Tracking technique for manoeuvring target with correlated measurement noises and unknown parameters

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Abstract: The problem of tracking a manoeuvring target with sequentially correlated measurement noise is considered in the paper. Using Singer's method to model the manoeuvring target, the correlated measurement noise can be decorrelated by reformulating the measurement equation such that the conventional Kalman filter can be directly applied in this tracking problem. An analytical error analysis for this processing is derived. If some of the parameters are unknown, the conventional innovation correlation method can usually be employed to estimate these parameters adaptively. This method assumes that the measurement noise is white. If the measurement noise is sequentially correlated, this technique is not valid and the parameters can not be estimated with sufficient accuracy to obtain the desired tracking performance. By considering the effect of noise correlation, a modified computationally efficient method known as a multiple-level estimator is presented to improve the performance in estimating the unknown parameters in the presence of correlated measurement noise.

1 Introduction

In tracking airborne or missile targets using noisy radar data, the measurement noise is usually assumed to be white, and a conventional Kalman filter is frequently used for tracking the nonmanoeuvring target. If the target is manoeuvring, a situation when the target is suddenly accelerated by the pilot or missile guidance program, the conventional Kalman filter should be modified to maintain the tracking performance. There have been several approaches to this modification so far [1–6]. In this paper, Singer's method [3] is employed to treat the manoeuvring problem. The method is simple and has a moderate tracking performance if the measurement noise is white.

In practice, the measurement noise is sequentially correlated, and this is often referred to as coloured noise, within a bandwidth of typically a few hertz [7, 8]. When the measurement frequency is much lower than the error bandwidth, the successive errors are essentially uncorrelated and can be treated as white noise. This is often the case in the classical track-while-scan system. However, in many modern radar systems, the measurement frequency is usually high enough so that the correlation can not be ignored. Rogers [8] described the correlated noise as a first-order Markov process in the nonmanoeuvring case. By reformulating the measurement equation, the noise may be decorrelated so that the conventional Kalman filter can be directly applied. In this paper, this concept is extended to the manoeuvring target by using Singer's model [3] in modelling the manoeuvring target.

Usually, the modified Kalman filter works well if all the system parameters are known. However, often this is not the case, and some parameters may be unknown. Several adaptive filtering techniques [9–14] can be applied to estimate these parameters adaptively. Among them, the innovation correlation method [11–14] that utilises the properties of the autocorrelations of the innovation to estimate the parameters is a very effective approach. However, this approach assumes that the measurement noise is white, in which case good performance may be achieved. If the measurement noise is correlated, the innovation correlation method should be modified; otherwise, very poor results may be obtained. In this paper, a modified innovation correlation technique to estimate the unknown parameters for the manoeuvring target with correlated measurement noise is presented.

2 Manoeuvring target model

In this Section, Singer's work in modelling the manoeuvring target is reviewed briefly. The target state is defined in the measurement vector (such as range, bearing and elevation in radar system) direction. Then the tracking filter may work separately in each direction approximately. Only single direction operation is described in the following.

Let X_k and W_k be the target state and the process noise, respectively, which are defined below

$$X_k = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}_k = \begin{bmatrix} \text{target position at time instant } k \\ \text{target velocity at time instant } k \\ \text{target acceleration at time instant } k \end{bmatrix} \qquad (1)$$

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$$W_k = \begin{bmatrix} w \\ w' \\ w'' \end{bmatrix}_k = \begin{bmatrix} \text{process noise in position at time instant } k \\ \text{process noise in velocity at time instant } k \\ \text{process noise in acceleration at time instant } k \end{bmatrix} \quad (2)$$

By modelling the manoeuvre as a first-order autoregressive process, the manoeuvring target dynamics can be derived to a standard form as follows [3]:

$$X_{k+1} = \phi X_k + W_k \quad (3)$$

where the transition matrix ϕ is given by

$$\phi = \begin{bmatrix} 1 & T & \frac{1}{\alpha^2}(-1 + \alpha T + e^{-\alpha T}) \\ 0 & 1 & \frac{1}{\alpha}(1 - e^{-\alpha T}) \\ 0 & 0 & e^{-\alpha T} \end{bmatrix} \quad (4)$$

the parameters T and α are the data sampling time and the reciprocal of manoeuvre time constant, respectively.

The process noise W_k is a vector of zero-mean white noise sequence. The covariance matrix of W_k is given by

$$Q = E[W_k W_k^T] = 2\alpha\sigma_m^2 \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \quad (5)$$

where the elements q_{ij} ($i, j = 1, 2, 3$) are functions of the parameters α and T, that can be found in Reference 3. The parameter σ_m is the manoeuvre standard deviation.

For the target state being defined in the measurement vector directions, the measurement equation would be a linear function given by

$$z_k = HX_k + v_k \quad (6)$$

where $H = [1 \ 0 \ 0]$, z_k and v_k are the measurement data and the measurement noise, respectively.

If the measurement noise v_k is white, the system including the target dynamic eqn. 3 and the measurement eqn. 6 can be processed by a conventional Kalman filter.

3 Correlated noise and decorrelation

When the measurement frequency is high, the measurement noise is sequentially correlated significantly. Assume that it can be modelled as a first-order Markov process [8] as

$$v_k = \lambda v_{k-1} + \nu_k \quad (7)$$

where the correlation coefficient $\lambda = e^{-\beta T}$, the parameter β is the correlation coefficient in the continuous form. The noise ν_k is a zero-mean white Gaussian noise. If the variance of v_k is r, then the variance of ν_k can be obtained from eqn. 7 to be $(1 - \lambda^2)r$.

