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Conformal Gravity with Torsion

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We study a conformal effective theory in the presence of the torsion field. A power-law inflationary solution in the constant gauge is obtained in $k = 0$ Friedman-Robertson-Walker space. If $k \neq 0$, a formal result is also presented. Some technique details are discussed and remarked. It appears that the low energy relic from the torsion field deserves more study.

We will study the following conformally (Weyl) invariant effective action[1] with tosrion field[2] conformally coupled to the metric field:

$$S = \int d^4x\sqrt{g}\Big[-\frac{1}{12}\phi^2 R + \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{6\phi^2}F_{abc}F^{abc} - V(\phi)\Big], \tag{1}$$

where ϕ is a real scalar field with a negative kinetic energy coupling. Throughout this paper, we will use a, b, c, . . . to denote space-time indices and i, j, k, . . . to denote spatial indices, i.e. a = 0,1,2,3 while i = 1,2,3 etc... Note that the fist two terms in (1) can actually be induced from $S \equiv \int \sqrt{g}\,[-1/12R]$ by replacing all g_{ab} by $\bar{g}_{ab} \equiv \phi^2 g_{ab}$ such that $S(g) = \int \sqrt{g}\,[-1/12\,\phi^2 R + 1/2\,\partial_a\phi\partial^a\phi]$. Therefore it is obvious that (1) is invariant under the following scale transformation: $\phi' = s^{-1}\phi$ and $g_{ab}' = s^2 g_{ab}$ with $s = s(x)$ being a local scale parameter. Note also that $V(\phi)$ has to be $\lambda/8\,\phi^4$ in order to preserve scale symmetry. In fact, we will find out that the dynamics of the system will force the self coupling of ϕ to take the following form: $V(\phi) = \lambda/8\,\phi^4$.

Moreover, F_{abc} is the curvature tensor of the torsion field A_{ab}. Writing $F \equiv F_{abc}dx^a dx^b dx^c$ and $A \equiv A_{ab}dx^a dx^b$ as three-form and two-form respectively, we find that $F = dA$. The F^2 term (without ϕ^{-2} coupling) is known as the Kalb-Ramond term which is present in the ten dimensional supergravity action.[3] In this paper we will study (1) in four dimensions and discuss the effect of the A_{ab} and ϕ fields on the evolution of the early universe.[4]

The equation of motion for A_{ab} can be derived directly from (1) by the least action principle. The result reads

$$D_a(\phi^{-2}F^{abc}) = 0 \tag{2}$$

Observing that F_{abc} is a totally skew-symmetric type T(0,3) tensor, we can hence write it as $F_{abc} \equiv \varepsilon_{abcd}T^d$ for some type T(1,0) vector T^d with the help of the totally skew-symmetric type T(0,4) Levi-Civita tensor ε_{abcd}. Writing $T \equiv T_a dx^a$ (a one-form), (2) can be written as $*d(\phi^{-2}T) = 0$, thus

$$d(\phi^{-2}T) = 0 \tag{3}$$

in four dimensional space. Here * is the Hodge star operator on differential forms.' If our (pseudo-) Riemannian manifold M belongs to the class for which the first cohomology group is trivial (i.e. $H^1(M) = 0$), one can show that all closed forms on M are exact. Consequently, there exists some scalar field χ such that $\phi^{-2}T = d\chi$. Furthermore, in order to make all coupling constants dimensionless (this is the reason why we want to study scale invariant theory), one can write χ as $\log \eta^l$ for some scalar field η of dimension one. Here l is some dimensionless constant to be determined. Finally,

$$F_{abc} = l\epsilon_{abcd}\phi^2\frac{\partial^d\eta}{\eta} \tag{4}$$

Consequently, the torsion field&b can be read off from (4). We will, however, not need to know Aab explicitly. (4) will be sufficient for our purpose.

