

# Self-induced transparency in a dispersive and nonlinear Kerr host medium

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A steady-state solution that describes pulse propagation in inhomogeneously broadening two-level atoms embedded in a dispersive and nonlinear Kerr lossless host medium has been found. The solution is of an unchirped hyperbolic-secant form with a linear spatial phase delay. The features of the solution are (i) the group velocity, the amplitude, and the width of the pulse are uniquely determined for given parameters of the medium and the carrier frequency; (ii) the group velocity is independent of the density of the resonant atoms; and (iii) the solution can exist in the positive dispersion region.

Self-induced transparency was first investigated by McCall and Hahn,<sup>1</sup> who found that a short light pulse above a critical input energy can propagate through resonant atoms without loss and distortion. Eberly and Matulic<sup>2</sup> extended McCall and Hahn's theory to include the Kerr effect in the host medium. They found that an undistorted lossless single pulse is still possible, but the pulse is necessarily chirped. On the other hand, Hasegawa and Tappert<sup>3</sup> have proposed the utilization of the nonlinear dependence of the index of refraction on intensity to overcome the dispersive spreading, so that it is possible to have a distortionless optical pulse propagating in fibers. In this Letter we consider the pulse propagation in resonant atoms embedded in a dispersive and nonlinear Kerr host, i.e., a composite medium with combined characteristics considered previously. We have found that the steady-state solution is an unchirped hyperbolic-secant form with linear spatial phase delay. Its amplitude, pulse width, and propagation velocity are determined uniquely if the medium parameter value is given.

The electric field of the optical pulse propagating along the  $z$  direction in such a medium is assumed to be linearly polarized in the  $x$  direction and can be expressed as

$$\mathbf{E}(z, t) = \hat{x} \frac{1}{2} q(z, t) \exp\{i[k_0 z - \omega_0 t + \phi(z, t)]\} + \text{c.c.}, \quad (1)$$

where  $q(z, t)$  and  $\phi(z, t)$  are the amplitude and phase, respectively,  $\omega_0$  is the carrier frequency, and  $k_0$  is the wave number at  $\omega_0$ . The macroscopic polarization due to the resonant atoms is taken to be

$$\mathbf{P}_r(z, t) = \hat{x} \frac{1}{2} [U(z, t) + iV(z, t)] \times \exp\{i[k_0 z - \omega_0 t + \phi(z, t)]\} + \text{c.c.}, \quad (2)$$

where  $U$  and  $V$  correspond to the dispersion (in phase) and the absorption (in quadrature) components, respectively. The wave equation for this problem can be written as

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu_0 \left( \frac{\partial^2 \mathbf{D}_l}{\partial t^2} + \frac{\partial^2 \mathbf{D}_{nl}}{\partial t^2} + \frac{\partial^2 \mathbf{P}_r}{\partial t^2} \right), \quad (3)$$

where  $\mathbf{D}_l$  is the linear displacement and  $\mathbf{D}_l = n^2(\omega)\epsilon_0 \mathbf{E}$ , with  $n(\omega)$  the refractive index of the dispersive medium;  $\mathbf{D}_{nl}$  is the nonlinear displacement for the nonresonant host and  $\mathbf{D}_{nl} = 2n_2 n_0 \epsilon_0 |\mathbf{E}|^2 \mathbf{E}$ , with  $n_0 = n(\omega_0)$ ,  $\epsilon_0$  and  $\mu_0$  denote the permittivity and permeability in a vacuum, respectively, and  $n_2$  is the Kerr coefficient. The resonant atom is considered to have two energy levels with transition frequency  $\omega_r$ . If we define  $\Delta\omega = \omega_r - \omega_0$ , the macroscopic polarization components contributed from atoms of the whole range of  $\Delta\omega$  are

$$U(z, t) = \int_{-\infty}^{\infty} u(\Delta\omega, z, t) g(\Delta\omega) d(\Delta\omega), \quad (4a)$$

$$V(z, t) = \int_{-\infty}^{\infty} v(\Delta\omega, z, t) g(\Delta\omega) d(\Delta\omega), \quad (4b)$$

