized to a GFRW metric such that the exclusive Friedmann equation can be reproduced accordingly.

Note that, throughout this Brief Report, the curvature tensor $R^{\ \ \ \beta}_{\mu\nu\alpha}(g_{\mu\nu})$ is defined by the equation

$$[D_\mu,D_\nu]A_\alpha=R^{\ \ \ \beta}_{\alpha\nu\mu}A_\beta\ , \tag{1}$$

i.e., $R^{\ \ \ \beta}_{\mu\nu\alpha}=-\partial_\alpha\Gamma^\beta_{\mu\nu}-\Gamma^\lambda_{\mu\nu}\Gamma^\beta_{\alpha\lambda}-(\nu\leftrightarrow\alpha)$. Here $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol (or spin connection of the covariant derivative, i.e., $D_\mu A_\nu\equiv\partial_\mu A_\nu-\Gamma^\alpha_{\mu\nu}A_\alpha$). To be more specific, $\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu}+\partial_\nu g_{\beta\mu}-\partial_\beta g_{\mu\nu})$. Also, the Ricci tensor $R_{\mu\nu}$ is defined as

$$R_{\mu\nu}=R^{\ \ \ \alpha}_{\mu\nu\alpha}\ , \tag{2}$$

and the scalar curvature R is defined as the trace of the Ricci tensor, i.e., $R\equiv g^{\mu\nu}R_{\mu\nu}$.

II. FRIEDMANN-ROBERTSON-WALKER METRIC AND REDUNDANCY

Consider the action given by

$$S=\int d^4x\sqrt{g}\,\mathcal{L}=\int d^4x\sqrt{g}\,(-R-2\Lambda)\ , \tag{3}$$

where $\mathcal{L}$ is the scalar Lagrangian. The equation of motion (EOM) for the general covariant theory (3) takes the form

$$G_{\mu\nu}\equiv\tfrac{1}{2}g_{\mu\nu}R-R_{\mu\nu}=-\Lambda g_{\mu\nu}\ . \tag{4}$$

In standard cosmology we are considering a spatially homogeneous and isotropic universe which is indicated by gravitational observations [1] as well as some philosophical considerations. It can thus be shown that all spatially isotropic and homogeneous spaces can be described [2–6] by the well-known FRW spaces. The FRW metric can be read off from the equation

$$ds^2\equiv g_{\mu\nu}dx^\mu dx^\nu=-dt^2+a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2d\Omega\right]\ . \tag{5}$$

Here $d\Omega$ is the solid angle $d\Omega=d\theta^2+\sin^2\theta\,d\chi^2$, and $k=0,\pm1$ stands for a flat, closed, or open universe, respectively.

The explicit form of all nonvanishing Ricci tensor components can be obtained by substituting the FRW metric (5) into the definition of the curvature tensor (2). After some algebra one obtains the following nonvanishing components of $\Gamma^\alpha_{\mu\nu}$:

$$\Gamma^t_{ij}=aa'h_{ij}\ , \tag{6}$$

$$\Gamma^t_{ij}=\frac{a'}{a}\delta^i_j\ , \tag{7}$$

$$\Gamma^i_{jk}(g_{ij})=\Gamma^i_{jk}(h_{ij})\ . \tag{8}$$

Here we have listed all nonvanishing spin-connection components. Also, a prime denotes differentiation with respect to the argument t. We have also written $h_{ij}\equiv g_{ij}/a^2$ (hence $h^{ij}=a^2g^{ij}$ such that $h^{ij}h_{jk}=\delta^i_k$). One can also compute all nonvanishing components of the

Ricci tensor $R_{\mu\nu}$:

$$R_{tt}=3\frac{a''}{a}\ , \tag{9}$$

$$R_{ij}=-(aa''+2a'^2+2k)h_{ij}\ . \tag{10}$$

Note that in deriving (10) we have used the identity $R_{ij}(h_{kl})=-2kh_{ij}$, which follows from the fact that h_{ij} corresponds to a three-dimensional maximally symmetric space. Moreover, by taking the trace of the Ricci tensor, one obtains the scalar curvature as

$$R=-6\frac{aa''+a'^2+k}{a^2}\ . \tag{11}$$

Now (4) becomes

$$\frac{(a')^2+k}{a^2}=\frac{\Lambda}{3}\ , \tag{12}$$

$$2\frac{a''}{a}+\frac{(a')^2+k}{a^2}=\Lambda\ , \tag{13}$$

after the FRW metric is substituted. The solution to (12) and (13) can hence be obtained by straightforward algebra. The result reads

$$a=a_0e^{\sqrt{(\Lambda/3)}t}+\frac{3k}{4\Lambda a_0}e^{-\sqrt{(\Lambda/3)}t}\ . \tag{14}$$

Here a_0 is a free parameter. Indeed, from (13) minus (12), we obtain the linear ordinary differential equation (ODE)

$$a''=\frac{\Lambda}{3}a\ , \tag{15}$$

which can be solved rather easily.