To decorrelate the correlated measurement noise v_k, a new measurement data y_k ($= z_k - \lambda z_{k-1}$), denoted as artificial measurement, can be obtained [15] as

$$y_k = H^* X_k + v_k^* \quad (8)$$

where

$$H^* = H - \lambda H\phi^{-1}$$
$$B = \lambda H\phi^{-1} = H - H^*$$
$$v_k^* = BW_{k-1} + \nu_k$$
$$r^* = E\{v_k^* v_k^*\} = BQB^T + (1 - \lambda^2)r \quad (9)$$

The new measurement noise v_k^* in eqn. 8 is white, but it is correlated with the process noise W_{k-1}. By reformulating the target dynamic eqn. 3, the process noise W_{k-1}. By reformulating the target dynamic eqn. 3, the process noise can be made to be uncorrelated with the measurement noise [15]. In most practical cases, this process can be omitted with little degradation in performance since the variance of the zero-mean item BW_{k-1} in eqn. 9 is generally very small. Thus, only a few simple substitutions in the measurement equation are required for decorrelating the system.

4 Autocorrelation of innovation

If some of the parameters, including λ and r, are unknown, these parameters should be estimated adaptively so that the decorrelation process mentioned in the preceding Section can work well. Since the autocorrelations of the innovation contain much information about the unknown parameters, they are very popular data in performing this estimation. Estimating the parameters in this way is known as the innovation correlation method in Reference 10. The technique is most suitable for constant coefficient systems in steady state.

For a system operating with a non-optimal Kalman gain K_k, the gain K_k can be computed from the following covariance update equations of Kalman filter:

$$P^*_{k|k-1} = \bar{\phi} P^*_{k-1|k-1} \bar{\phi}^T + \bar{Q} \quad (10)$$

$$K_k = P^*_{k|k-1} \bar{H}^T [\bar{H} P^*_{k|k-1} \bar{H}^T + \bar{r}]^{-1} \quad (11)$$

$$P^*_{k|k} = [I - K_k \bar{H}] P^*_{k|k-1} \quad (12)$$

where $P^*_{k|k-1}$ and $P^*_{k|k}$ are the predicted and estimated error covariance matrices, respectively, in the Kalman filtering procedure. The variables (or vectors, matrices) with bars over denote the preset, or the estimated, values used in the filter computation. In this paper, the parameters ϕ and H are assumed to be known, and Q, r and λ are unknown parameters. The value of the nonoptimal Kalman gain in steady state (K_∞) will be frequently used in the evaluation of the autocorrelations of the innovation. For simplicity, the notation K, the subscript ∞ is omitted, is used instead of K_∞ in the following expressions.

Let ε_k be the innovation process of a decorrelated system, where the measurement noise is decorrelated but some of the parameters including λ may be preset inaccurately, and ρ_j ($j = 0, 1, \ldots$) be the jth order autocorrelation of ε_k in steady state. Then ε_k and ρ_j can be expressed as

$$\varepsilon_k = y_k - \bar{H}\hat{X}_{k|k-1} \quad (13)$$

$$\rho_j = E\{\varepsilon_k \varepsilon_{k-j}\}_{k=\infty} \quad j = 0, 1, \ldots \quad (14)$$

For the case of the target model shown in eqn. 3 and the scalar measurement in eqns. 6 and 7, from the derivation in Appendixes 11.1 and 11.2, the autocorrelations ρ_j ($j = 0, 1, \ldots$) can be obtained as a linear function of manoeuvre variance s ($= \sigma_m^2$) and noise variance r ($= \sigma_r^2$) and a nonlinear function of noise correlation λ.

$$\rho_0 = f_{m,0}\, s + f_{r,0}(\lambda) r \quad (15)$$

$$f_{m,0} = \bar{H}\alpha_0\bar{H}^T + BQ_1B^T + \bar{H}Q_1B^T + BQ_1\bar{H}^T \quad (16)$$

$$f_{r,0}(\lambda) = \bar{H}\beta_0\bar{H}^T + \lambda_a + \bar{H}P_a + P_b\bar{H}^T \quad (17)$$

$$\rho_j = f_{m,j}\, s + f_{r,j}(\lambda) r \quad j = 1, 2, \ldots \quad (18)$$

$$f_{m,j} = \bar{H}\Psi^j\alpha_0\bar{H}^T - \bar{H}\Psi^{j-1}\phi KBQ_1\bar{H}^T + \bar{H}\Psi^j Q_1 B^T - \bar{H}\Psi^{j-1}\phi KBQ_1B^T \quad (19)$$

$$
\begin{aligned}
f_{r,j}(\lambda) = {} & \bar{H}\Psi^j\beta_0\bar{H}^T - \bar{H}(\lambda^{j-1}I + \lambda^{j-2}\Psi \\
& + \cdots + \Psi^{j-1})\phi K P_b \bar{H}^T + \lambda^{j-1}\lambda_b \\
& + \bar{H}\Psi^j P_a - \lambda_b \bar{H}(\lambda^{j-2}I + \lambda^{j-3}\Psi \\
& + \cdots + \Psi^{j-2})\phi K - \lambda_a \bar{H}\Psi^{j-1}\phi K \\
& + \lambda^j P_b \bar{H}^T
\end{aligned} \tag{20}
$$

where

$$Q_1 = Q/s \tag{21}$$

$$\bar{H} = H - \bar{\lambda} H \phi^{-1} \tag{22}$$

$$B = H - \bar{H} \tag{23}$$

$$\Psi = \phi(I - K\bar{H}) \tag{24}$$

$$\lambda_a = 1 - 2\lambda\bar{\lambda} + \bar{\lambda}^2 \tag{25}$$

$$\lambda_b = (\lambda - \bar{\lambda})(1 - \lambda\bar{\lambda}) \tag{26}$$

$$P_a = -\lambda_b(I - \lambda\Psi)^{-1}\phi K \tag{27}$$

$$P_b = -\lambda_b K^T\phi^T(I - \lambda\Psi^T)^{-1} \tag{28}$$

α_0, β_0 are matrices, defined by $P' = \alpha_0 s + \beta_0 r$ and can be solved from the following equation:

$$
\begin{aligned}
P' = {} & \Psi P'\Psi^T + (Q_1 + \phi K B Q_1 B^T K^T \phi^T \\
& - \Psi Q_1 B^T K^T \phi^T - \phi K B Q_1 \Psi^T)s \\
& + (\lambda_a \phi K K^T \phi^T - \Psi P_a K^T \phi^T - \phi K P_b \Psi^T)r
\end{aligned} \tag{29}
$$

