Approximate Solutions for Transient Response of a Shell and Tube Heat Exchanger

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Approximate solutions for the transient response of a shell and tube heat exchanger to step change in shell fluid temperature and tube fluid velocity are derived. It has been shown that other previously published limited or approximate solutions are derivable from the exact solutions. The limiting asymptotic "no wall" solutions for both response to shell fluid temperature and tube fluid velocity change are also deducible from a limiting tube wall time constant solution. Some of the approximate solutions presented include the use of simple exponential forms of equations that enable the quick estimation of the time required to reach a given degree of approach to the final steady state. The error between the exact and the approximate exponential form of solutions are in most cases less than 2%. All the previous derived exact and approximate solutions presented here are applicable to crosscurrent, cocurrent, or countercurrent flow heat exchangers with one infinite thermal capacitance rate fluid.

Introduction

The design, operation, and control of heat exchangers would be facilitated by equations describing the transient response. Unfortunately the basic dynamic equations involve simultaneous partial differential equations for which the analytic solutions are complex and difficult to obtain. In most cases, numerical methods such as method of characteristics (Tan and Spinner, 1984) are used to calculate the transient response for heat exchanger with cocurrent, crosscurrent, or countercurrent flow.

For the case of heat exchangers with one infinite capacitance rate fluid, London et al. (1959) present some results from analog simulation. Myers et al. (1970) gave extensive results from finite difference calculations. Rizika (1956) gave an exact analytic solution for this case, but the solution is applicable only for time less than one residence time. Limiting and approximate solutions were also presented by London et al. (1959) and Myers et al. (1970). All the above studies deal only with response to fluid temperature change.

Exact analytic solutions for the transient response of a shell and tube heat exchanger to change in both fluid temperature and tube fluid velocity changes have been previously published (Tan and Spinner, 1978). Shah (1981) has stated that these solutions are equally applicable to a heat exchanger having an infinite capacitance rate fluid. Graphs prepared from these exact solutions have also been published (Tan and Spinner, 1978; Shah, 1981).

The purpose of this paper is to present some useful approximate solutions which give results that are in good agreement with the exact solutions. The applicability of these solutions with regard to the values of the design parameters will be investigated, and the magnitude of error introduced compared to the exact solution will be discussed. Limiting asymptotic solutions as applied to the case in which the product of Cf (limiting dimensionless tube wall time constant) becomes very small will be examined. It will be shown that all the previous published approximate solutions for response to shell fluid temperature change are directly derivable from these exact solutions. It will be shown that the limiting "no wall" solution is obtainable from the approximate solution for a small value of the product Cf.

The use of a simple exponential form of equation as an approximate solution not only facilitates the evaluation of the transient history of the exchanger but also provides a simple method of estimating the time required to reach a given degree of attainment of the final steady state. For compactness, all the derived new approximate solutions will be expressed in terms of the previous notation (Tan and Spinner, 1978).

Transient Response to Step Change in Shell Fluid Temperature

In a previous paper Tan and Spinner (1978) derived the following exact solution to the transient response of step change in shell fluid temperature:

$$U = \frac{1}{T^{\infty}}\left[1 - \frac{R_1}{R_1 - R_2}\exp(-R_2\alpha\theta^*) + \frac{R_2}{R_1 - R_2} \times \exp(-R_1\alpha\theta^*) + H(\theta^* - 1)\frac{R_1}{R_1 - R_2}\exp(-R_2\alpha\theta^*)\phi_1\left[\left(\frac{1}{Cf} - R_2\right)(\theta^* - 1)\alpha, \frac{f\alpha}{1 - CfR_2}\right] - H(\theta^* - 1) \times \exp[-(1 - f)\alpha]\phi_1\left[\frac{(\theta^* - 1)\alpha}{Cf}, f\alpha\right] + H(\theta^* - 1)\frac{R_2}{R_1 - R_2} \times \exp\left[-\alpha - \left(\frac{2}{Cf} - R_1\right)(\theta^* - 1)\alpha + \frac{f\alpha}{CfR_1 - 1}\right]\psi\left[\left(R_1 - \frac{1}{Cf}\right)(\theta^* - 1)\alpha, \frac{f\alpha}{CfR_1 - 1}\right]\right] \quad (1)$$

where U is the fractional attainment of the final steady state, with

$$R_1 = \frac{1}{2}\left[\left(1 + \frac{1}{Cf}\right) + \left[\left(1 + \frac{1}{Cf}\right)^2 - \frac{4(1 - f)}{Cf}\right]^{1/2}\right]$$

$$R_2 = \frac{1}{2}\left[\left(1 + \frac{1}{Cf}\right) - \left[\left(1 + \frac{1}{Cf}\right)^2 - \frac{4(1 - f)}{Cf}\right]^{1/2}\right]$$

Since C is the dimensionless thermal capacity ratio of tube wall to tube fluid and f is the fractional heat-transfer resistance on the shell side, then Cf is the dimensionless tube wall time constant and $1/R_1$ and $1/R_2$ are the "effective time constants" for the coupled tube fluid and wall system.

The normalized final steady state value is given by

$$\bar{T}^{\infty} = 1 - \exp[-(1 - f)\alpha]$$

The two mathematical functions ϕ_1 and ψ are defined by

$$\phi_1(x,y) = \int_0^x \phi_0(\lambda,y)\, d\lambda = 1 - J(x,y)$$

where $\phi_0(x,y) = \exp(-x - y)I_0[2(xy)^{1/2}]$

$$\psi(x,y) = \int_0^x \exp[-2(x - \lambda)]\phi_0(\lambda,y)\, d\lambda$$

Both J and ψ can be easily generated by the following simple algorithms:

$$\phi_1(x,y) = \exp(-y)\sum_0^{\infty}\frac{y^k X_k(x)}{k!k!}$$

$X_0 = 1 - \exp(-x)$ and $X_k = kX_{k-1} - x^k \exp(-x)$ for $k \geq 1$

$$\psi(x,y) = \exp(-x - y)\sum_0^{\infty}\frac{y^k X_k(x)}{k!k!}$$

$X_0 = 1 - \exp(-x)$ and $X_k = x^k - kX_{k-1}$ for $k \geq 1$

Note that J is a well-known tabulated function whose values are readily obtainable (Helfferich, 1962; Sherwood et al., 1975). It was mentioned in the previous publication (Tan and Spinner, 1978) that neglecting the term containing the ψ function would lead to at most 2% relative error for a practical range of parameters. More specifically, the range was $1 < \alpha < 3$, $C < 3$, $f < 0.2$. Thus eq 1 without the ψ term is a good approximation for design purposes since J or ϕ_1 are known tabulated functions and the numerical evaluation could be readily carried out if tables were available. Alternatively the J function could be easily calculated with a programmable calculator or microcomputer using the algorithms given.