It is known that because of the Bianchi identity $D_\mu G^{\mu\nu}=0$, (13) is actually derivable from the Friedmann equation (12). This can be checked as follows. (i) $\partial_t(12)$ gives $a'a''=(\Lambda/3)aa'$. If $a'\neq 0$, we should have $a''=(\Lambda/3)a$. (ii) One has $2aa''=(2\Lambda/3)a^2$ from $2a$ times the preceding equation. (iii) Adding (12) to the preceding equation, one obtains $2aa''+a'^2+k=\Lambda a^2$, which is exactly Eq. (13). It is, however, impossible to derive (12) from (13) alone. A formal proof will be given shortly.

Therefore, the redundancy due to the Bianchi identity does apply to Eq. (13), but does not apply to the exclusive Friedmann equation (12). This can also be understood by observing that the Friedmann equation (12) is a first-order ODE in contrast with (13), which is a second-order one.

Note, however, that the redundant field equation [e.g., (13)] is usually very useful in obtaining a solution even if it can be ignored initially. Also, the exclusive role played by the Friedmann equation is due to the differential structure of the gravitational BI. This point will be emphasized again in Sec. IV.

III. REDUCED PROBLEM AND THE GENERALIZED FRIEDMANN-ROBERTSON-WALKER METRIC

On the other hand, the reduced action $\tilde{S}(a(t))\equiv S(g_{\mu\nu}(a(t)))$, with $g_{\mu\nu}(a(t))$ denoting the FRW metric given in (5), can be shown to be

$$\tilde{S}(a(t))=N\int dt[6(a^2a''+aa'^2+ka)-2\Lambda a^3]\ . \tag{16}$$

Here $N\equiv\int\sqrt{\det h_{ij}}\,d^3x$ is the time-independent factor.

The reduced problem can thus be defined as the reduced one-dimensional Lagrangian system given by (16). The variational equation $\delta\tilde{S}(a)/\delta a=0$ gives, however, only Eq. (13): $2a''/a+[(a')^2+k]/a^2=\Lambda$. The Friedmann equation (12) cannot, unfortunately, be obtained from the reduced action (16).

A remedy for this flaw can be found by introducing a $b(t)$ field variable in the (GFRW) metric $\tilde{g}_{\mu\nu}$ given by

$$ds^2\equiv\tilde{g}_{\mu\nu}dx^\mu dx^\nu$$
$$=-b^2(t)dt^2+a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2d\Omega\right]\ . \tag{17}$$

Defining the modified reduced problem from the reduced action $\tilde{S}(b(t),a(t))\equiv S(\tilde{g}(b(t),a(t)))$, the variational equation

$$\left.\frac{\delta\tilde{S}(b(t),a(t))}{\delta b(t)}\right|_{b=1}=0 \tag{18}$$

will thus reproduce the Friedmann equation (12). Indeed, it can be shown that the spin-connection components are

$$\Gamma^t_{ij}=\frac{aa'}{b^2}h_{ij}\ , \tag{19}$$

$$\Gamma^t_{tt}=\frac{b'}{b}\ , \tag{20}$$

after substituting the GFRW metric (17). The remaining spin-connection components are the same as listed in (7) and (8). Similarly, all nonvanishing $R_{\mu\nu}$ components become

$$R_{tt}=3\frac{a''b-a'b'}{ab}\ , \tag{21}$$

$$R_{ij}=-\left[\frac{aa''b+2a'^2b-aa'b'+2kb^3}{b^3}\right]h_{ij}\ . \tag{22}$$

Also, the scalar curvature changes to

$$R=-6\left[\frac{aa''+a'^2+kb^2}{a^2b^2}-\frac{a'b'}{ab^3}\right]\ . \tag{23}$$

Hence, the reduced action $\tilde{S}$ becomes

$$\tilde{S}(b(t),a(t))$$
$$=N\int dt\left[6\left[\frac{a^2a''+aa'^2+kab^2}{b}-\frac{a^2a'b'}{b^2}\right]-2\Lambda a^3b\right]\ . \tag{24}$$

It is then straightforward to derive the Friedmann equation from the variational equation of $b(t)$. Indeed straightforward algebra shows that the variational equation (18) reproduces the Friedmann equation as promised.