5 Parameter estimation

From the statistical relationship between the autocorrelations of the innovation and the unknown parameters (λ, s, r), these parameters can be estimated adaptively during the Kalman filtering process if the following time-average autocorrelations of the innovations are employed to approximate the statistical autocorrelations of the innovations

$$\hat{\rho}_j = (1/N) \sum_{q=k-N+1}^{k} \varepsilon_q \varepsilon_{q-j} \quad j = 0, 1, \ldots \tag{30}$$

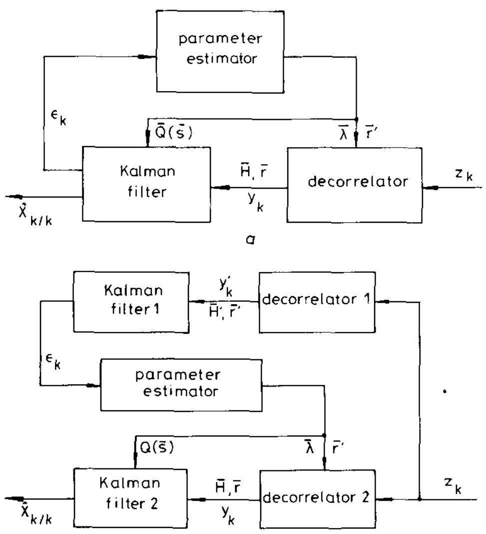

Fig. 1 *Adaptive tracking systems*
a System 1 *b* System 2

Fig. 1 shows two adaptive tracking systems. The first system needs only one Kalman filter to generate the innovation ε_k as the state estimate $\hat{X}_{k|k}$. The innovation ε_k is applied in the parameter estimator to estimate the unknown parameters. Using the new parameters obtained from the parameter estimator, the decorrelator and the Kalman filter will work more accurately. Two Kalman filters are employed in the second system. Kalman filter 1 and decorrelator 1 work with fixed parameters to generate the innovations for the parameter estimator. The parameter estimator then calculates the estimated parameters to update the operations of decorrelator 2 and Kalman filter 2 to get more accurate state estimates. This structure has an advantage in stationary noise environment owing to the fact that a large number of innovations can be collected to generate better autocorrelations and obtain more accurate parameter estimates.

To estimate the parameters, a nonlinear programming problem will be encountered because the autocorrelations of the innovation are nonlinear functions of λ. If the zeroth to Lth order autocorrelations are computed and the least square criterion is used in estimating the parameters, the following nonlinear programming problem must be solved

$$\min_{\lambda, s, r} \sum_{i=0}^{L} [\hat{\rho}_i - (f_{m,i}s + f_{r,i}(\lambda)r)]^2$$

$$\text{subject to} \quad 0 \leqslant \lambda < 1, s \geqslant 0, r \geqslant 0 \tag{31}$$

Many complicated computations would be involved in solving this problem. Sometimes a severe numerical problem will make it difficult to be solved. To overcome these difficulties, a structure called a multiple level estimator, as shown in Fig. 2, is proposed. In this structure, M linear least-square estimators work in parallel. The

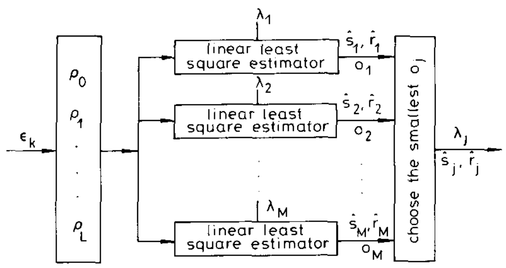

Fig. 2 *Multiple level estimator*

parameter λ is partitioned into M values within the region [0, 1), and every estimator corresponds to one of these values. For each estimator, the parameter λ is a known value (λ_q) such that the autocorrelations of the innovation are only linear functions of the parameters (s, r). Let

$$Z = [\hat{\rho}_0 \quad \hat{\rho}_1 \quad \cdots \quad \hat{\rho}_L]^T \tag{32}$$

$$A_q = \begin{bmatrix} f_{m,0} & f_{r,0}(\lambda_q) \\ f_{m,1} & f_{r,1}(\lambda_q) \\ \vdots & \vdots \\ f_{m,L} & f_{r,L}(\lambda_q) \end{bmatrix} \tag{33}$$

If $\lambda_q = \lambda$, the following equation can be obtained from eqns. 32, 15 and 18:

$$Z = A_q \begin{bmatrix} s \\ r \end{bmatrix} + (\text{error}) \tag{34}$$

where the error term has zero-mean because $\rho_j = E\{\hat{\rho}_j\}$.

From eqn. 34, the least square estimates of the parameters (s, r) and an objective function can be obtained as

$$\begin{bmatrix} s \\ r \end{bmatrix}_q = (A_q^T A_q)^{-1} A_q^T Z \tag{35}$$

$$o_q = \sum_{i=0}^{L} [\hat{\rho}_i - (f_{m,i} s_q + f_{r,i}(\lambda_q) r_q)]^2 \tag{36}$$

It can be seen from eqns. 35 and 34 that eqn. 35 is an unbiased estimate

$$\left(\text{i.e.} \quad E\begin{bmatrix} s \\ r \end{bmatrix}_q = \begin{bmatrix} s \\ r \end{bmatrix}\right) \quad \text{if} \quad \lambda_q = \lambda$$

Then the most likely set of parameters (λ, s, r) can be obtained from the objective function in eqn. 36. Comparing values of the objective functions over all M estimators, the estimator having the least objective function is selected. The value of λ corresponding to this estimator and the values of the parameters (s, r) that output from this estimator will be the desired estimated parameters. This structure needs M linear estimators to avoid the difficult nonlinear programming problem. The value of M is not necessary to be large because λ is confined in a small region $[0, 1)$. Section 7 will show that the system with a moderate value of M (e.g. 20) may have a rather good performance in estimating the parameters.