For engineering application, it would be worthwhile to consider the possibility of using an equation that is in a simpler form. Myers et al. (1970) had successfully used a first-order exponential function to approximate the transient response for $\theta^* \geq 1$. They compared 27 cases of solution with those obtained from a finite difference numerical method. The difference between the exponential form of equation and the finite difference solution was almost indistinguishable on the graphs presented. Since exact analytic solutions are available, it would not be difficult to compare the approximate solution for an extensive range of values of the parameters C, f, and α. The method used by Myers et al. (1970) in formulating an exponential form of equation as an approximating solution was followed. However, a more compact equation for the evaluation of the constant used in the exponential function was obtained. The derivation by Myers et al. was based on the exact solution for transient response for the first time domain as presented by Rizika (1956). In this regard it should be noted that the first three terms of eq 1 which provide the first time domain ($\theta^* < 1$) response are readily rearranged in the following form:

$$U = \frac{1}{\bar{T}^{\infty}}(1 - \exp(-X)[\sinh(X/Y) + \cosh(X/Y)]) \quad (2)$$

$$X = \frac{(R_1 + R_2)\alpha\theta^*}{2} = \frac{1 + Cf}{2Cf}\alpha\theta^*$$

$$Y = \frac{R_1 + R_2}{R_1 - R_2} = \frac{1 + Cf}{Cf}\left[\left(1 + \frac{1}{Cf}\right)^2 - \frac{4(1 - f)}{Cf}\right]^{-1/2}$$

The first three terms of eq 1 are thus identical with the analytical solution derived by Rizika (1956). However, eq 2 not only provides the exact solution for the first time domain ($\theta^* < 1$) but also constitutes part of the solution for the entire time domain. The approximate solution proposed by Myers et al. could thus be expressed in the following form using the present authors' notation:

$$U = 1 - (1 - U_1)\exp[-K(\theta^* - 1)] \quad (3)$$

An equivalent form of eq 3 is

$$U = U_1 + (1 - U_1)(1 - \exp[-K(\theta^* - 1)]) \quad (3A)$$

with

$$U_1 = \frac{1}{\bar{T}^{\infty}}\left[1 - \frac{R_1}{R_1 - R_2}\exp(-R_2\alpha) + \frac{R_2}{R_1 - R_2}\exp(-R_1\alpha)\right]$$

where U_1 is the fractional attainment of the final steady state at one throughput time.

Since $1 - U_1$ represents the fraction of remaining transient after one throughput time has elapsed, the second term of eq 3A can be viewed as a simple exponential growth function that describes the transients of the term $1 - U_1$ for the time region of $\theta^* \geq 1$. Equation 3 contains only one adjustable parameter, K, and therefore can be readily determined from one independent relationship. Myers et al. obtained the value of K by matching the derivative of U with respect to θ^* at $\theta^* = 1$. Since eq 3 is to be applied for the time domain of $\theta^* \geq 1$, it would be more appropriate to use the derivative evaluated from eq 1 for matching purposes. After differentiating eq 1 for $\theta^* \geq 1$ and letting θ^* approach 1, the following value of K was obtained:

$$K = \frac{R_1R_2[\exp(-R_2\alpha) - \exp(-R_1\alpha)]\alpha}{\bar{T}^{\infty}(1 - U_1)(R_1 - R_2)} \quad (4)$$

By substituting the values for U_1 and $\bar{T}^{\infty}$

$$K = \frac{R_1R_2[\exp(-R_2\alpha) - \exp(-R_1\alpha)]\alpha}{R_1\exp(-R_2\alpha) - R_2\exp(-R_1\alpha) - (R_1 - R_2)\exp[-(1 - f)\alpha]} \quad (4A)$$

Myers et al. (1970) used Rizika's exact solution for the first time domain for determining the slope at $\theta^* = 1$. The value of K expressed in the same hyperbolic functions but using the present authors' notation is

$$K = \frac{2C(1 - f)\alpha}{\bar{T}^{\infty}(1 - U_1)(1 + Cf)}Y\exp(-Z)\sinh(Z/Y) \quad (4B)$$

where

$$Z = \frac{1 + Cf}{2Cf}\alpha$$

and Y is defined in eq 2.

Equations 4 and 4B are identical; however, it is felt that the form of eq 4 is more convenient to use. It is noted that in eq 4 the "effective time constant" of the approximate equation consists of $\bar{T}^{\infty}(1 - U_1)$, which is the fraction of remaining transient after $\theta^* \geq 1$ and the quantity, $(R_1 - R_2)/R_1R_2$, which is the difference of the two "effective time constants", $1/R_1$ and $1/R_2$. The constant K incorporates all the parameters C, f, and α. The fact that the derivatives $dU/d\theta^*|_{\theta^*=1}$ are identical when derived from both time domain solutions (as shown in Appendix 1) verifies that there is continuity in both U and its derivative at $\theta^* = 1$. The advantage of eq 3 is its simplicity. For engineering application, it is useful to obtain a rough estimation of the time required to reach a certain degree of fractional attainment of the final steady state. From eq 3 the time required to obtain U fraction of the final steady state is

$$\theta^* = 1 + \frac{1}{K} \ln \left(\frac{1 - U_1}{1 - U} \right) \quad (5)$$

For large values of α and Cf, e.g., $\alpha > 10$, $Cf > 2$, and $f < 0.5$, it can be shown that K is approximately equal to $R_2\alpha$, thus simplifying the evaluation of value of K for eq 3. Extensive calculations with eq 3 for more than 100 cases of different values of C, f, and α showed very good agreement with the exact solution. Discussion of these calculated results and the criteria for using this approximate design equation will be covered in a later section. The drawback in using eq 3 is the need for preliminary calculation of R_1, R_2, $\bar{T}^\infty$, and $1 - U_1$ before K can be determined. In addition, the basic parameter effects cannot be readily determined. Ideally if the values of R_1 and R_2 could be related to C and f without use of the quadratic form, the quantitative effects of C and f could be more easily interpreted.