where  $g(\Delta\omega)$  is the inhomogeneous broadening normalized line-shape function with  $\int_{-\infty}^{\infty} g(\Delta\omega) d(\Delta\omega) = 1$ , and  $u$  and  $v$  are the polarization components of the atoms with frequency  $\Delta\omega$  detuned from  $\omega_0$ . Following the same procedure of Ref. 4 for derivation of the  $\partial^2 \mathbf{D}_l / \partial t^2$  term, using the slowly varying envelope approximation for the  $\partial^2 \mathbf{D}_{nl} / \partial t^2$  and  $\partial^2 \mathbf{P}_r / \partial t^2$  terms, and making the coordinate transformation  $\xi = z$ ,  $\tau = t - z/v_g$ , we can obtain from Eqs. (1)–(4) the following differential equations:

$$\begin{aligned} -q \frac{\partial \phi}{\partial \xi} + \alpha q \frac{\partial \phi}{\partial \tau} + \beta \left[ \frac{\partial^2 q}{\partial \tau^2} - q \left( \frac{\partial \phi}{\partial \tau} \right)^2 \right] + r q^3 \\ = -s \int_{-\infty}^{\infty} u(\Delta\omega, \xi, \tau) g(\Delta\omega) d\Delta\omega, \end{aligned} \quad (5a)$$

$$\begin{aligned} \frac{\partial q}{\partial \xi} - \alpha \frac{\partial q}{\partial \tau} + \beta \left( 2 \frac{\partial q}{\partial \tau} \frac{\partial \phi}{\partial \tau} + q \frac{\partial^2 \phi}{\partial \tau^2} \right) \\ = -s \int_{-\infty}^{\infty} v(\Delta\omega, \xi, \tau) g(\Delta\omega) d\Delta\omega, \end{aligned} \quad (5b)$$

where

$$k_0' = \frac{\partial k}{\partial \omega} \Big|_{\omega=\omega_0}, \quad k_0'' = \frac{\partial^2 k}{\partial \omega^2} \Big|_{\omega=\omega_0},$$

$$\alpha = \frac{1}{v_g} - k_0', \quad \beta = \left( \frac{1}{v_g^2} - k_0 k_0'' - k_0'^2 \right) / 2k_0,$$

$$r = \frac{n_2 \omega_0}{c}, \quad s = \mu_0 \omega_0^2 / 2k_0.$$

The equations of motion of  $u$  and  $v$  and the population difference  $w$  are Bloch equations,<sup>1</sup>

$$\frac{\partial u}{\partial \tau} = v \left( \frac{\partial \phi}{\partial \tau} + \Delta \omega \right), \quad (6a)$$

$$\frac{\partial v}{\partial \tau} = -u \left( \frac{\partial \phi}{\partial \tau} + \Delta \omega \right) + \frac{\mu}{\hbar} q w, \quad (6b)$$

$$\frac{\partial w}{\partial \tau} = -\frac{\mu}{\hbar} q v, \quad (6c)$$

where  $w$  is in the unit of  $\mu$  per unit volume and  $\mu$  is the dipole matrix element of the individual atom. Here we assume that phenomenological relaxation times are long compared with the pulse duration and may be ignored.

In order to obtain the analytic solution, we assume that  $v(\Delta\omega, z, t)$  is in a factorized form

$$v(\Delta\omega, z, t) = v_1(\xi, \tau) f(\Delta\omega), \quad (7)$$

where  $f(\Delta\omega)$  is known as the dipole spectra response function<sup>3</sup> and is normalized as  $f(0) = 1$ . Integrating Eq. (6a), we have

$$u(\Delta\omega, \xi, \tau) = [u_1(\xi, \tau) + u_2(\xi, \tau) \Delta\omega] f(\Delta\omega), \quad (8)$$

where  $u_1$  and  $u_2$  are defined by  $\partial u_1 / \partial \tau = v_1(\partial \phi / \partial \tau)$  and  $\partial u_2 / \partial \tau = v_1$ , respectively. Similarly, by integrating Eq. (6c), we obtain