IV. APPLICATIONS

The use of the GFRW metric can reduce the labor in deriving the equations of motion in gravitational theories in which FRW spaces are of major concern. Especially when we encounter a theory with matter fields coupled directly to the metric field, the general Einstein field equations are usually very difficult to obtain without a thorough knowledge of covariant operations. It is also a very complicated exercise to work out the field equations, which require delicate care in handling the factors and signs. The advantage of the GFRW metric is to reduce the computation from a general covariant theory to a simple one-dimensional Lagrangian system. This reduced problem can normally be handled without knowing any geometric concepts of the gravitational theory. An undergraduate student should be able to handle the reduced problem by interpreting the reduced problem as a constrained motion [i.e., setting $b(t)=1$ after deriving the variational equation]. For example, let us analyze the following general covariant theory involving a scalar-metric coupling:

$$S=\int d^4x\sqrt{g}\,[-\tfrac{1}{2}\epsilon\phi^2R-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-V(\phi)]\ . \tag{25}$$

Here ϕ denotes a real scalar field, while ϵ is a dimensionless coupling constant. Equation (25) also provides a natural explanation for a universe with dimensional constants such as gravitational and cosmological "constants."

For later convenience we will define $\phi\equiv e^{\varphi/2}$ and $k_1\equiv 1/4\epsilon$. In terms of this set of new variables and parameters, one obtains the following variational equations for φ and metric g_{ab}, respectively [6]:

$$G_{\mu\nu}=\partial_\mu\varphi\,\partial_\nu\varphi+D_\mu\partial_\nu\varphi-g_{\mu\nu}(\partial_\alpha\varphi\,\partial^\alpha\varphi+D_\alpha\partial^\alpha\varphi)-T_{\mu\nu}(\varphi)\ , \tag{26}$$

$$R=k_1(\partial_\mu\varphi\,\partial^\mu\varphi+2D^\mu\partial_\mu\varphi)-\frac{2}{\epsilon}\frac{\partial V}{\partial\varphi}e^{-\varphi}\ , \tag{27}$$

after some straightforward calculations. The generalized energy-momentum tensor $T_{\mu\nu}(\varphi)$ is defined as

$$T_{\mu\nu}(\varphi)\equiv k_1\left[\frac{1}{2}g_{\mu\nu}\partial_\alpha\varphi\,\partial^\alpha\varphi-\partial_\mu\varphi\,\partial_\nu\varphi\right]+\frac{1}{\epsilon}Ve^{-\varphi}g_{\mu\nu}\ . \tag{28}$$

Assuming that $\varphi=\varphi(t)$, one has the following generalized Friedmann and scalar field equations:

$$\alpha'^2+\frac{k}{a^2}+\alpha'\varphi'=\frac{k_1}{6}\varphi'^2+\frac{V}{3\epsilon}e^{-\varphi}\ , \tag{29}$$

$$\varphi''+3\alpha'\varphi'+\varphi'^2=\frac{2}{(3+2k_1)\epsilon}e^{-\varphi}\left[2V-\frac{\partial V}{\partial\varphi}\right]\ . \tag{30}$$

Here we have written $a=e^\alpha$. Equation (30) is obtained by comparing the scalar curvature term R in Eqs. (26) and (27). The R for (26) can be obtained by taking the trace of (26). Also, the redundant G_{ij} equations in (26) are not listed since they are related by the Bianchi identity.