As pointed out previously, the terms $1/R_1$ and $1/R_2$ are the "effective time constants" for the coupled tube fluid and wall system. The values of these effective time constants depend on both f and the product Cf. From the expressions given for R_1 and R_2, it can be shown that the limiting values of R_1 are given by $1 < R_1 < 1 + 1/Cf$. Since $R_1R_2 = (1 - f)/Cf$, the limiting values of R_2 are determined by $(1 - f)/CfR_1$. For relatively small values of Cf, R_1 is approximately equal to $1/Cf + f$ and R_2 is approximately equal to $(1 - f)/(1 + Cf^2)$. The error in the estimation of R_1 and R_2 using these approximations is less than 1%. The coefficients of the exponential term $\exp(-R_1\alpha\theta^*)$ and $\exp(-R_2\alpha\theta^*)$ in eq 1 can also be simplified to $[Cf(1 - f)]/(1 + Cf^2)$ and $(1 + Cf)/(1 + Cf^2)$, respectively. Hence for very small value of Cf, say $Cf < 0.1$, the first time domain ($\theta^* < 1$) solution can be approximated by

$$U_1 = \frac{1}{\bar{T}^\infty}\left(1 - \frac{1 + Cf}{1 + Cf^2} \exp\left[-\left(\frac{1 - f}{1 + Cf^2}\right)\alpha\theta^*\right] + \frac{Cf - Cf^2}{1 + Cf^2} \exp\left[-\left(\frac{1 + Cf^2}{Cf}\right)\alpha\theta^*\right]\right) \quad (6)$$

Although in most cases for $Cf < 0.1$ more than 90% of the final steady state value is reached at $\theta^* = 1$, eq 3 can be used to calculate the remaining transient if desired. Here U_1 in eq 3 is calculated from

$$U_1 = \frac{1}{\bar{T}^\infty}\left(1 - \frac{1 + Cf}{1 + Cf^2} \exp\left[-\left(\frac{1 - f}{1 + Cf^2}\right)\alpha\right] + \frac{Cf - Cf^2}{1 + Cf^2} \exp\left[-\left(\frac{1 + Cf^2}{Cf}\right)\alpha\right]\right)$$

and the value of K in eq 3 is calculated from

$$K = \frac{\alpha}{\bar{T}^\infty} \frac{(1 - f)}{(1 - U_1)}\left(\exp\left[-\left(\frac{1 - f}{1 + Cf^2}\right)\alpha\right] - \exp\left[-\left(\frac{1}{Cf} + f\right)\alpha\right]\right) \quad (4C)$$

Calculations using eq 6 (for $\theta^* < 1$) and eq 3 (for $\theta^* \geq 1$) for $\alpha > 0.5$ and $Cf < 0.1$ result in at most 2% error.

Limiting "No Wall" and Small *Cf* Solutions. For extremely small values of C and f where both C and f are finite, practically no transient exists after the first time domain. In such a case the third term in eq 6 could become negligible and a two-term equation would be sufficient to describe the dynamic response. Thus, for $Cf < 0.05$, the third term in eq 1 is negligible. In this case the first "effective time constant", $1/R_1$, has become very much smaller than the second "effective time constant", $1/R_2$. Finally, for Cf approaching zero, the solution reduces to the "no wall" solution. This latter solution was obtained (Tan and Spinner, 1978) on the assumption that either C or f is zero, i.e., conditions different from the cases where the value $1/Cf$ is large but both C and f are finite. The final "no wall solution" can thus be obtained by substituting either C or f equal to zero in eq 6.

Asymptotic Solution for Large Value of *C* and *Cf*. For very large C ($C > 100$) and assuming f is not too small ($Cf > 20$), the value of R_1 approaches 1 and R_2 approaches $(1 - f)/Cf$. Since R_2 is extremely small compared to R_1, the coefficients of exponential terms $\exp(-R_1\alpha\theta^*)$ and $\exp(-R_2\alpha\theta^*)$ are approximately equal to zero and unity, respectively. The value of $1/Cf - R_2$ can be shown to approach $1/C$. Thus, for large values of Cf, the asymptotic solution as obtained from eq 1 is

$$U = \frac{1}{\bar{T}^\infty}\left(1 - \exp\left[-\frac{(1 - f)\alpha\theta^*}{Cf}\right] - H(\theta^* - 1)\exp[-(1 - f)\alpha]\phi_1\left(\frac{\theta^* - 1}{Cf}\alpha, f\alpha\right) + H(\theta^* - 1)\exp\left[-\frac{(1 - f)\alpha\theta^*}{Cf}\right]\phi_1\left[\frac{\theta^* - 1)}{C}\alpha, \alpha\right]\right) \quad (7)$$

Myers et al. (1970) derived an analytic equation for the transient response for shell fluid temperature change for very large values of C, by neglecting the transients for the first time domain. The derived equation was identical with eq 7 except for the second term in which $\theta^* - 1$ rather than θ^* appeared in the argument of exponential term. It should be noted that eq 7 contains the solution of $\theta^* < 1$ even though the transient for large values of C and Cf is small. For large values of C and Cf, large values of θ^* are required before the attainment of the final steady state. Thus eq 7 and the analytic equation of Myers et al. give almost identical results of response for large C, Cf, and θ^*.

If eq 3 is used to calculate transients for large values of C, the expression for K can further be simplified. For $C > 100$ and $Cf > 20$, an approximate expression for K is

$$K = \frac{[(1 - f)/f][1 - \exp(-\alpha)]\alpha}{\bar{T}^\infty C} \quad (4D)$$

with the value of U_1 calculated from the first two terms of eq 7.

Examination of eqs 3 and 4D shows that for large values of C and Cf there is justification for normalizing the time variable in terms of $(\theta^* - 1)/C$ in the manner used by Myers et al.

Transient Response to Step Change in Tube Fluid Velocity

Following a procedure similar to that used for the response to shell fluid temperature change, an exponential form of equation can be formulated as an approximate solution for response to step change in tube fluid velocity. Unlike the case for response to shell fluid temperature change, the derivative of U with respect to θ^* for the time domain of $\theta^* < 1$ and $\theta^* \geq 1$ do not match at $\theta^* = 1$ (see

Appendix 2 for the exact solution and the evaluation of its time derivative). Matching the slope of the exponential form of equation at $\theta^* = 1$ with that obtained from the second time domain solution, the following exponential constant was obtained:

$$K = \frac{[AR_4 \exp(-R_4\alpha) - BR_3 \exp(-R_3\alpha)]\alpha}{(1 - U_1)\bar{T}^\infty} + \frac{(BR_3 - AR_4)\alpha}{(1 - U_1)\bar{T}^\infty} \exp[-\alpha - \alpha(1 - f^*)V^n/V] \quad (4E)$$

with

$$U_1 = \frac{1}{\bar{T}^\infty}[1 - A \exp(R_4\alpha\theta^*) + B \exp(R_3\alpha\theta^*)]$$

where U_1 is the fractional attainment of the final steady state and A, B, R_3, R_4 and $\bar{T}^\infty$ are defined in Appendix 2.

