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The minimum size of critical sets in latin squares

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Abstract

A critical set C of order n is a partial latin square of order n which is uniquely completable to a latin square, and omitting any entry of the partial latin square destroys this property. The size $s(C)$ of a critical set C is the number of filled cells in the partial latin square. The size of a minimum critical set of order n is $s(n)$. It is likely that $s(n)$ is approximately $\frac{1}{4}n^2$, though to date the best-known lower bound is that $s(n) \geqslant n+1$. In this paper, we obtain some conditions on C which force $s(C) \geqslant \lfloor (n-1)/2 \rfloor^2$. For $n > 20$, this is used to show that in general $s(n) \geqslant \lfloor (7n-3)/6 \rfloor$, thus improving the best-known result. © 1997 Elsevier Science B.V.

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1. Introduction

A *(partial) latin square* of *order n* is an $n \times n$ array with entries in $S = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that each cell contains (at most) one symbol, and each element of S occurs (at most) once in each row and (at most) once in each column. An *incomplete latin square L* of *order n* on the symbols $\sigma_1, \ldots, \sigma_v$ is an $n \times n$ array in which each cell contains exactly one of the symbols (possibly some symbols occur in no cells), and each row and column contains each symbol at most once; so clearly $n \leqslant v$, and if $n = v$ then L is a latin square. If L is a partial or incomplete latin square, then let $L(i,j)$ be the symbol in cell (i,j) of L (if one exists). A *critical set C* of *order n* and *size* $s(C)$ is a partial latin square of order n, with exactly $s(C)$ filled cells, which satisfies

(1) C is contained in a unique latin square $L(C)$ of order n, and

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(2) if any entry of C is removed then the resulting partial latin square is contained in at least two latin squares of order n.

A critical set of order n with minimum size is called a *minimum critical set*, and its size is denoted by $s(n)$.

The problem of determining $s(n)$ turns out to be very interesting, and is probably a difficult one (Colbourn, 1984). It is well known that every critical set must contain at least one filled cell in each row except possibly for at most one (and in each column except possibly for at most one, and must contain each symbol except possibly for at most one); for if C contains no cells in two rows then in any completion $L(C)$ these two rows can be interchanged. So clearly $s(n) \geqslant n-1$. This lower bound for $s(n)$ was later improved (Donovan et al., to appear) to $s(n) \geqslant n$ for $n \geqslant 4$. More recently, it was independently shown (Cooper et al., 1994; Fu et al., 1995) that $s(n) \geqslant n+1$ for $n \geqslant 5$.

In this paper, we shall further improve the lower bound to $s(n) \geqslant \lfloor \frac{1}{6}(7n-3) \rfloor$ for $n > 20$. Our result is a corollary of a proposition we prove which gives conditions on a critical set C of order n that imply that C has size at least $\lfloor \frac{1}{2}(n-1) \rfloor^2$.

2. The main results

To obtain our results, we will rely on the following theorem.

Theorem 2.1 (Rodger, 1984). *Let $n > 2v$ and $v \geqslant 10$, so $n > 20$. An incomplete latin square A of order v on symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be embedded in a latin square L of order n with completely prescribed diagonal D outside A if and only if*

(1) *$N_A(\sigma_j) \geqslant 2v - n + N_D(\sigma_j)$ for $1 \leqslant j \leqslant n$,*

(2) *for $1 \leqslant j \leqslant n$, if $N_A(\sigma_j) = v$ then $N_A(\sigma_j) + N_D(\sigma_j) \neq n-1$, and*

(3) *if A is a latin square on v symbols, say $\sigma_1, \ldots, \sigma_v$, and if $n = 2v+1$, then $\sum_{j=1}^{v} N_D(\sigma_j) \neq 1$,*

where $N_A(\sigma_j)$ and $N_D(\sigma_j)$ are the numbers of times σ_j occurs in A and D, respectively.

(It is worth noting that a companion to Theorem 2.1 in which D has at least one empty cell (so $\sum_{i=1}^{n} N_D(\sigma_i) < n - v$) has also been proved by Andersen et al. (1980); see also Rodger, 1984, Theorem 4.)

Two partial latin squares are *isotopic* if one can be transformed into the other by permuting rows, permuting columns and renaming the symbols. That is, two partial latin squares L and L' are *isotopic* if there exists an ordered triple (α, β, γ) of permutations such that $L(i,j) = \sigma_k$ if and only if $L'(i\alpha, j\beta) = \sigma_{k\gamma}$; the ordered triple (α, β, γ) is called an *isotopism*. Thus, the following result is clear.

Lemma 2.2 (Donovan et al., to appear). *Let C be a critical set of order n, and let L be the unique latin square of order n containing C. Let (α, β, γ) be an isotopism from the critical set C onto C'. Then C' is a critical set in the latin square L' obtained from L by applying the isotopism (α, β, γ).*

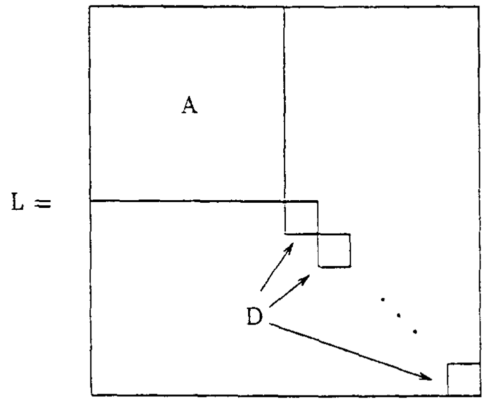

Fig. 1. D is the diagonal of L outside A.