$$w(\Delta\omega, \xi, \tau) = w_0 - w_1(\xi, \tau) f(\Delta\omega), \quad (9)$$

where  $w_1(\xi, \tau)$  is defined by  $\partial w_1 / \partial \tau = (\mu/\hbar) q v_1$  and  $w_0$  is the initial population difference and is assumed in the ground state, i.e.,  $w_0 = N_0 \mu$ , with  $N_0$  the density of the resonant atoms. Substituting Eqs. (7)–(9) into Eq. (6b), we have

$$\frac{\partial v_1}{\partial \tau} = \left[ -u_2 \Delta \omega^2 - \left( u_1 + u_2 \frac{\partial \phi}{\partial \tau} \right) \Delta \omega + \frac{\mu w_0}{\hbar} q / f(\Delta \omega) \right]$$

$$- u_1 \frac{\partial \phi}{\partial \tau} - \frac{\mu}{\hbar} q w_1. \quad (10)$$

The terms in the square brackets in Eq. (10) should be independent of  $\Delta\omega$  because the other terms in Eq. (10) are independent of  $\Delta\omega$ . Therefore, we have  $f(\Delta\omega) = (1 + c_1 \Delta\omega + c_2 \Delta\omega^2)^{-1}$ ,  $u_1 = c_1 (\mu w_0 / \hbar) q - u_2 (\partial \phi / \partial \tau)$ , and  $u_2 = c_2 (\mu w_0 / \hbar) q$ , where  $c_1$  and  $c_2$  are constants to be determined. Then Eq. (10) reduces to

$$\frac{\partial v_1}{\partial \tau} = \frac{\mu}{\hbar} (w_0 - w_1) q - u_1 \frac{\partial \phi}{\partial \tau}. \quad (11)$$

The population difference can be found by substituting  $v_1 = (\partial u_2 / \partial \tau)$  and  $u_2 = c_2 (\mu w_0 / \hbar) q$  into Eq. (6c),

$$w = w_0 - c_2 \frac{\mu^2 w_0}{2(\hbar)^2} q^2 f(\Delta\omega). \quad (12)$$

In what follows, we consider the solution of distortionless amplitude, i.e.,  $q$  is independent of  $\xi$  but the phase  $\phi$  is allowed to depend on  $\tau$  and  $\xi$ . With this assumption, we can write  $\partial q / \partial \tau = dq/d\tau = \dot{q}$ ,  $\partial^2 q / \partial \tau^2 = d^2 q / d\tau^2 = \ddot{q}$ , and  $\partial q / \partial \xi = 0$ . By substituting  $w_1(\xi, \tau)$ ,  $v_1(\xi, \tau)$ , and  $u_1(\xi, \tau)$  as functions of  $q(\tau)$  derived above into Eq. (11), we have

$$c_2 \frac{d^2 q}{d\tau^2} = \left[ 1 - c_1 \frac{\partial \phi}{\partial \tau} + c_2 \left( \frac{\partial \phi}{\partial \tau} \right)^2 \right] q - \frac{c_2 \mu^2}{2(\hbar)^2} q^3. \quad (13)$$

From Eq. (13) we know that  $\partial \phi / \partial \tau$  is a function of  $\tau$  only; from Eq. (5a) for distortionless propagation, it is seen that  $\partial \phi / \partial \xi$  is also a function of  $\tau$  only. Since  $\partial^2 \phi / \partial \tau \partial \xi = (\partial / \partial \tau) (\partial \phi / \partial \xi) = (\partial / \partial \xi) (\partial \phi / \partial \tau) = 0$ , which implies that  $\partial \phi / \partial \xi$  must be a constant, we write  $\partial \phi / \partial \xi = \Delta k_0$ . For the distortionless amplitude solution, Eqs. (5a) and (5b) reduce to

$$-\Delta k_0 q + \alpha q \frac{\partial \phi}{\partial \tau} + \beta \left[ \frac{d^2 q}{d\tau^2} - q \left( \frac{\partial \phi}{\partial \tau} \right)^2 \right] + r q^3$$

