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A GLOBAL PINCHING THEOREM FOR COMPACT MINIMAL SURFACES IN S^3

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ABSTRACT. Let M be a compact minimally immersed surface in the unit sphere S^3, and let S denote the square of the length of the second fundamental form of M. We prove that if $\|S\|_2 \leq 2\sqrt{2}\pi$, then M is either the equatorial sphere or the Clifford torus.

Let M be a compact minimally immersed hypersurface in the unit sphere S^{n+1}. Denote by S the square of the length of the second fundamental form of M. It is well known that if $0 \leq S \leq n$, then M is either the equatorial sphere or a Clifford torus [1]. Recently, C. L. Shen [4, Theorem 2] proved that if M is a compact embedded minimal surface of nonnegative Gauss curvature in the unit sphere S^3 with $\|S\|_2 < 1/(6912\sqrt{2\pi(g+1)})$, then M is the equatorial sphere, where g denotes the genus of M. The purpose of this note is to improve this theorem and obtain the best constant. The following is our main result:

Theorem. *Let M be a compact minimally immersed surface in the unit sphere S^3. Then $\|S\|_2 \geq 2\sqrt{2g}\pi$. The equality sign holds if and only if M is either the equatorial sphere or the Clifford torus. In particular, if $\|S\|_2 \leq 2\sqrt{2}\pi$, then M is either the equatorial sphere or the Clifford torus.*

1. NOTATIONS AND AUXILIARY RESULTS

Let M be a compact connected minimally immersed surface in the unit sphere S^3. Following the notations of [1], denote by $h = (h_{ij})$ the second fundamental form of M, and by S the square of the length of h, $S = \sum h_{ij}^2$. We need the following auxiliary results.

Lemma 1 [1]. *$\frac{1}{2}\Delta S = S(2-S) + \sum h_{ijk}^2$, where h_{ijk} denote the covariant derivatives of h_{ij}.*

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Lemma 2 [3]. *The set of all zeros of* S *is either the whole space* M *or at most a finite set of points.*

Lemma 3. $|\nabla S|^2 = 2S \sum h_{ijk}^2$.

From Lemmas 1 and 3, we see that if S is constant, then either $S \equiv 0$ or $S \equiv 2$.

Lemma 4. *If* $g \geq 1$, *then*

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{k} \int_{\partial B\varepsilon(p_i)} \frac{S_r}{S} = 16\pi(g-1)$$

where $p_1, p_2, \ldots, p_k$ *constitute all the zeros of* S *and* S_r *denotes the derivative of* S *on* $\partial B\varepsilon(p_i)$ *in the radial direction from* p_i.

Proof. At the points where S is positive, by Lemma 3, we get

$$\Delta \log S = 2(2-S). \tag{1}$$

Integrating (1) over $M\varepsilon = M \setminus \bigcup_{i=1}^{k} B\varepsilon(p_i)$, we get from the Gauss equation

$$2K = 2 - S, \tag{2}$$

where K is the Gauss curvature of M, the assertion by Stokes's theorem and the theorem of Gauss-Bonnet. □

Lemma 5.

$$\int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right)\frac{2-S}{2} \geq (g+1)\pi^2.$$

Proof. Regard M as an immersed surface of $\mathbf{R}^4$. Then the total absolute curvature of M, in the sense of [2], is given by

$$\int_M \int_0^{2\pi} \left|(\sin\theta)^2 - \frac{S}{2}(\cos\theta)^2\right| d\theta\, dV$$
$$= \int_M 2\sqrt{2S} + \left(\pi - 4\sin^{-1}\sqrt{\frac{S}{2+S}}\right)\frac{2-S}{2}.$$

By a well-known inequality of Chern-Lashof [2], we have

$$\int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1}\sqrt{\frac{S}{2+S}}\right)\frac{2-S}{2} \geq \frac{\pi^2}{2}(b_0 + b_1 + b_2),$$

where b_i denotes the ith Betti number relative to the real field, for $i = 0, 1, 2$. Since M is of two-dimensional, $b_0 = 1$, $b_1 = 2g$, and $b_2 = 1$. □

2. Proof of theorem

We may assume that S is positive except possibly at a finite set of points (Lemma 2). By using (1) and Lemmas 3 and 4, we get

$$\begin{aligned}
&\int_M 1+\frac{1}{4}h_{ijk}^2-\sqrt{\frac{S}{2}}-\left(\frac{\pi}{4}-\sin^{-1}\sqrt{\frac{S}{2+S}}\right)\frac{2-S}{2} \\
&= \int_M \left[\frac{1}{\sqrt{2}(\sqrt{2}+\sqrt{S})}-\frac{1}{2}\left(\frac{\pi}{4}-\sin^{-1}\sqrt{\frac{S}{2+S}}\right)\right](2-S)+\frac{1}{4}h_{ijk}^2 \\
&= \lim_{\varepsilon\to 0}\int_{M\varepsilon}\frac{1}{2}\left[\frac{1}{\sqrt{2}(\sqrt{2}+\sqrt{S})}-\frac{1}{2}\left(\frac{\pi}{4}-\sin^{-1}\sqrt{\frac{S}{2+S}}\right)\right]\Delta\log S+\frac{|\nabla S|^2}{8S} \\
&= \lim_{\varepsilon\to 0}\int_{M\varepsilon}\frac{1}{2}\nabla\left[\frac{1}{2}\left(\frac{\pi}{4}-\sin^{-1}\sqrt{\frac{S}{2+S}}\right)-\frac{1}{\sqrt{2}(\sqrt{2}+\sqrt{S})}\right]\nabla\log S+\frac{|\nabla S|^2}{8S} \\
&\quad -\lim_{\varepsilon\to 0}\int_{\partial M\varepsilon}\frac{1}{2}\left[\frac{1}{\sqrt{2}(\sqrt{2}+\sqrt{S})}-\frac{1}{2}\left(\frac{\pi}{4}-\sin^{-1}\sqrt{\frac{S}{2+S}}\right)\right]\frac{S_r}{S} \\
&= -(4-\pi)\pi(g-1)+\lim_{\varepsilon\to 0}\int_{M\varepsilon}\left(\frac{1}{4}-\frac{1}{(2+S)(\sqrt{2}+\sqrt{S})^2}\right)\frac{|\nabla S|^2}{2S} \\
&= -(4-\pi)\pi(g-1)+\int_M\left(\frac{1}{4}-\frac{1}{(2+S)(\sqrt{2}+\sqrt{S})^2}\right)h_{ijk}^2 \\
&\geq -(4-\pi)\pi(g-1),
\end{aligned}$$

where the equality sign holds if and only if S is constant. According to Lemma 5, we get

(3)

$$\begin{aligned}
A+\frac{1}{4}\int_M h_{ijk}^2 &\geq -(4-\pi)\pi(g-1)+\int_M\sqrt{\frac{S}{2}}+\left(\frac{\pi}{4}-\sin^{-1}\sqrt{\frac{S}{2+S}}\right)\frac{2-S}{2} \\
&\geq 2g\pi^2-4\pi(g-1),
\end{aligned}$$

where A denotes the area of M. By combining (2) with the inequality (3), it follows that

$$\int_M 2S+h_{ijk}^2=4A+16\pi(g-1)+\int_M h_{ijk}^2\geq 8g\pi^2.$$

The desired inequality now follows from Lemma 1.

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