6 Performance analysis

In this Section, some numerical analysis of the tracking performance before and after decorrelation will be given. Measurement noise is white (or after perfect decorrelation process) and all the preset (or estimated) parameters are accurate, the Kalman gain K_k will be (approx.) optimal and the estimated error covariance $P^*_{k|k}$ computed from 10–12 will (approx.) be equal to the actual error covariance $P_{k|k}$ $(= E\{(X_k - \hat{X}_{k|k})(X_k - \hat{X}_{k|k})^T\})$. However, when the system works with some inaccurate parameters or the correlated measurement noise is not decorrelated perfectly (i.e. $\bar{\lambda} \neq \lambda$), $P_{k|k}$ will differ from $P^*_{k|k}$. In Appendix 11.2, the analytical solution of $P_{k|k}$ has been derived in eqn. 80 and eqns. 21–29. From these equations, the performances of the (perfectly) decorrelated system ($\bar{\lambda} = \lambda$) and the undecorrelated system ($\bar{\lambda} = 0$) are demonstrated and compared below.

Three cases with noise correlation $\lambda = 0.6$, 0.8 and 0.9, corresponding to data sampling time $T = 0.25$, 0.1092 and 0.0516 s, respectively, are tested. Assume that the following parameters in the system are fixed:

$$\begin{aligned} &\text{manoeuvre time constant} && 1/\alpha = 20\ \text{s} \\ &\text{noise correlation coefficient} && \beta = 2.0433\ \text{s}^{-1} \\ &\text{actual variance of measurement noise} && r = 100^2\ \text{ft}^2 \end{aligned} \tag{37}$$

In Figs. 3*a–c*, the performances of the decorrelated and undecorrelated systems are evaluated when the manoeuvre parameter σ_m is preset (or estimated) inaccurately. Assume that the preset variance of measurement noise is accurate (i.e. $\bar{r} = r = 100\ \text{ft}^2$). The actual manoeuvre parameter σ_m is fixed at 100 (ft/s^2), while the corresponding preset value $\bar{\sigma}_m$ used in the Kalman filter may be larger or smaller than σ_m.

In the overpreset case ($\bar{\sigma}_m > \sigma_m$), part of the measurement noise will be absorbed and considered as manoeuvre in the undecorrelated system because both the noise and manoeuvre are sequentially correlated, but the effect is much milder in the decorrelated system where the measurement noise has been decorrelated. Thus significant improvement can be expected by the decorrelation process, as shown in Figs. 3*a–c*. The advantage obtained from the decorrelation process increases as the parameter $\bar{\sigma}_m$ increases, and is prominent in the case with highly correlated measurement noise. It can also be seen from Figs. 3*a–c* that, in the undecorrelated system, the performance is velocity and acceleration estimations degrades very fast as $\bar{\sigma}_m$ increases. The performance in position estimation is not so sensitive to $\bar{\sigma}_m$ since the position is a double (single) integral of the acceleration (velocity), and the variation from the acceleration and velocity errors will be smoothed. Similarly to the case of overpresetting σ_m, the undecorrelated system with underpresetting σ_r $(=\sqrt{r})$ will increase the Kalman gain (see eqn. 11), and so most noise will be absorbed and considered as manoeuvre.

In the underpreset case ($\bar{\sigma}_m < \sigma_m$), the advantage due to the decorrelation process decreases and the performance of the decorrelated system may be worse than that of the undecorrelated system in severe cases ($\bar{\sigma}_m \ll \sigma_m$). This is because the false manoeuvre from correlated measurement noise is reduced and the decorrelated system responds more sensitively to the error caused by underpresetting the manoeuvre.

Next, the degradation in the performance of parameter estimation is investigated when the correlation of the measurement noise is partially or completely ignored. Assuming that the time-average autocorrelations of the innovation are noise-free, the estimates of the parameters s $(=\sigma_m^2)$ and r $(=\sigma_r^2)$, that are estimated in an imperfectly decorrelated system ($\bar{\lambda}$ may equal to λ or not) by a linear least square estimator, can be computed from eqns. 15, 18, 32, 33 and 35. Using the parameters specified in eqn. 37, Figs. 4*a* and *b* and Fig. 5, the results as shown in Figs. 4*a* and *b* and Fig. 5 can be obtained.

From Figs. 4*a* and *b*, it is found that the parameter s will be overestimated and the parameter r will be underestimated if the measurement noise is not decorrelated enough. These effects are more significant in the case with highly correlated measurement noise. When the noise correlation is completely ignored, an overestimate in s and an underestimate in r are very evident. Figs. 3*a–c* show that the over- and under-estimate will cause the tracking system to have very poor performance in velocity and acceleration estimations.

Fig. 4*a* shows that the estimate of the parameter s is very sensitive to the preset parameters $\bar{s}$ and $\bar{r}$. It is often highly overestimated except where a very small $\bar{s}$ (or a very large $\bar{r}$) is used. Using too small an $\bar{s}$ (or too large an $\bar{r}$) in the system will have a drawback in that a very long period is necessary to reach steady state. On the other hand, the estimate of the parameter r is not sensitive to those preset parameters, as shown in Fig. 4*b*. The effects of overestimation and underestimation can be reduced if the order L of the autocorrelation involving the estimation increases. But, as shown in Fig. 5, this improvement is still limited.