It should be noted that the first part of eq 4E corresponds to the value obtained by matching the time derivative of eq 3 with that obtained by differentiating the exact solution for $\theta^* < 1$. The discontinuity in $\mathrm{d}U/\mathrm{d}\theta^*$ of the exact solution at $\theta^* = 1$ is indicated by the presence of the second part of eq 4E. Since the response to tube fluid velocity change involves more parameter effects compared to the response to shell fluid temperature change, an exponential fit with one single constant would not be expected to be valid for as wide a range of parameters. For practical purposes, only the cases for which C and α are not greater than 10 have been investigated. A large number of applications are within this range of parameter values, and hence the approximate solution could be very useful provided that the error introduced is tolerable. To use eq 3 as an approximation solution, we evaluate the value of K from eq 4E. Preliminary calculations indicate that use of the value of only the first part of K in eq 4E would lead to large error. Because of the unrealistic assumption of constant heat transfer coefficient, the study was confined to the case of $n = 0.8$. Extensive calculations showed that good agreement with the exact solution was obtainable for many practical ranges of design parameters. The calculated results and the constraints on the parameter values will be discussed later. In the previous paper (Tan and Spinner, 1978) numerous calculations for the range of $C < 3$ and $f < 0.2$ and for $\alpha > 1$ were carried out. In all those calculations it was found that dropping the ψ-function term in the exact solution resulted in at most 2% relative error. In this work, we found that dropping of this term was also justified for other ranges of parameters (see later section).

Small Cf and Limiting "No Wall" Solution. Like the response to shell fluid temperature change, the transient response to the velocity change depends on the values of the two "effective time constants", $1/R_3$ and $1/R_4$. For small values of Cf, it can be shown that R_3 approaches the value of $1 - (1 - f^*)V^n/V$ while R_4 approaches the value of $[1 - f - (1 - f^*)V^n/V]/[Cf[1 - (1 - f^*)V^n/V]]$. Likewise the coefficients of $\exp(-R_3\alpha\theta^*)$ and $\exp(-R_4\alpha\theta^*)$ approach zero and unity, respectively. Thus for small values of Cf, say $Cf < 0.1$, the following approximate equation can be used for $\theta^* < 1$:

$$U = \frac{1}{\bar{T}^\infty}\left(1 - \exp\left[-\frac{1 - f - (1 - f^*)V^n/V}{1 + Cf^2}\alpha\theta^*\right]\right) \quad (8)$$

For small values of Cf, the final steady state is essentially reached at the end of one throughput time. However, if transients for $\theta^* \geq 1$ are needed, then eq 3 can also be used. In this case the values of U_1 and K obtained from eq 8 are

$$U_1 = \frac{1}{\bar{T}^\infty}\left(1 - \exp\left[-\frac{1 - f - (1 - f^*)V^n/V}{1 + Cf^2}\alpha\right]\right) \quad (8A)$$

$$K = \frac{\alpha}{\bar{T}^\infty}\left[\frac{1 - f - (1 - f^*)V^n/V}{1 + Cf^2}\right] \times \exp\left[-\frac{1 - f - (1 - f^*)V^n/V}{1 + Cf^2}\alpha\right] \quad (4F)$$

Calculation indicates that for $Cf < 0.1$, $V < 1.5$, and $\alpha > 1$ the relative error is at most 2%.

For either C or f approaching zero as the other remains finite, eq 8 reduces to the "no wall" solution. Thus even with both C and f finite, a solution similar to the "no wall" solution can be obtained. For $Cf < 0.01$, the exact "no wall" solution (Tan and Spinner, 1978) is applicable since substituting zero for Cf instead of any values less than 0.01 in eq 8 would hardly give any significant error.

Comparison of Approximate and Exact Solutions

Response to Step Change in Shell Fluid Temperature Change. For large values of f, dropping out the ψ term in the exact solution provides an approximate solution with a small error. For example, for $f < 0.9$, the relative error is less than 1% if $\alpha > 3$ and 5% if $\alpha > 2$.

Calculations for values of Cf less than 0.1 using the approximate solution of eq 6 showed at most a relative error of about 2%. The approximate solutions of eq 7 for large values of C and Cf were also checked against the exact solution. For $C > 100$ and $Cf > 20$, it was found that the relative error was about 3%. Because of the simplicity of evaluating K, it is recommended that eq 3 instead of eq 7 to be used for large values of C and Cf.

We have found that eq 3 could be used for any values of α and C with $f < 0.5$. The resulting relative error compared to the exact solution is at most 2%. A maximum relative error of 5% is obtained for $0.5 < f < 0.7$, 7% for $0.7 < f < 0.8$, 10% for $0.8 < f < 0.9$, and up to 18% for $0.9 < f < 0.99$. The error is smaller when α is small. Thus, for any value of C and for $f < 0.9$, the relative error is no more than 2%, if $\alpha < 1$ and 5% if $1 < \alpha < 2$. It was found that the largest error occurred when f was close to 1 and α was between 3 and 8. Some typical calculated results abstracted from more than 100 calculations using the semiempirical exponential form of eq 3 are given in Table I.

Response to Tube Fluid Velocity Change. For small values of C, large values of α^*, and either small or large values of f, neglecting the ψ-function term in the exact solution (Appendix 2) gave very small error. For $\alpha^* > 1$, $f^* < 0.2$, and $C < 5$, the relative error was less than 2%, while for $\alpha^* > 2$, $f^* < 0.9$, and $C < 5$, the relative error was less than 1%.

For values of Cf less than 0.1, use of the asymptotic solution given in eq 8 for $\theta^* < 1$ and the approximate solution of eqs 8A and 4F for $\theta^* \geq 1$ led to about 2% relative error.

The use of the semiempirical exponential form of equation led to at most 5% relative error for a large range of practical parameters values. For $1.5 < \alpha^* < 4$, $C < 10$, $f^* < 0.5$, and $V < 1.5$, the relative error is at most 2%. The range of applicability for α^* and f^* can be extended when smaller values of C are used. For example, when $C < 5$, $f^* < 0.9$, and $0.5 < \alpha^* < 2.5$, less than 5% error was observed. Some typical calculated results using eq 3 for the range of parameters examined are presented in Table II.