The Euler-Lagrange equations for ϕ and g_{ab} read

$$\phi^2[\frac{1}{2}g_{ab}R - R_{ab}]\ t\ [-\nabla_a\partial_b\phi^2\ t\ \nabla_c\partial^c\phi^2 g_{ab}] + 6g_{ab}V$$
$$= (3g_{ab}\partial_c\phi\partial^c\phi - 6\partial_a\phi\partial_a\phi] - \frac{1}{\phi^2}[g_{ab}F^2 - 6F_{acd}F_b^{cd}] \tag{5}$$

$$\frac{1}{6}\phi R + \nabla_a\partial^a\phi - \frac{F^2}{3\phi^3} + \frac{\partial V}{\partial\phi} = 0. \tag{6}$$

In order to write the equations of motion for g_{ab} and ϕ in a more compact form, we will set $\phi \equiv e^{\varphi/2}$ and $\eta \equiv e^{\theta/2}$. After some algebra we have

$$\begin{aligned} G_{ab} &= \partial_a\varphi\partial_b\varphi + D_a\partial_b\varphi - g_{ab}(\partial_c\varphi\partial^c\varphi + D_c\partial^c\varphi) \\ &\quad -T_{ab}(\varphi) - T_{ab}(\theta), \end{aligned} \tag{7}$$

$$R = -\frac{3}{2}(\partial_a\varphi\partial^a\varphi + 2D^a\partial_a\varphi) - 3l^2(\partial_a\theta\partial^a\theta) - 12\frac{\partial V}{\partial\psi}, \tag{8}$$

after inserting the explicit form of F_{abc} as given in (4). Here the Einstein tensor G_{ab}, the generalized momentum tensor of φ and θ (i.e. $T_{ab}(\varphi)$ and $T_{ab}(\theta)$) are defined as $G_{ab} \equiv 1/2\, g_{ab}R - R_{ab}$, $T_{ab}(\varphi) \equiv -3/2[1/2\, g_{ab}\partial_c\varphi\partial^c\varphi - \partial_a\varphi\partial_b\varphi] + 6g^{ab}V/\psi$ and $T_{ab}(\theta) \equiv 3l^2[1/2\, g_{ab}\partial_c\theta\partial^c\theta - \partial_a\theta\partial_b\theta]$ respectively. Also $\psi \equiv \phi^2 = e^{\varphi}$. As we promised before, we find that the potential term V must obey the following equation:

$$(\psi\partial_\psi - 2)V = 0 \tag{9}$$

Note that (9) followed directly from comparing the trace equation of (7) and R given in (8). This means that V = $\lambda/8\,\psi^2$ since $\psi\partial_\psi$ is the ψ power counting operator. Note also that equation (9) is nothing but a dynamical constraint born with the conformal symmetry of (1). The presence of the ϕ^4 coupling will complicate our computations. At the moment, we are not able to make any definite conclusions if $\lambda \neq 0$. Therefore, in the remainder of this letter, we will simply assume λ = 0 (hence V = 0). Consequently the theory (1) has effectively a zero cosmological constant. We will not address the problem of the cosmological constant here. It is, however, very natural to assume that (1) is merely an effective Weyl invariant action with torsion and vanishing cosmological constant. We will ignore the V term by assuming that the vanishing of the cosmological constant was achieved by some other mechanism well before (1) became effective. We hope that (1) has to do with the evolution of our early universe.

Furthermore, the Bianchi identity D_aG^{ab} = 0 will put a constraint on the field equation (7). After some algebra we find

$$\partial_a\theta\partial^a\varphi + D_a\partial^a\theta = 0. \tag{10}$$

Note that the Robertson-Walker metric[4] is defined by

$$ds^2 \equiv g_{ab}dx^a dx^b = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right). \tag{11}$$

Here $d\Omega = d\theta^2 + \sin^2\theta d\chi^2$ denotes the solid angle, and k = 0, ± 1 stands for a flat, closed or open universe respectively. Note that the dx^a defined above is a bosonic differential in contrast to the differential form used earlier in this paper.

One can show that θ and φ can only be a function of t if θ and φ are assumed to have maximal symmetry.[4,6] Writing a = e^α, we can write the G_{00}-part of (7) and (10) as

$$\alpha'^2 + \alpha'\varphi' + \frac{k}{a^2} = -\frac{1}{4}\varphi'^2 + \frac{l^2}{2}\theta'^2, \tag{12}$$

$$\theta'' + 3\alpha'\theta' + \theta'\varphi' = 0, \tag{13}$$

after some algebra. It is known that four (in fact one, due to the symmetry of the R-W metric) out of the equations (5-6) is redundant due to the Bianchi identity D_aG^{ab} = 0. But, a careful analysis shows that every equation is equally redudant except the tt component of (5), which is known as the generalized Friedman equation (12). This is readily understood by observing that the Friedman equation is in fact a first order ODE, in contrast to other equations which are all second order ones. In fact, (7) takes the form H_{ab} = 0 where $H_{ab} \equiv G_{ab} - K_{ab}$ with K_{ab} denoting what appear on the right hand side of (7). Consequently, the Bianchi identity can be rephrased as