7 Simulation results

Some Monte Carlo simulations with 50 runs in each simulation are performed in this Section for further demonstrations. The accuracy of the parameters estimated by the multiple level estimator from noisy time-average autocorrelations of the innovation are checked first. The target is generated according to the target

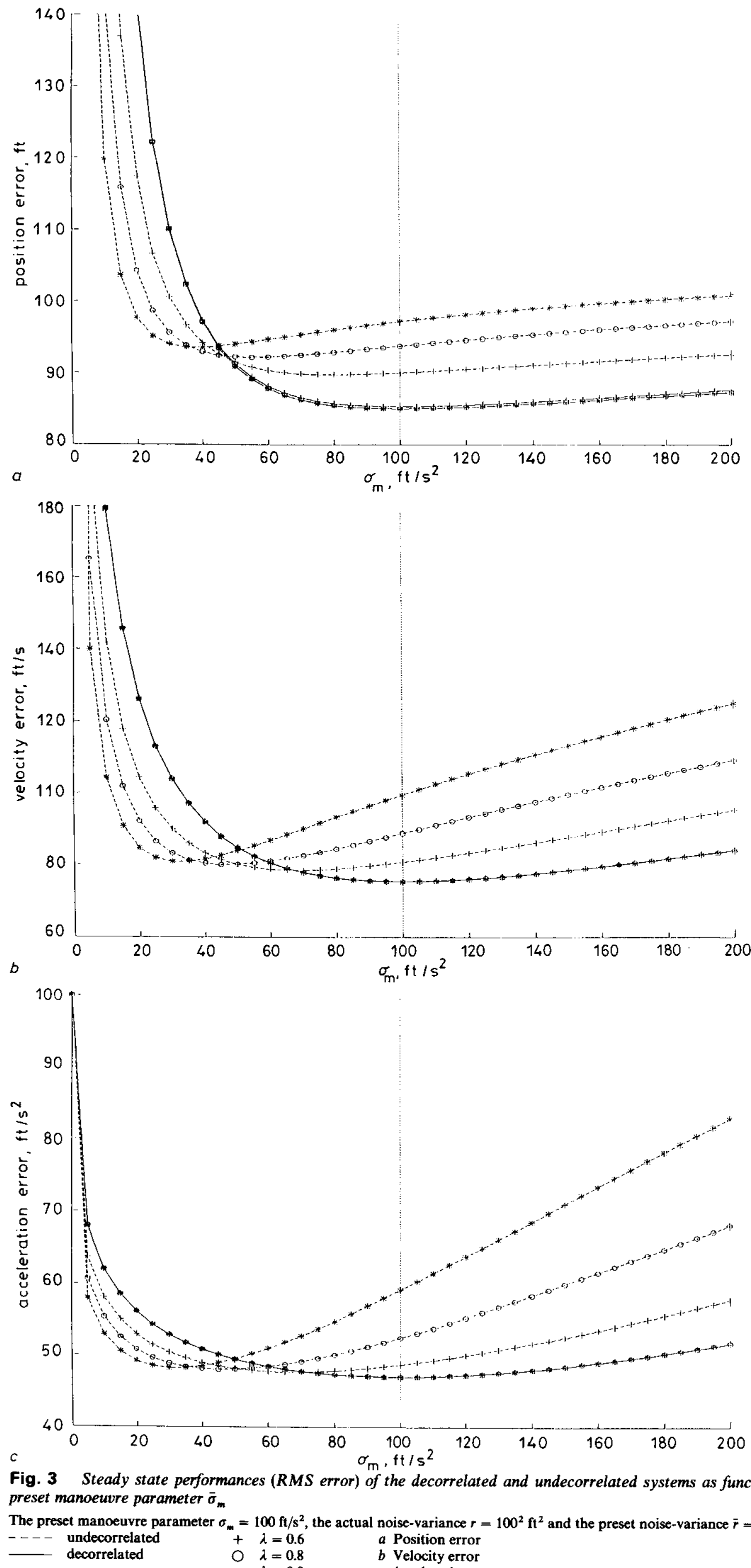

Fig. 3 *Steady state performances (RMS error) of the decorrelated and undecorrelated systems as functions of the actual noise correlation λ and the preset manoeuvre parameter $\bar{\sigma}_m$*

The preset manoeuvre parameter $\sigma_m = 100$ ft/s^2, the actual noise-variance $r = 100^2$ ft^2 and the preset noise-variance $\bar{r} = 100^2$ ft^2

- - - - undecorrelated	+ $\lambda = 0.6$	*a* Position error
——— decorrelated	○ $\lambda = 0.8$	*b* Velocity error
	* $\lambda = 0.9$	*c* Acceleration error

model, eqns. 3–5 and the measurement eqns. 6 and 7. Assume that the target is manoeuvring with $\sigma_m = 100$ (ft/s^2) in the whole tracking period. The target data is measured every $T = 0.1092$ s (corresponding to noise correlation $\lambda = 0.8$) and the parameters specified in eqn. 37 and Table 1*a* and *b* are used. This target is tracked adaptively by adaptive system 2 (Fig. 1*b*) with a multiple level estimator, Fig. 2.

Tables 1*a* and *b* show the performance of parameter estimation in this system when M (the number of estimators) is equal to 20 and N (the number of the innovations to be used to compute the autocorrelations) is equal to 200 and 400, respectively. From these Tables, it is found that the parameters λ and r can be estimated quite accurately in most cases. The performance of estimating s is not as good as that of estimating λ and r, and