It is known that four (in fact, one because of the symmetry of the Friedmann-Robertson-Walker metric) out of Eqs. (26) and (27) are redundant because of the Bianchi identity $D_\mu G^{\mu\nu}=0$. A careful analysis shows, however, that every equation is equally redundant except the tt component of Eq. (26) [i.e., Eq. (29), which is known as the generalized Friedmann equation]. This can be readily understood by observing that the generalized Friedmann equation (29) is in fact a first-order ODE in contrast with all other equations which are second-order ones. In fact, Eq. (26) takes the form $H_{\mu\nu}=0$. Here $H_{\mu\nu}\equiv G_{\mu\nu}-K_{\mu\nu}$, with $K_{\mu\nu}$ denoting what appears on the right-hand side of Eqs. (26). Consequently, the Bianchi identity can be rephrased, on shell, as

$$D_\mu H^{\mu\nu}=0\ . \tag{31}$$

This is because $D_\mu G^{\mu\nu}=0$ as a result of the Bianchi identity and $D_\mu K^{\mu\nu}=0$, serving as an on-shell constraint. Equation (31) becomes

$$(\partial_t+3\alpha')H_{00}+3aa'H=0\ , \tag{32}$$

as soon as the FRW metric is substituted into (31). Here $H\equiv\frac{1}{3}h^{ij}H_{ij}$ and $g_{ij}\equiv a^2h_{ij}$. It is now straightforward to show that $H_{ij}=Hh_{ij}$ in this theory under the constraint that φ is spatially independent. The exclusive role played by the tt equation of $H_{\mu\nu}=0$ can be readily checked at this moment. Indeed, (32) indicates that $H_{00}=0$ implies $H=0$ if $a'\neq 0$. On the other hand, $H=0$ implies, instead,

$$(\partial_t+3\alpha')H_{00}=0\ . \tag{33}$$

In fact, Eq. (33) can be integrated directly to give the result

$$a^3H_{00}=\text{const}\ , \tag{34}$$

which is not sufficient to imply the desired result $H_{00}=0$. Therefore, the generalized Friedmann equation (29) is indeed an exclusive equation of motion. We are thus free to exclude any redundant equations among the whole set of equations of motion, except the generalized Friedmann equation $H_{00}=0$. For later convenience we can hence stick to (29) and (30) by ignoring the redundant ij equation of (27) without loose ends.

Alternatively, we could deal instead with the reduced problem defined by the reduced action $\tilde{S}(b,a,\varphi)\equiv S(g_{\mu\nu}(b(t),a(t)),\varphi(t))$. Here $b(t)$ is treated as a constrained field variable. After some algebra we obtain the reduced action

$$\tilde{S}(b,a,\varphi)=N\int dt\, e^{\varphi}\left[3\epsilon\left\{\frac{a^2a''+aa'^2+kab^2}{b}-\frac{a^2a'b'}{b^2}\right\}+\frac{a^3}{8b}\varphi'^2-Ve^{-\varphi}ba^3\right]. \quad (35)$$

The generalized Friedmann equation can now be obtained by varying $\tilde{S}$ with respect to b and imposing the constraint $b=1$ afterward, i.e.,

$$\left.\frac{\delta\tilde{S}}{\delta b}\right|_{b=1}=0\,. \quad (36)$$

This correspondence can be verified directly. Indeed, we can write

$$\frac{\delta\tilde{S}}{\delta b}=\frac{\delta\tilde{S}}{\delta g_{00}}\frac{\delta g_{00}}{\delta b}=-2b\frac{\delta\tilde{S}}{\delta g_{00}}\,. \quad (37)$$

The constraint $\delta\tilde{S}/\delta b|_{b=1}=-2\delta S(g_{\mu\nu}^{\mathrm{FRW}})/\delta g_{00}=0$ gives us precisely the generalized Friedmann equation (29).

V. CONCLUSIONS

We have shown that the Friedmann equation is an exclusive equation which cannot be obtained from the reduced problem defined by the reduced action, e.g., (16). It is also shown that the nontrivial redundancy relation, which left the Friedmann equation as an exclusive nonredundant field equation, is due to the nontrivial differential structure of the gravitational Bianchi identity. This point was clarified in Sec. IV. Finally, it is found that a general Friedmann-Robertson-Walker metric can be employed to define a modified one-dimensional reduced problem. The reduced problem can hence be used to reproduce the desired Friedmann equation as its variational field equation. One can therefore greatly reduce the labor in deriving gravitational field equations on Friedmann-Robertson-Walker spaces.

The method introduced in this Brief Report can be applied to all theories that have to do with the cosmological evolution of our spatially isotropic and homogeneous universe. This includes all higher-dimensional gravity models such as Kaluza-Klein [11–14] theories. The most important point in defining a proper reduced problem is demonstrated in Eq. (37). Equation (37) shows how to introduce appropriate constrained field variables in order to recover the lost field equation due to the symmetry of the ansatz (e.g., the FRW metric). This method can hence be generalized to similar systems.

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