Table I. Typical Calculated Results for Response to Shell Fluid Temperature Change: Comparison of Exact Solution and Approximate Solution of Eq 3

$C = 1.0, f = 0.2, \alpha = 1.0$			$C = 3.0, f = 0.7, \alpha = 3.0$			$C = 5.0, f = 0.5, \alpha = 5.0$		
θ^*	eq 3	exact	θ^*	eq 3	exact	θ^*	eq 3	exact
1.00	0.827 18	0.827 18	1.00	0.353 08	0.353 08	1.00	0.531 01	0.531 01
1.05	0.860 81	0.860 84	1.50	0.527 86	0.532 78	1.25	0.630 86	0.631 88
1.10	0.887 89	0.888 00	2.00	0.655 42	0.668 94	1.50	0.709 44	0.712 62
1.15	0.909 71	0.909 89	2.50	0.748 51	0.769 30	2.00	0.819 99	0.827 57
1.20	0.927 28	0.927 54	3.00	0.816 46	0.841 58	2.50	0.888 47	0.898 56
1.25	0.941 43	0.941 75	4.00	0.902 24	0.928 08	3.00	0.930 90	0.941 38
1.35	0.962 01	0.962 40	5.50	0.961 99	0.979 64	4.00	0.973 48	0.981 35
$C = 10, f = 0.6, \alpha = 8$			$C = 20.0, f = 0.6, \alpha = 4.0$			$C = 500.0, f = 0.5, \alpha = 6.0$		
θ^*	eq 3	exact	θ^*	eq 3	exact	θ^*	eq 3	exact
1.00	0.361 66	0.361 66	2.0	0.246 54	0.247 59	30.0	0.312 92	0.314 49
1.50	0.506 33	0.507 31	4.0	0.453 28	0.459 94	60.0	0.528 89	0.533 42
2.00	0.618 22	0.621 35	6.0	0.603 30	0.616 31	90.0	0.676 98	0.684 11
3.00	0.771 66	0.779 57	8.0	0.712 15	0.730 01	120.0	0.778 52	0.787 29
4.00	0.863 43	0.874 37	10.0	0.791 13	0.811 71	150.0	0.848 14	0.857 55
5.00	0.918 32	0.929 98	14.0	0.890 03	0.910 64	180.0	0.895 87	0.905 11
6.50	0.962 22	0.972 07	22.0	0.969 51	0.981 50	210.0	0.928 60	0.937 14

Table II. Typical Calculated Results for Response to Tube Fluid Velocity Change: Comparison of Exact Solution and Approximate Solution of Eq 3

$C = 1.0, f^* = 0.50, \alpha^* = 5.0$; $V = 0.8, f = 0.455, \alpha = 5.23$			$C = 3.0, f^* = 0.40, \alpha^* = 6.0$; $V = 1.10, f = 0.418, \alpha = 5.89$			$C = 4.0, f^* = 0.50, \alpha^* = 8.0$; $V = 0.70, f = 0.429, \alpha = 8.59$		
θ^*	eq 3	exact	θ^*	eq 3	exact	θ^*	eq 3	exact
1.00	0.816 35	0.816 35	1.00	0.601 91	0.601 91	1.00	0.634 54	0.634 54
1.05	0.848 92	0.850 26	1.10	0.660 93	0.663 00	1.20	0.724 30	0.729 39
1.10	0.875 71	0.879 91	1.20	0.711 20	0.717 96	1.40	0.792 02	0.806 62
1.15	0.897 75	0.905 11	1.50	0.821 54	0.844 07	1.60	0.843 10	0.866 24
1.20	0.915 88	0.926 00	1.80	0.889 73	0.919 72	1.80	0.881 64	0.910 15
1.30	0.943 07	0.956 54	2.00	0.920 00	0.950 05	2.00	0.910 71	0.941 22
1.45	0.968 30	0.981 80	2.40	0.957 90	0.981 82	2.60	0.961 67	0.985 59
$C = 5.0, f^* = 0.50, \alpha^* = 3.0$; $V = 1.2, f = 0.536, \alpha = 2.89$			$C = 7.0, f^* = 0.60, \alpha^* = 2.0$; $V = 0.70, f = 0.530, \alpha = 2.15$			$C = 10.0, f^* = 0.30, \alpha^* = 4.0$; $V = 1.50, f = 0.372, \alpha = 3.69$		
θ	eq 3	exact	θ^*	eq 3	exact	θ^*	eq 3	exact
1.00	0.374 97	0.374 97	1.0	0.319 89	0.319 89	1.00	0.403 03	0.403 03
1.50	0.521 01	0.524 63	2.0	0.519 89	0.523 05	1.50	0.539 63	0.542 69
2.00	0.632 92	0.642 94	3.0	0.661 07	0.668 81	2.00	0.644 98	0.653 23
3.00	0.784 42	0.804 73	4.0	0.760 74	0.772 01	2.50	0.726 21	0.739 38
4.00	0.873 39	0.896 78	5.0	0.831 10	0.844 24	3.00	0.788 86	0.805 66
5.00	0.925 64	0.946 91	6.0	0.880 77	0.894 32	5.00	0.925 32	0.943 60
7.00	0.974 35	0.986 84	10.0	0.970 39	0.978 82	7.00	0.973 59	0.984 85

Application of Exact and Approximate Solutions to Heat Exchangers with One Infinite Capacitance Rate Fluid

As pointed out by Shah (1981), the previous derived exact solutions (Tan and Spinner, 1978) are applicable to heat exchangers with one infinite thermal capacitance rate fluid. Thus for this case the exact and approximate solutions are applicable to crosscurrent, cocurrent, and countercurrent flow heat exchangers. The response to the shell fluid temperature change corresponds to a C_{max} (maximum of C_c or C_h) fluid temperature change. The response to tube fluid velocity change corresponds to a C_{min} (minimum of C_c or C_h) fluid velocity change. It is noted that the notation used in the mechanical engineering literature (Kays and London, 1964; Myers et al., 1970; Shah, 1981) is quite different from that used here. To facilitate the conversion to the nomenclature used in mechanical engineering publications, a table for conversion is provided in Table III.

