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# **A GLOBAL PINCHING THEOREM**  FOR COMPACT MINIMAL SURFACES IN  $S^3$

#### **YI-JUNG HSU**

**(Communicated by Jonathan M. Rosenberg)** 

**ABSTRACT. Let M be a compact minimally immersed surface in the unit sphere S3 ,and let S denote the square of the length of the second fundamental form**  of M. We prove that if  $||S||_2 \leq 2\sqrt{2}\pi$ , then M is either the equatorial sphere **or the Clifford torus.** 

**Let M be a compact minimally immersed hypersurface in the unit sphere**   $S^{n+1}$ . Denote by S the square of the length of the second fundamental form of M. It is well known that if  $0 \leq S \leq n$ , then M is either the equatorial sphere **or a Clifford torus [1]. Recently, C. L. Shen [4, Theorem 2] proved that if M is a compact embedded minimal surface of nonnegative Gauss curvature in the**  unit sphere  $S^3$  with  $||S||_2 < 1/(6912\sqrt{2\pi(g+1)})$ , then M is the equatorial sphere, where  $g$  denotes the genus of  $M$ . The purpose of this note is to **improve this theorem and obtain the best constant. The following is our main result:** 

**Theorem. Let M be a compact minimally immersed surface in the unit sphere S**<sup>3</sup>. Then  $||S||_2 \geq 2\sqrt{2g}\pi$ . The equality sign holds if and only if M is either *the equatorial sphere or the Clifford torus. In particular, if* $||S||_2 \leq 2\sqrt{2}\pi$ **, then M is either the equatorial sphere or the Clifford torus.** 

#### **1. NOTATIONS AND AUXILIARY RESULTS**

Let M be a compact connected minimally immersed surface in the unit sphere  $S^3$ . Following the notations of [1], denote by  $h = (h_{ij})$  the second fundamental form of M, and by S the square of the length of  $h$ ,  $S = \sum h_{ij}^2$ . **We need the following auxiliary results.** 

**Lemma 1** [1].  $\frac{1}{2}\Delta S = S(2-S) + \sum h_{ijk}^2$ , where  $h_{ijk}$  denote the covariant derivatives of  $h_{ij}$ .

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**Lemma 2 [3]. The set of all zeros of S is either the whole space M or at most a finite set of points.** 

**Lemma 3.**  $|\nabla S|^2 = 2S \sum h_{ijk}^2$ .

From Lemmas 1 and 3, we see that if S is constant, then either  $S = 0$  or  $S \equiv 2$ .

**Lemma 4.** If  $g \ge 1$ , then

$$
\lim_{\varepsilon \to 0} \sum_{i=1}^k \int_{\partial B\varepsilon(p_i)} \frac{S_r}{S} = 16\pi(g-1)
$$

where  $p_1, p_2, \ldots, p_k$  constitute all the zeros of S and S<sub>r</sub> denotes the derivative of S on  $\partial B\varepsilon(p_i)$  in the radial direction from  $p_i$ .

**Proof. At the points where S is positive, by Lemma 3, we get** 

(1) 
$$
\Delta \log S = 2(2 - S).
$$

Integrating (1) over  $M\epsilon = M\setminus\bigcup_{i=1}^k B\epsilon(p_i)$ , we get from the Gauss equation

$$
(2) \t\t\t 2K = 2 - S,
$$

where  $K$  is the Gauss curvature of  $M$ , the assertion by Stokes's theorem and **the theorem of Gauss-Bonnet.**  $\Box$ 

## **Lemma 5.**

$$
\int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \frac{2-S}{2} \ge (g+1)\pi^2.
$$

*Proof.* Regard  $M$  as an immersed surface of  $\mathbb{R}^4$ . Then the total absolute curvature of  $M$ , in the sense of [2], is given by

$$
\int_M \int_0^{2\pi} \left| (\sin \theta)^2 - \frac{S}{2} (\cos \theta)^2 \right| d\theta dV
$$
  
= 
$$
\int_M 2\sqrt{2S} + \left( \pi - 4 \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2}
$$

**By a well-known inequality of Chern-Lashof [2], we have** 

$$
\int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \frac{2-S}{2} \ge \frac{\pi^2}{2} (b_0 + b_1 + b_2),
$$

where  $b_i$  denotes the *i*th Betti number relative to the real field, for  $i = 0, 1, 2$ . Since  $\dot{M}$  is of two-dimensional,  $b_0 = 1$ ,  $b_1 = 2g$ , and  $b_2 = 1$ .  $\Box$ 

#### **2. PROOF OF THEOREM**

We may assume that S is positive except possibly at a finite set of points **(Lemma 2). By using (1) and Lemmas 3 and 4, we get** 

$$
\int_{M} 1 + \frac{1}{4} h_{ijk}^{2} - \sqrt{\frac{S}{2}} - \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \frac{2-S}{2}
$$
\n
$$
= \int_{M} \left[ \frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \right] (2-S) + \frac{1}{4} h_{ijk}^{2}
$$
\n
$$
= \lim_{\epsilon \to 0} \int_{M\epsilon} \frac{1}{2} \left[ \frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \right] \Delta \log S + \frac{|\nabla S|^{2}}{8S}
$$
\n
$$
= \lim_{\epsilon \to 0} \int_{M\epsilon} \frac{1}{2} \nabla \left[ \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) - \frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} \right] \nabla \log S + \frac{|\nabla S|^{2}}{8S}
$$
\n
$$
- \lim_{\epsilon \to 0} \int_{\partial M\epsilon} \frac{1}{2} \left[ \frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \right] \frac{S}{S}
$$
\n
$$
= -(4 - \pi)\pi (g - 1) + \lim_{\epsilon \to 0} \int_{M\epsilon} \left(\frac{1}{4} - \frac{1}{(2 + S)(\sqrt{2} + \sqrt{S})^{2}}\right) \frac{|\nabla S|^{2}}{2S}
$$
\n
$$
= -(4 - \pi)\pi (g - 1) + \int_{M} \left(\frac{1}{4} - \frac{1}{(2 + S)(\sqrt{2} + \sqrt{S})^{2}}\right) h_{ijk}^{2}
$$
\n
$$
\geq -(4 - \pi)\pi (g - 1),
$$

**where the equality sign holds if and only if S is constant. According to Lemma 5, we get** 

**(3)** 

$$
A + \frac{1}{4} \int_M h_{ijk}^2 \ge -(4 - \pi)\pi(g - 1) + \int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right) \frac{2-S}{2}
$$
  
 
$$
\ge 2g\pi^2 - 4\pi(g - 1),
$$

where A denotes the area of  $M$ . By combining (2) with the inequality (3), it **follows that** 

$$
\int_M 2S + h_{ijk}^2 = 4A + 16\pi (g - 1) + \int_M h_{ijk}^2 \ge 8g\pi^2.
$$

**The desired inequality now follows from Lemma 1.** 

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