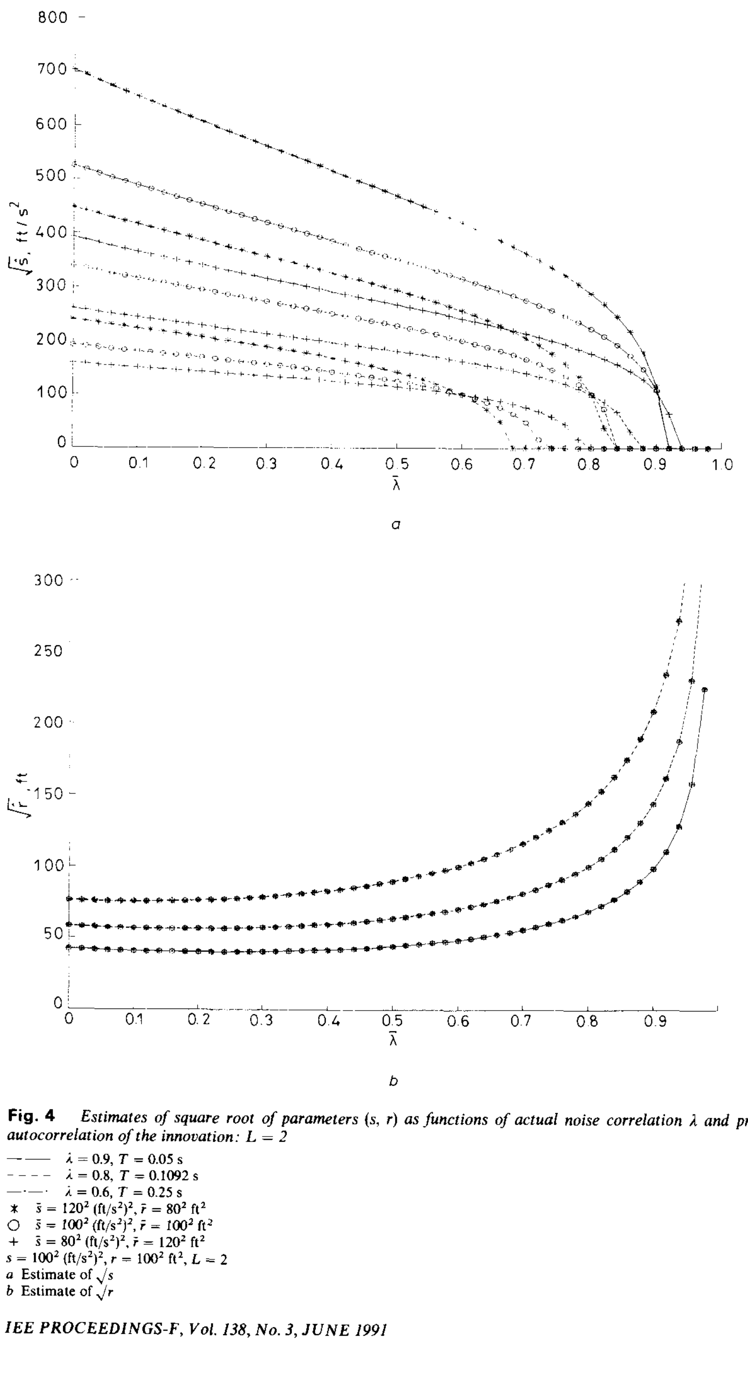

Fig. 4 *Estimates of square root of parameters (s, r) as functions of actual noise correlation λ and preset parameters ($\bar{\lambda}$, $\bar{s}$, $\bar{r}$) with the order of the autocorrelation of the innovation: $L = 2$*

—— $\lambda = 0.9$, $T = 0.05$ s
- - - - $\lambda = 0.8$, $T = 0.1092$ s
—·— $\lambda = 0.6$, $T = 0.25$ s
✱ $\bar{s} = 120^2$ (ft/s^2)^2, $\bar{r} = 80^2$ ft^2
○ $\bar{s} = 100^2$ (ft/s^2)^2, $\bar{r} = 100^2$ ft^2
\+ $\bar{s} = 80^2$ (ft/s^2)^2, $\bar{r} = 120^2$ ft^2
$s = 100^2$ (ft/s^2)^2, $r = 100^2$ ft^2, $L = 2$
a Estimate of $\sqrt{s}$
b Estimate of $\sqrt{r}$

it is worse when the preset parameter $\bar{s}$ (or $\bar{r}$) is large (or small). Using a small $\bar{s}$ (or a large $\bar{r}$) may result in a more accurate estimation for s and will degrade the performances of λ and r, slightly. However, a very small $\bar{s}$ (or a very large $\bar{r}$) will make the estimation for λ and R very difficult since it needs a long time to reach steady state. The estimation accuracy can be enhanced if we increase the values of L and N.

It can also be seen from these Tables that in estimating the parameters, the system performing the decorrelation process before parameter estimation ($\bar{\lambda} = 0.8$) does not offer an obvious advantage over the system without the decorrelation process ($\bar{\lambda} = 0$). Thus, decorrelator 1 in adaptive system 2 can be omitted. The special case with $\bar{\lambda} = \lambda = 0.8$, $\bar{s} = s = 100^2$ $(ft/s^2)^2$, $\bar{r} = r = 100^2$ (ft^2), will have good estimates for λ and r. The estimate for s is not

Table 1: Inaccuracy (RMS error) of the parameters estimated by the multiple level estimator for different preset parameters ($\bar{\lambda}, \bar{s}, \bar{r}, L$)

The number of estimators is $M = 20$ and the number of the innovations to be used to compute the autocorrelations is N. In Table 1a, $N = 200$, and in Table 1b, $N = 400$.

(a)

		$\bar{\lambda}$					
		0.0			0.8		
		errors					
L	$\sqrt{(\bar{S})}$	$\Delta\lambda$	$\Delta\sqrt{(r)}$	$\Delta\sqrt{(s)}$	$\Delta\lambda$	$\Delta\sqrt{(r)}$	$\Delta\sqrt{(s)}$
3	2	0.2503	111.2668	33.3363	0.2535	166.1640	41.7025
	5	0.1225	83.4079	35.2699	0.1765	144.3708	40.3405
	30	0.0858	37.6905	53.8022	0.1542	78.1198	50.6745
	100	0.0905	19.0795	83.3192	0.1429	42.1443	91.6577
	300	0.1047	37.1925	190.9192	0.1448	34.6672	219.0691
	1000	0.1195	69.8353	466.5911	0.1544	68.3530	593.1141
6	2	0.2954	91.2484	38.1151	0.2707	165.3239	41.4192
	5	0.1081	62.4267	33.2368	0.1535	130.0967	39.5385
	30	0.0758	27.9381	52.7926	0.1153	86.3603	46.0665
	100	0.0800	18.2348	84.5359	0.0966	32.4786	77.1396
	300	0.0947	40.1728	162.8400	0.0987	42.3866	132.8785
	1000	0.1064	63.4095	386.6643	0.1132	77.4166	312.8109
10	2	0.3334	79.6061	37.8439	0.2972	172.3286	41.4214
	5	0.0944	31.5446	33.5838	0.1492	106.0950	38.9790
	30	0.0817	22.6408	44.9822	0.0869	52.0274	37.5330
	100	0.0804	19.0549	80.3623	0.0773	22.9732	60.9135
	300	0.0920	29.0784	137.4935	0.0853	29.4031	108.1117
	1000	0.1033	62.9533	278.5101	0.1009	72.1496	248.4385