Table III. Table for Conversion to Mechanical Engineering Notation

notation in this paper	mech eng notations
f	$\dfrac{1}{R^* + 1}$
f^*	$\dfrac{1}{R^*(0) + 1}$ (Shah, 1981)
α	$\dfrac{(R^* + 1)N}{R^*}\dfrac{x}{L}$
α^*	$N_{tu,t}(0)$ (Shah, 1981)
$\dfrac{\theta}{\alpha}$	θ^*
θ	$\dfrac{(R^* + 1)N}{R^*}\theta^*$
C	C^*_w
V	$u_m/u_m(0)$ (Shah, 1981)

Summary and Conclusions

In this paper, we have presented approximate solutions for the transient response of shell and tube heat exchangers. These solutions are obtained from the exact solutions in our previous publication. It has been shown that the limiting "no wall" solutions can be deduced from the approximate solution for small values of Cf. For the response to shell fluid temperature change, the previous published analytic solution for large values of C (Myers et al., 1970) is also directly derivable from the exact solution. Rizika's (1956) analytic solution in terms of hyperbolic functions was shown to be identical with the first three terms of eq 1. The case of $f = 1$ for which London et al. (1959) gave an exact solution can also be derived starting with our model equations (Tan and Spinner, 1978) and letting $\hat{T}$

Table IV. Summary of Constraints and Errors for Using Eq 3

Response to Step Change in Shell Fluid Temperature for $\theta^* \geq 1$

range of param	max rel error, %
any C and α, $f < 0.5$	2
any C and α, $0.5 < f < 0.7$	5
any C and α, $0.7 < f < 0.8$	7
any C and α, $0.8 < f < 0.9$	10

Response to Step Change in Tube Fluid Velocity for $\theta^* \geq 1$

range of param	max rel error, %
$V < 1.5$, $f^* < 0.5$, $C < 10$	
$1.5 < \alpha^* < 4$	5
$4 < \alpha^* < 6$	10
$6 < \alpha^* < 8$	15
$V < 1.5$, $f^* < 0.9$, $C < 5$	
$0.5 < \alpha^* < 2.5$	5

$= \hat{T}_w$. It is noted that the previously derived exact solutions and the approximate solutions presented here are also applicable to crosscurrent, cocurrent, or countercurrent flow heat exchangers with an infinite capacitance rate fluid.

In the previous paper (Tan and Spinner, 1978) and in this work it has been found that neglecting the ψ-function term in the exact solution leads to small error for $C < 5$ and $\alpha > 2$. Thus the exact solution without the ψ function provides a good approximation for many practical values of C and α. Thus only tabulated values of the known mathematical function J or ϕ_1 in addition to the exponential terms will be required for the numerical evaluation of the transient response. For engineering design calculations, it is desirable to use a simple form of equation. Thus we have formulated a simple exponential form of equation for approximating the transient response. It is noted that the constant present in the exponential form of equations is obtained by matching the slope of the response curve of the exact solutions. Hence the resulting approximate solutions are not entirely empirical. For the response to shell fluid temperature, the exponential approximate solution gave excellent agreement with the exact solution for any values of C and α, except for $f > 0.5$. For the response to tube fluid velocity change, the approximate exponential form of solution resulted in at most 5% relative error if $1 < \alpha^* < 4$, $f^* < 0.5$, $C < 10$, and $V < 1.5$. Table IV summarizes the range of constraints and the maximum relative error for the use of eq 3. A further practical use of this exponential form of approximate solution is the estimation of the time required to reach a given fractional attainment of final steady state given the magnitude of C, f, and α. In addition, it can be easily adopted for the study of the dynamic response to time-dependent disturbances.

Nomenclature

C = thermal capacity ratio of tube wall to tube fluid, dimensionless
C^* = thermal capacitance rate ratio of C_h/C_c, dimensionless
C_h = thermal capacitance rate of hot fluid, W/°C
C_c = thermal capacitance rate of cold fluid, W/°C
f^* = initial fractional heat-transfer resistance on the shell side
f = final fractional heat-transfer resistance on the shell side
H = Heaviside function with $H(\theta^*<1) = 0$ and $H(\theta^*\geq 1) = 1$
I_0 = modified Bessel function of the first kind of order zero
J = mathematical function, defined in the text
L = heat exchanger flow length, m
n = exponent in velocity-dependence heat-transfer correlation equation
N = number of heat-transfer units, dimensionless
R^* = ratio of fluid heat-transfer resistance of C_c and C_h fluid
$\hat{T}^\infty$ = normalized final steady state tube fluid temperature, dimensionless
U = fractional attainment of final steady state
U_1 = fractional attainment of final steady state at $\theta^* = 1$
V = normalized linear velocity of tube fluid, dimensionless
x = length coordinate in flow direction, m

Greek Letters

α = normalized exchanger length variable, dimensionless
θ = normalized time variable, dimensionless
θ^* = normalized throughput time, dimensionless
ϕ_0 = mathematical function, defined in text
ϕ_1 = mathematical function = $1 - J$, defined in text
ψ = mathematical function, defined in text

Appendix 1. Evaluation of Time Derivative from Exact Solution for Response to Step Change in Shell Fluid Temperature

For $\theta^* < 1$

$$\frac{\partial U}{\partial \theta^*} = \frac{R_1R_2}{R_1 - R_2}[\exp(-R_2\alpha\theta^*) - \exp(-R_1\alpha\theta^*)]\frac{\alpha}{\hat{T}^\infty}$$

For $\theta^* \geq 1$

$$\frac{\partial U}{\partial \theta^*} = \frac{R_1R_2}{R_1 - R_2}[\exp(-R_2\alpha\theta^*) - \exp(-R_1\alpha\theta^*)]\frac{\alpha}{\hat{T}^\infty} - \frac{R_1}{R_1 - R_2}\exp(-R_2\alpha\theta^*)\frac{1}{\hat{T}^\infty}\left(R_2\alpha\phi_1\left[\left(\frac{1}{Cf} - R_2\right)(\theta^* - 1)\alpha, \frac{f\alpha}{1 - CfR_2}\right] - \left(\frac{1}{Cf} - R_2\right)\alpha\phi_0\left[\left(\frac{1}{Cf} - R_2\right)(\theta^* - 1)\alpha, \frac{f\alpha}{1 - CfR_2}\right]\right) + \frac{R_2}{R_1 - R_2}\exp\left[-\alpha - \left(\frac{2}{Cf} - R_1\right)(\theta^* - 1)\alpha + \frac{f\alpha}{CfR_1 - 1}\right]\left(\frac{\partial}{\partial\theta^*}\psi\left[(R_1 - \frac{1}{Cf})(\theta^* - 1)\alpha, \frac{f\alpha}{CfR_1 - 1}\right] - \left(\frac{2}{Cf} - R_1\right)\alpha\psi\left[\left(R_1 - \frac{1}{Cf}\right)(\theta^* - 1)\alpha, \frac{f\alpha}{CfR_1 - 1}\right]\right)\frac{1}{\hat{T}^\infty} - \exp[-(1 - f)\alpha]\frac{\alpha}{Cf}\phi_0\left[\frac{(\theta^* - 1)\alpha}{Cf}, f\alpha\right]\frac{1}{\hat{T}^\infty}$$