$$= -s \left[ I_1 \left( c_1 - c_2 \frac{\partial \phi}{\partial \tau} \right) + c_2 I_2 \right] \frac{\mu w_0}{\hbar} q, \quad (14a)$$

$$-\alpha \frac{dq}{d\tau} + \beta \left( 2 \frac{dq}{d\tau} \frac{\partial \phi}{\partial \tau} + q \frac{\partial^2 \phi}{\partial \tau^2} \right) = -s c_2 \frac{\mu w_0}{\hbar} I_1 \frac{dq}{d\tau}, \quad (14b)$$

where

$$I_1 = \int_{-\infty}^{\infty} f(\Delta\omega) g(\Delta\omega) d(\Delta\omega),$$

$$I_2 = \int_{-\infty}^{\infty} \Delta\omega f(\Delta\omega) g(\Delta\omega) d(\Delta\omega).$$

Integrating Eq. (14b), we obtain  $\partial \phi / \partial \tau = \Omega + d_1 q^{-2}$ , where  $\Omega$  and  $d_1$  are constants. By using this relation and substituting Eq. (13) into Eq. (14a), Eq. (14a) becomes a polynomial equation including only the terms  $q^{-3}$ ,  $q^{-1}$ ,  $q^1$ , and  $q^3$ . This polynomial equation is valid only when the coefficient of each term is zero. These conditions give the following results:

(i)  $d_1 = 0$ , i.e.,

$$\frac{\partial \phi}{\partial \tau} = \Omega. \quad (15a)$$

This means that the carrier frequency of the pulse is shifted to  $\omega = \omega_0 - \Omega$  and the pulse is unchirped.

(ii)  $\beta(1 - \Omega^2 c_2) = (\Delta k_0 - \alpha \Omega + \beta \Omega^2) c_2$

$$- s(\Omega I_1 + I_2) \frac{\mu w_0}{\hbar} c_2^2. \quad (15b)$$

Here we have used the relation  $c_1 = 2\Omega c_2$ , which can be obtained by definitions of  $u_1$  and  $u_2$ .

$$(iii) \quad r = \frac{\beta \mu^2}{2(\hbar)^2}. \quad (15c)$$

From Eq. (15c), the pulse propagation group velocity

is given by

$$v_g = \left[ \frac{4n_2\omega_0(\hbar)^2 k_0}{\mu^2 c} + k_0'^2 + k_0 k_0'' \right]^{-1/2}. \quad (16)$$

The group velocity is uniquely determined by the parameters of the medium and the carrier frequency, and it is independent of the density of the resonant atoms. This is different from McCall and Hahn's finding that the group velocity varies with the pulse width and depends on the density of the resonant atoms. In a recent paper, Brains *et al.*<sup>5</sup> have found that the group velocities of the self-induced transparency beyond the slowly varying envelope approximation in a linear host medium form a discrete set. In our system, only a single velocity is allowed. From Eq. (16), we know that it is possible to have distortionless propagation at  $k_0'' > 0$ ; this is different from the soliton propagation in optical fibers, for which only dark solitons can exist in the positive dispersion region.<sup>6</sup> From Eqs. (14b) and (15a), we have

$$-\alpha + 2\beta\Omega + \frac{\mu}{\hbar} c_2 s \omega_0 I_1 = 0. \quad (17)$$

The constants  $\Delta k_0$  and  $c_2$  can be determined from Eqs. (15b) and (17). The solution of the electric field can be obtained from Eq. (13),

$$\mathbf{E}(z, t) = \hat{x} \frac{2\hbar}{\mu T_0} \operatorname{sech}\left(\frac{t - z/v_g}{T_0}\right) \exp\{i[(k_0 + \Delta k_0)z - \omega_0 t + \Omega(t - z/v_g)]\}, \quad (18)$$

where  $T_0 = (1/c_2 - \Omega^2)^{-1/2}$  is the pulse width. We can substitute Eq. (18) into expressions for  $u$ ,  $v$ , and  $w$  and obtain