In order to apply Theorem 2.1, we need to arrange for the entries of a critical set in a latin square L to occur inside A and D (see Fig. 1), which we can sometimes do using Lemma 2.2.

Proposition 2.3. *Let $n > 20$ and $v = \lfloor \frac{1}{2}(n-1) \rfloor$. Suppose that C is a critical set of order n, and suppose that C is isotopic to a partial latin square in which if row (column) i contains at least two filled cells then $i \leqslant v$ and for such rows (columns), all filled cells occur in columns (rows) $1, \ldots, v$. Then $s(C) \geqslant v^2$.*

Proof. By Lemma 2.2 and the assumption in this proposition, we can assume that rows and columns $v+1, v+2, \ldots, n$ each contain at most one filled cell of C, and if these rows and columns contain a filled cell then it is a diagonal cell. Let $L = L(C)$ be the unique latin square of order n that contains C. Let A be the incomplete latin square of order v consisting of the first v rows and columns of L, and let D be the diagonal of L outside A (see Fig. 1). Since A is embedded in L, it follows that (1)–(3) of Theorem 2.1 all hold.

Suppose $s(C) < v^2$. Since C contains at least $n - v - 1$ filled cells of D (for any critical set C' there is at most one row containing no filled cells of C'), at most $v^2 - 1 - (n - v - 1)$ cells of A are filled cells in C. So A contains at least $n - v$ cells that are not filled in C.

The number of cells in A containing symbols that are needed to make sure that condition (1) of Theorem 2.1 is satisfied is at most $\sum_{j=1}^{n} \max\{0, N_D(\sigma_j) - 1\} \leqslant n - v - 1$ (since $\sum_{j=1}^{n} N_D(\sigma_j) \leqslant n - v$). So there exists a cell that is not filled in C, and such that if the symbol in this cell in A is replaced by any other symbol to form an incomplete latin square A', then A' also satisfies (1); let c be such a cell. By this choice of c, the symbol σ_c in cell c of A cannot satisfy $N_A(\sigma_c) + N_D(\sigma_c) = n$.

At most $2v - 2$ symbols other than σ_c occur in the row or column of A containing c, and $n \geqslant 2v + 1$, so there is a choice of at least two symbols that can be used to replace the symbol in cell c of A to produce an incomplete latin square A' of order v. Therefore, since clearly at most one symbol σ_j can satisfy $N_A(\sigma_j)+N_D(\sigma_j)=n-2$ (so if σ_j is used to replace σ_c in c to form A' then $N_{A'}(\sigma_j)+N_D(\sigma_j)=n-1$), since when $\sum_{i=1}^{v} N_D(\sigma_i)=1$, at most one symbol σ_k could make A' a latin square when used to replace σ_i in c, and since these two cases cannot happen simultaneously, it is easy to see that we can choose the symbol to replace σ_c in cell c so that A' satisfies (1)–(3) of Theorem 2.1.

Using Theorem 2.1 we can embed A' in a latin square L' of order n in which the diagonal of L' outside A' is D. This is a contradiction, since $L \neq L'$ (they differ at cell c), yet both contain C which is a critical set. $\square$

With the above theorem for $n>20$ it is clear that all the previously known results on the lower bound of the size of critical sets are easy to obtain. We can increase the lower bound on $s(n)$ with the following result.

Theorem 2.4. *Let $n>20$. Then any critical set of order n has size at least $\lfloor \frac{1}{6}(7n-3)\rfloor/6$.*

Proof. Suppose C is a critical set of size at most $\alpha = \lfloor \frac{1}{6}(7n-9)\rfloor$. Since C is a critical set, at least $n-1$ rows and columns in C must contain a filled cell. So at most $\alpha-(n-1)$ rows of C contain at least two filled cells, and each of at least $n-1-(\alpha-(n-1))=2n-2-\alpha$ rows of C contains exactly one symbol. Similarly, there are at most $2(\alpha-(n-1))$ symbols that occur in columns that contain at least one other symbol. Therefore, there is a set D of at least $2n-2-\alpha-2(\alpha-(n-1))=4n-4-3\lfloor(7n-9)/6\rfloor \geqslant 4n-4-(7n-9)/2=(n+1)/2$ cells in C, each of which is the only filled cell in the row and column containing it. So there exists an isotopic partial latin square C' in which the cells in D are permuted to the cells $(n-i,n-i)$ for $0 \leqslant i < |D|$, where $|D| \geqslant \lceil \frac{1}{2}(n+1)\rceil$. It follows that if row or column i of C' contains more than one symbol then $i \leqslant n-|D| \leqslant \lfloor \frac{1}{2}(n-1)\rfloor = v$. Therefore, we can apply Proposition 2.3 to obtain a contradiction, since clearly $\alpha < v^2$. $\square$

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