$\lambda = 0.8$, $s = 100^2$, $r = 100^2$, $\bar{r} = 100^2$, $M = 20$, $N = 200$; s is in $(ft/s^2)^2$ and r in ft^2

(b)

		$\bar{\lambda}$					
		0.0			0.8		
		errors					
L	$\sqrt{(\bar{S})}$	$\Delta\lambda$	$\Delta\sqrt{(r)}$	$\Delta\sqrt{(s)}$	$\Delta\lambda$	$\Delta\sqrt{(r)}$	$\Delta\sqrt{(s)}$
3	2	0.1869	48.3244	28.3600	0.2147	158.8280	32.3238
	5	0.0728	26.5129	24.0972	0.1695	143.2868	31.4085
	30	0.0595	16.4608	41.1434	0.1474	56.3109	50.6722
	100	0.0567	11.8626	73.4693	0.1297	23.0243	88.7624
	300	0.0641	12.8652	170.1908	0.1304	21.9253	213.6532
	1000	0.0735	15.6680	444.5929	0.1374	24.2419	581.0601
6	2	0.2395	50.2461	28.1586	0.1979	117.2941	31.8559
	5	0.0713	16.9940	22.8186	0.0852	54.3718	26.2717
	30	0.0584	13.6861	39.8119	0.0773	32.1688	34.6705
	100	0.0578	10.7487	70.5894	0.0593	15.8126	67.0567
	300	0.0642	11.6287	145.2246	0.0573	13.2012	115.4844
	1000	0.0772	18.6386	329.8669	0.0661	18.5228	274.2830
10	2	0.2674	56.1126	28.3105	0.2242	118.1059	31.9460
	5	0.0756	16.8239	22.8482	0.1015	53.0106	29.1954
	30	0.0539	12.7899	30.7396	0.0608	18.2561	29.1954
	100	0.0531	11.5229	68.7874	0.0537	13.4453	57.0401
	300	0.0587	12.1663	119.3452	0.0533	12.5351	93.9162
	1000	0.0757	24.3937	290.9872	0.0678	23.1329	214.7093

$\lambda = 0.8$, $s = 100^2$, $r = 100^2$, $\bar{r} = 100^2$, $M = 20$, $N = 400$; s is in $(ft/s^2)^2$ and r in ft^2

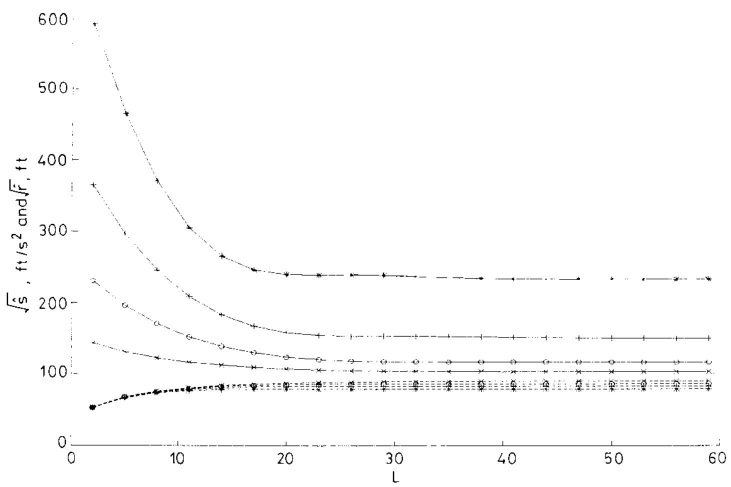

Fig. 5 *Estimates of square root of parameters (s, r) as functions of order L of the autocorrelation of the innovation and preset parameters* $(\bar{s}, \bar{r})$
The actual noise correlation $\lambda = 0.8$ and the preset noise correlation $\bar{\lambda} = 0$.
——— $\sqrt{\hat{s}}$
- - - $\sqrt{\hat{r}}$
* $\bar{s} = 200^2$ (ft/s^2)^2, $\bar{r} = 100^2$ ft^2
+ $\bar{s} = 100^2$ (ft/s^2)^2, $\bar{r} = 100^2$ ft^2
○ $\bar{s} = 50^2$ (ft/s^2)^2, $\bar{r} = 100^2$ ft^2
× $\bar{s} = 20^2$ (ft/s^2)^2, $\bar{r} = 100^2$ ft^2,
$s = 100$ (ft/s^2)^2, $r = 100^2$ ft^2
$\lambda = 0.8$, $\bar{\lambda} = 0$

good in this case, though it is the best condition for adaptive system 1 in steady state. When a system is working on these parameters, a very small value of $f_{m,j}$ relative to $f_{r,j}$ is usually obtained, where $f_{m,j}$ and $f_{r,j}$ are the coefficients of s and r, respectively, in eqns. 15 and 18. Thus, the estimation for s will be very sensitive to the variation of the noisy autocorrelation ρ_j. Better performance can be obtained from adaptive system 2 with properly preset parameters (a small $\bar{s}$ or a large $\bar{r}$).

In the last simulation the performance of target state estimation in the adaptive systems, with the consideration of correlation in the measurement noise, is illustrated. The target is generated according to eqns. 3–7 in manoeuvring state with $\sigma_m = 100$ (ft/s^2) in the whole tracking period. Some of the parameters are specified in eqn. 37 and the target is measured every $T = 0.1092$ s (corresponding to noise correlation $\lambda = 0.8$). The system considering the correlation works under the condition $(\bar{\lambda}, \bar{s}, \bar{r}, L, M) = (0, 30^2$ (ft/s^2)^2, 100^2 (ft^2), 10, 20), while the system ignoring the correlation works with $(\bar{\lambda}, \bar{s}, \bar{r}, L, M) = (0, 100^2$ (ft/s^2)^2, 100^2 (ft^2), 10, 1). Figs. 6*a–c* are the performances obtained in this simulation. It can be seen from these figures that the performances, especially in velocity and acceleration estimations, of the system considering the correlation is much better than those of the system ignoring the correlation. In Figs. 6*a–c*, the improvements in position, velocity and acceleration estimations are about 10, 40 and 47%, respectively.