For $\theta^* < 1$

$$\left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} = \frac{R_1R_2}{R_1 - R_2}[\exp(-R_2\alpha) - \exp(-R_1\alpha)]\frac{\alpha}{\hat{T}^\infty}$$

For $\theta^* \geq 1$

$$\left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} = \frac{R_1R_2}{R_1 - R_2}[\exp(-R_2\alpha) - \exp(-R_1\alpha)]\frac{\alpha}{\hat{T}^\infty} + \frac{R_1}{R_1 - R_2}\left(\frac{1}{Cf} - R_2\right)\frac{\alpha}{\hat{T}^\infty}\exp\left(-R_2\alpha - \frac{f\alpha}{1 - CfR_2}\right) + \frac{R_2}{R_1 - R_2}\left(R_1 - \frac{1}{Cf}\right)\exp(-\alpha)\frac{\alpha}{\hat{T}^\infty} - \frac{\alpha}{Cf\hat{T}^\infty}\exp(-\alpha)$$

$$= \frac{R_1R_2}{R_1 - R_2}[\exp(-R_2\alpha) - \exp(-R_1\alpha)]\frac{\alpha}{\hat{T}^\infty} + \left[\frac{R_1}{R_1 - R_2}\left(\frac{1}{Cf} - R_2\right) + \frac{R_2}{R_1 - R_2}\left(R_1 - \frac{1}{Cf}\right) - \frac{1}{Cf}\right]\frac{\alpha}{\hat{T}^\infty}\exp(-\alpha)$$

$$= \frac{R_1R_2}{R_1 - R_2}[\exp(-R_2\alpha) - \exp(-R_1\alpha)]\frac{\alpha}{\hat{T}^\infty}$$

Note that

$$\exp\left(-R_2\alpha - \frac{f\alpha}{1 - CfR_2}\right) = \exp(-\alpha)$$

Thus we have shown that

$$\left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} \text{ for } \theta^* < 1 = \left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} \text{ for } \theta^* \geq 1$$

Appendix 2. Exact Solution for Response to Step Change in Tube Fluid Velocity and Evaluation of Its Time Derivatives

Exact Solution for Response to Step Flow Rate Change.

$$U = \frac{1}{T^\infty}\Bigg(1 - A\exp(-R_4\alpha\theta^*) + B\exp(-R_3\alpha\theta^*) + H(\theta^* - 1)B\exp\left(-R_3\alpha + \frac{2f\alpha}{CfR_3 - 1} - \left(\frac{2}{Cf} - R_3\right)(\theta^* - 1)\alpha\right]\varphi\left[\left(R_3 - \frac{1}{Cf}\right)(\theta^* - 1)\alpha, \frac{f\alpha}{CfR_3 - 1}\right] + H(\theta^* - 1)A\exp(-R_4\alpha\theta^*)\phi_1\left[\left(\frac{1}{Cf} - R_4\right)(\theta^* - 1)\alpha, \frac{f\alpha}{1 - CfR_4}\right] - H(\theta^* - 1)\exp[-(1 - f)\alpha + (1 - f^*)\alpha^*]\phi_1\left[\frac{(\theta^* - 1)\alpha}{Cf}, f\alpha\right]\Bigg)$$

where

$$A = \frac{1 - f^*}{R_3 - R_4}\left[\frac{F_2}{R_4} - \left(\frac{1}{CfR_4} - 1\right)F_1\right]$$

$$B = A - 1 = \frac{1 - f^*}{R_3 - R_4}\left[\frac{F_2}{R_3} + \left(1 - \frac{1}{CfR_3}\right)F_1\right]$$

$$R_3 = \frac{1}{2}\left[\left[1 + \frac{1}{Cf} - (1 - f^*)\frac{V}{V^n}\right] + \left[\left[1 + \frac{1}{Cf} - (1 - f^*)\frac{V}{V^n}\right]^2 - \frac{4}{Cf}\left[1 - f - (1 - f^*)\frac{V}{V^n}\right]\right]^{1/2}\right]$$

$$R_4 = \frac{1}{2}\left[\left[1 + \frac{1}{Cf} - (1 - f^*)\frac{V}{V^n}\right] - \left[\left[1 + \frac{1}{Cf} - (1 - f^*)\frac{V}{V^n}\right]^2 - \frac{4}{Cf}\left[1 - f - (1 - f^*)\frac{V}{V^n}\right]\right]^{1/2}\right]$$

$$R_3 > \frac{1}{Cf} > R_4 > 0 \quad \text{for } V < 1$$

$$R_3 > \frac{1}{Cf} > 0 > R_4 \quad \text{for } V > 1$$

The two velocity forcing functions are

$$F_1 = \frac{V}{V^n} - 1, \quad F_2 = \left(\frac{1}{V^n} - 1\right)\Big/ C$$

The normalized final steady state is $T^\infty = 1 - \exp[-(1 - f)\alpha + (1 - f^*)\alpha^*]$.