$$u(\Delta\omega, z, t) = f(\Delta\omega) (\Omega + \Delta\omega) \frac{2N_0\mu}{T_0(T_0^{-2} + \Omega^2)} \times \operatorname{sech}\left(\frac{t - z/v_g}{T_0}\right), \quad (19a)$$

$$v(\Delta\omega, z, t) = -f(\Delta\omega) \frac{2N_0\mu}{T_0^2(T_0^{-2} + \Omega^2)} \times \operatorname{sech}\left(\frac{t - z/v_g}{T_0}\right) \tanh\left(\frac{t - z/v_g}{T_0}\right), \quad (19b)$$

$$w(\Delta\omega, z, t) = \omega_0 - f(\Delta\omega) \frac{2N_0\mu}{T_0^2(T_0^{-2} + \Omega^2)} \times \operatorname{sech}^2\left(\frac{t - z/v_g}{T_0}\right), \quad (19c)$$

where

$$f(\Delta\omega) = \frac{1}{1 + 2\Omega(T_0^{-2} + \Omega^2)\Delta\omega + (T_0^{-2} + \Omega^2)\Delta\omega^2}.$$

For simplicity, we consider the case of  $\Omega = 0$  and Lorentzian line shape inhomogeneous broadening, i.e.,  $g(\Delta\omega) = (\Delta\omega_a/2\pi)/[\Delta\omega^2 + (\Delta\omega_a/2)^2]$ , where  $\Delta\omega_a$  is the FWHM of  $g(\Delta\omega)$ . Then from Eqs. (15)–(17) we have

$$T_0^2 = \omega_t^{-2} \left\{ \left[ 1 + \frac{k_0}{k_0'^2} (k_0'' + k_e'') \right]^{1/2} - 1 \right\} \times \left( 1 + \frac{\Delta\omega_a T_0}{2} \right), \quad (20a)$$

where

$$\omega_t = \left( \frac{N_0 \mu^2 \mu_0 \omega_0^2}{2k_0 \hbar k_0'} \right)^{1/2}, \quad k_e'' = \frac{4n_2\omega_0(\hbar)^2}{\mu^2 c},$$

and

$$\Delta k_0 = \frac{2n_2\omega_0(\hbar)^2}{\mu^2 c T_0^2}. \quad (20b)$$

There exists a minimum  $k_0'' = -k_e''$  such that when  $k_0'' \leq -k_e''$ , the steady-state pulse cannot exist. The existence of a soliton solution in the positive dispersion region can be seen from the term  $\beta(\ddot{q} - \dot{q}\phi^2)$  in Eq. (14a), where  $\beta > 0$  when  $k_0'' > -k_e''$  and  $\beta(\ddot{q} - \dot{q}\phi^2)$  is equivalent to the negative dispersion term although  $k_0''$  is positive. Eberly and Matulic<sup>2</sup> have found that a distortionless pulse is possible in a nonlinear, nondispersive medium, but the pulse is necessarily chirped. The case considered by them is a special case of ours when  $k_0'' = 0$ . When  $k_0'' = 0$ , we still have an unchirped solution. The difference is because they have neglected the  $\beta(\ddot{q} - \dot{q}\phi^2)$  term, which is important in our treatment.

For the case  $\partial\phi/\partial\tau = \Omega \neq 0$  and homogeneous broadening, we can obtain the pulse width  $T_0$  as

$$T_0 = \left\{ \frac{\omega_t^2}{[1 + k_0/k_0'^2(k_0'' + k_e'')]^{1/2} - \Omega k_e''/k_0' - 1} - \Omega^2 \right\}^{-1/2}. \quad (21)$$

In summary, we have found a steady-state solution for the pulse propagating in resonant atoms embedded in a dispersive and nonlinear Kerr host medium. The pulse is of an unchirped hyperbolic-secant form with a linear spatial phase delay. There are several salient features in this solution that are different from the results obtained previously. First, for the given parameters of the medium and carrier frequency, the group velocity, the amplitude, and the pulse width are uniquely determined. Second, the group velocity is independent of the density of the resonant atoms. Third, the solution exists in the positive dispersion region, which is different from solitons in fibers.

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