$$D_a H^{ab} = 0, \tag{14}$$

on shell. This is because $D_a G^{ab}$ = C due to the Bianchi identity and $D_a K^{ab} = 0$ as an on shell constraint. The equation (14) will become

$$(\partial_t + 3\alpha')H^{00} + 3aa'H = 0. \tag{15}$$

as soon as the R-W metric is substituted into (14). Here $H \equiv 1/3\, h^{ij} H_{ij}$ while $g_{ij} \equiv a^2 h_{ij}$. In fact, it is straightforward to show that $H_{ij} = H h_{ij}$ in this theory under the constraint that θ and φ are both spatial independent. The exclusive role played by the tt equation of $H_{ab} = 0$ can be readily checked at this moment. Indeed, (15) indicates that: $H_{00} = 0$ will imply $H = 0$ if $a' \neq 0$. On the other hand, $H = 0$ implies

$$(\partial_t + 3\alpha')H_{00} = 0. \tag{16}$$

(16) can hence be integrated directly to give

$$a^3 H_{00} = \text{constant}, \tag{17}$$

which is not sufficient to deduce the desired result $H_{00} = 0$. Therefore the generalized Friedman equation (12) is indeed an exclusive equation of motion. Hence we are free to exclude **any** redundant one among the whole set of equations of motion except the generalized Friedman equation. For later convenience, we will stick to (12-13) by ignoring the ij equation of (7) without loose ends.

Note also that in this theory the EOM (8) is in fact redundant which is equivalent to the constraint (9). Therefore we have effectively three variables and two equations. This is a general feature of conformal actions which indicates that we have an extra freedom of gauge choice. Moreover, the torsion field actually can not be determined completely by the dynamics of the system. It is a rather unusual situation that θ (hence φ and A_{ab}) can only be determined by phenomenological contraints (see below). Note that (13) can be integrated to give

$$\theta' = k_1 e^{-3\alpha-\varphi} \tag{18}$$

where k_1 is an integration constant to be determined. Moreover writing $\beta = a + \varphi/2$, (12) becomes

$$\beta'^2 = \frac{k_1^2 l^2}{2} e^{-6\beta-\varphi} - k e^{-2\beta-\varphi}. \tag{19}$$

Let $ds/dt = e^{\varphi/2}$, (19) can be simplified as

$$(\frac{d\beta}{ds})^2 = \frac{k_1^2 l^2}{2} e^{-6\beta} - k e^{-2\beta}, \tag{20}$$

by assuming $\varphi = \varphi(s(t))$ and $\beta = \beta(s(t))$. Let $b \equiv e^{\beta(s)}$, then (20) becomes

$$b'^2 = \frac{k_1^2 l^2}{2} b^{-4} - k, \tag{21}$$

where $b' \equiv \partial b/\partial s$ from now on. If k = 0, one fmds that $b' = \pm k_1 l/\sqrt{2}\, b^{-2}$. Hence

$$b = \left[b_0^3 \pm \frac{3}{\sqrt{2}} k_1 l s\right]^{\frac{1}{3}}, \tag{22}$$

if k = 0. Here b_0 is an integration constant. If $k \neq 0$ one has

$$s(t) = \pm(\frac{k_1^2 l^2}{2})^{\frac{1}{4}} \int_{b_0(\frac{k_1^2 l^2}{2})^{-\frac{1}{4}}}^{b(\frac{k_1^2 l^2}{2})^{-\frac{1}{4}}} dy \frac{y^2}{\sqrt{1 - k y^4}}. \tag{23}$$

Here $y \equiv (2/k_1^2 l^2)^{1/4} b$. Note that the integration in (23) can be integrated to give combinations of elliptic functions.[7] Since the explicit form of b(s) is not easy to express, we will instead put (23) in a more compact form:

$$s(t) = \pm(\frac{k_1^2 l^2}{32})^{\frac{1}{4}} \int_{f^{-1}(\frac{\sqrt{2} b_0^2}{k_1 l})}^{f^{-1}(\frac{\sqrt{2} b^2}{k_1 l})} dz \sqrt{f(z)} + s_0, \tag{24}$$