The relationships of f and f^*, α and α^* are

$$f = \frac{f^* V^n}{1 + (V^n - 1)f^*}, \quad \alpha = \frac{V^n\alpha^*}{V}$$

Evaluation of Time Derivatives. For $\theta^* < 1$

$$\frac{\partial U}{\partial \theta^*} = [AR_4\exp(-R_2\alpha\theta^*) - BR_3\exp(-R_1\alpha\theta^*)]\frac{1}{T^\infty}$$

For $\theta^* \geq 1$

$$\frac{\partial U}{\partial \theta^*} = [AR_4\exp(-R_2\alpha\theta^*) - BR_3\exp(-R_1\alpha\theta^*)]\frac{1}{T^\infty} + \frac{B}{T^\infty}\exp\left[-R_3\alpha + \frac{2f\alpha}{CfR_3 - 1} - \left(\frac{2}{Cf} - R_3\right)(\theta^* - 1)\alpha\right]\left(\frac{\partial}{\partial\theta^*}\varphi\left[\left(R_3 - \frac{1}{Cf}\right)(\theta^* - 1)\alpha, \frac{f\alpha}{CfR_3 - 1}\right] - \left(\frac{2}{Cf} - R_3\right)\alpha\varphi\left[\left(R_3 - \frac{1}{Cf}\right)(\theta^* - 1)\alpha, \frac{f\alpha}{CfR_3 - 1}\right]\right) + \frac{A}{T^\infty}\exp(-R_4\alpha\theta^*)\left(\left(\frac{1}{Cf} - R_4\right)\alpha\phi_0\left[\left(\frac{1}{Cf} - R_4\right)(\theta^* - 1)\alpha, \frac{f\alpha}{1 - CfR_4}\right] - R_4\alpha\phi_1\left[\left(\frac{1}{Cf} - R_4\right)(\theta^* - 1)\alpha, \frac{f\alpha}{1 - CfR_4}\right]\right) - \frac{\alpha}{CfT^\infty}\exp[-(1 - f)\alpha + (1 - f^*)\alpha^*]\phi_0\left[\frac{(\theta^* - 1)\alpha}{Cf}, f\alpha\right]$$

For $\theta^* < 1$

$$\left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} = \frac{\alpha}{T^\infty}[AR_4\exp(-R_4\alpha) - BR_3\exp(-R_3\alpha)]$$

For $\theta^* \geq 1$

$$\left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} = \frac{\alpha}{T^\infty}[AR_4\exp(-R_4\alpha) - BR_3\exp(-R_3\alpha)] + \frac{A\alpha}{T^\infty}\left(\frac{1}{Cf} - R_4\right)\exp\left(-R_4\alpha - \frac{f\alpha}{1 - CfR_4}\right) + \frac{B\alpha}{T^\infty}\left(R_3 - \frac{1}{Cf}\right) \times \exp\left(-R_3\alpha + \frac{f\alpha}{CfR_3 - 1}\right) - \frac{\alpha}{CfT^\infty}\exp[-\alpha + (1 - f^*)\alpha^*]$$

$$= \frac{\alpha}{T^\infty}[AR_4\exp(-R_4\alpha) - BR_3\exp(-R_3\alpha)]\frac{1}{T^\infty} + \frac{\alpha}{T^\infty}\left[A\left(\frac{1}{Cf} - R_4\right) + \frac{B\alpha}{T^\infty}\left(R_3 - \frac{1}{Cf}\right) - \frac{1}{Cf}\right]\exp[-\alpha + (1 - f^*)\alpha^*]$$

$$= \frac{\alpha}{T^\infty}([AR_4\exp(-R_4\alpha) - BR_3\exp(-R_3\alpha)] + (BR_3 - AR_4)\exp[-\alpha + (1 - f^*)\alpha^*])$$

Note that

$$\exp\left(-R_4\alpha - \frac{f\alpha}{1 - CfR_4}\right) = \exp\left(-R_3\alpha + \frac{f\alpha}{CfR_3 - 1}\right) = \exp[-\alpha + (1 - f^*)\alpha^*]$$

Thus we have shown that

$$\left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} \text{ for } \theta^* < 1 \neq \left.\frac{\partial U}{\partial \theta^*}\right|_{\theta^*=1} \text{ for } \theta^* \geq 1$$

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Dynamics of Fluid Mixing Induced at a T-Junction. 2.[†] An Evaluation of a Mathematical Model with Existing Experimental Observations

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Mixing induced by flow through a T-junction is described with a mathematical model based on the flow geometry comprising the jet trajectory, the evolution of the jet cross section, and the three-dimensional velocity distribution constructed for the jet stream. The experimentally determined maximum tracer concentration across the main pipe reported in the literature is found to be closely represented by the present model. The predicted second moment is also found to agree with the existing correlation based on the integral analysis incorporating scaling laws and verified with extensive experimental observations.

I. Introduction

Flow through a T-junction has been considered a sensible means to promote mixing, and hence heat and mass transfer and chemical reaction (Kadotani and Goldstein, 1979; Forney and Kwon, 1979; Kim, 1985; Tosun, 1987). In the natural gas industry, effective utilization of a local production relies on mixing with a high-quality main stream via flow through a T-junction (Chen et al. (1990), referred to hereafter as part 1). Conceptually, mixing is caused by fluid flow that defines the fluid macroscale on which micromixing takes place via molecular diffusion. Thus, a problem involving mixing with simultaneous chemical and/or physical processes can be treated from a streamline perspective (e.g., Ou et al., 1985). Such an approach has its general appeal to a problem in which the underlying three-dimensional flow field could be established analytically or numerically from a fluid dynamic standpoint; laminar flow in a device with a simple geometry represents a manageable example (e.g., Lee et al., 1987). Mixing induced at a T-junction, in which turbulent flow occurs within a rather complex geometry, remains a challenging problem for which a solution of a fundamental nature is attempted.

In the work reported here the flow geometry is synthesized through considerations of the jet trajectory as well as the growth of the jet stream via entrainment (Hill, 1972). Mixing between the jet stream and the ambient fluid is thus envisioned to occur through the jet entrainment of the ambient fluid as well as turbulent mass transfer across the jet/ambient boundary. One of the unique features of our approach is the new physical insight into the evolution of the jet cross section that permits the three-dimensional flow field to be established for the jet stream. The treatment of a jet stream staying in contact with the main pipe wall is the other unique feature of the present study. The mathematical model is tested by comparing the predicted values for both the maximum and the second moment of the tracer concentration distribution across the main pipe to the experimental observations reported by Forney and his co-workers (Forney and Kwon, 1979; Forney and Lee, 1982; Sroka and Forney, 1989) over a wide range of flow conditions.

II. Mathematical Model

In the model to be presented, a passive tracer is injected as a side stream into a main flow containing no tracer. Both flow streams enter a T-junction at the same temperature, pressure, and essentially the same density for a relatively low tracer concentration. Flow conditions leading to both a free jet, defined here as a jet stream staying clear of the main pipe wall, and a wall jet, i.e., a jet stream staying in contact with the main pipe wall, are considered; the geometric features of both cases are depicted in Figure 1. Moreover, the jet-to-ambient flow velocity ratio $R_v \equiv v_{jo}/v_{ao}$ will be varied from 0.05 to 7, which covers a wide range of conditions, to permit the present model to be tested with existing experimental results for T-mixing (Forney and Kwon, 1979; Forney and

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