which is a formal expression ready for computer programming. Here f(z) = sin(z) if k = 1 and f(z) = sinh(z) if k = -1. Also $z \equiv y^2$. Though we have formally obtained all solutions regardless of the signature of k, it is not easy to analyse the behaviour of a if $k \neq 0$. We will only discuss the k = 0 case for the moment. Since $ds/dt = \exp(\varphi/2)$, we have $s(t) = \int_0^t \exp(\varphi(t')/2)dt' + s_0$, and therefore

$$a = e^{-\frac{\varphi(t)}{2}}[\exp(\frac{3\varphi_0}{2})a_0^3 \pm \frac{3}{\sqrt{2}} k_1 l (\int_0^t \exp(\frac{\varphi(t')}{2})dt' + s_0)]^{\frac{1}{3}} \tag{25}$$

Here $\varphi_0 \equiv \varphi(t = 0)$ and $a_0 \equiv a(t = 0)$. If $\exp(\varphi/2) = k_2 = \text{const}$, one has

$$a = [a_0^3 \pm \frac{3 k_1 l}{\sqrt{2} k_2^2} t]^{\frac{1}{3}} \tag{26}$$

This is a power-law inflationary solution if $k_1 l/k_2^2 >> 1$. If $k \neq 0$ and $\exp(\varphi/2) = k_2$, (24) gives

$$t = \pm(\frac{k_1^2 l^2}{32 k_2^4})^{\frac{1}{4}} \int_{f^{-1}(\frac{\sqrt{2} k_2^2 a_0^2}{k_1 l})}^{f^{-1}(\frac{\sqrt{2} k_2^2 a^2}{k_1 l})} dz \sqrt{f(z)} + s_0, \tag{27}$$

(27)

Note that different choice of φ corresponds to different choice of gauge in the theory (1). It appears that the gauge choice has to await more informations from cosmological observation to depict the most appropriate gauge choosing. Therefore the conformal symmetry breaking can only be fixed by comparing our theory with cosmological observation.

In summary, we study a conformal effective theory in the presence of the torsion field. A power-law inflationary solution in the constant gauge is obtained in k = 0 Friedman-Robertson-Walker space. If k $\neq$ 0, a formal result is also presented. Some technique details are discussed and remarked. It appears that the low energy relic from the torsion field deserves more study.

REFERENCES

1. L. Smolin, Nucl. Phys. **B160**, 253 (1979); A. Zee, Phys. Rev. Lett. 42,417 (1979); 44,703 (1980); F. S. Accetta, D. J. Zoller and M. S. Turner, Phys. Rev. **31**, 3046 (1985); A. S. Goncharov, A. D. Linde and V. F. Mukhanov, Int. J. Mod. Phys. **A2**, 561 (1987); H. Cheng, Phys. Rev. Lett. **61**, 2182 (1988); H. Cheng and W. F. Kao, "Consequences of Scale Invariance" (MIT preprint, 1988); W. F. Kao, Chinese J. Phys. 29, 1 (1991); Phys. Lett. **154A**, 1 (1991); Phys. Lett. **147A**, 165 (1990).
2. Y. S. Myung and Y. D. Kim, Phys. Lett. **145B**, 45 (1984); J. A. Stein-Schabes and M. Gleiser, Phys. Rev. **D34**, 3242 (1986).
3. G. F. Chapiine and N. S. Manton, Phys. Lett. **120B**, 105 (1983); C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Nucl. Phys. **B262**, 593 (1985).
4. S. Weinberg, "Gravitation And Cosmology" (Wiley, New York, 1972); C. W. Misner, K. Thorne and J. A. Wheeler, "Gravitation" (Freeman, SF, 1973); Robert M. Wald, "General Relativity", (Univ. of Chicago Press, Chicago, 1984); E. W. Kolb and M. S. Turner, "The Early Universe" (Addison-Wesley, 1990); E. W. Kolb and M. S. Turner, Ann. Rev. Nucl. Part. Sci. 33,645 (1983).
5. N. Steenrod, "The Topology of Fibre Bundle" (Priceton Univ. Press, 1970); T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rep. 66, 213 (1980); C. Nash and S. Sen "Topology And Geometry for Physicists" (Academic Press, 1983).
6. W. F. Kao, "Spatial Dependence of Inflaton in the Presence of the Torsion Field" CTUMP-7 (Preprint, 1991).
7. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series and Products", (Translated from Russian, 1965, to English by Scripta Technica, Inc., 1980, Edited by A. Jeffrey).