8 Conclusion

We have considered the tracking problem of manoeuvring target with correlated measurement noise, and correlation phenomenon can not be ignored when the radar measurement frequency is high enough. Using Singer's method to model the manoeuvring target, the correlated measurement noise can be decorrelated by reformulating the measurement equation so that the conventional Kalman filter can be directly applied in this tracking problem. An analytical error analysis for this processing is derived in this paper.

If some of the parameters, including the parameters of noise correlation, are unknown, these parameters should be estimated adaptively so that the decorrelation process can work well. The conventional innovation correlation method, that utilises the statistical relationship between the autocorrelations of the innovation and the unknown parameters, can be employed to estimate these parameters from the time-average autocorrelations of the innovation. This approach assumes that the measurement noise is white, in which case good performance may be achieved. However, if the measurement noise is correlated, this technique is not valid and the parameters can not be estimated with sufficient accuracy to obtain the desired tracking performance.

By considering the effect of noise correlation, the relationship between the autocorrelations of the innovation and the parameters is rederived and a modified innovation correlation method known as the multiple level method is presented. In this method, several linear estimators are employed in parallel. It is found from the computer simulations that a moderate number (e.g. 20) of linear estimators may be enough to provide good performance in estimating the unknown parameters in the presence of correlated measurement noise. This technique and the analytical error analysis are the main contribution of this paper.

9 Acknowledgment

The authors thank the referees for many helpful comments and suggestions on the presentation of the paper.

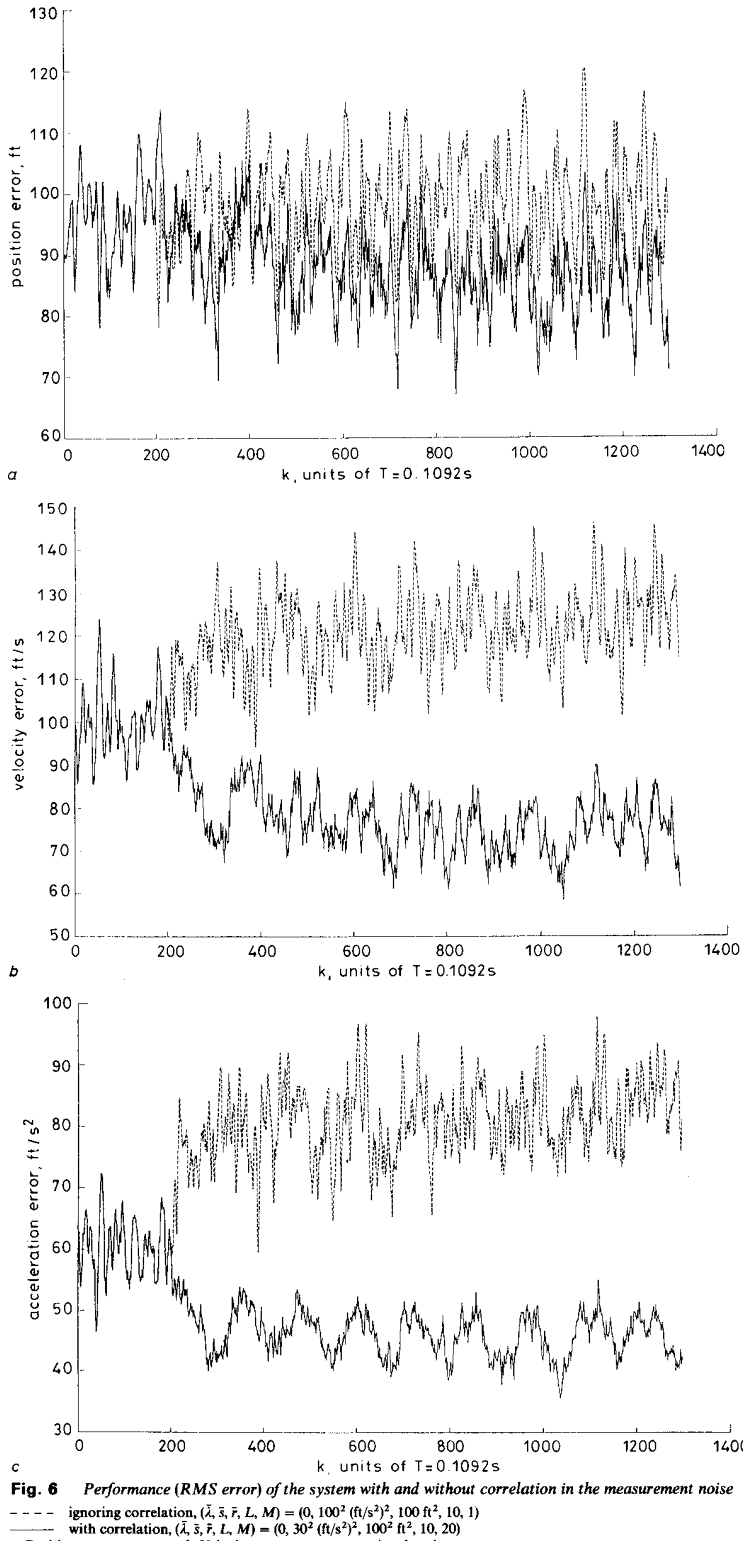

Fig. 6 *Performance (RMS error) of the system with and without correlation in the measurement noise*

- - - - ignoring correlation, $(\bar{\lambda}, \bar{s}, \bar{r}, L, M) = (0, 100^2\ (ft/s^2)^2, 100\ ft^2, 10, 1)$

——— with correlation, $(\bar{\lambda}, \bar{s}, \bar{r}, L, M) = (0, 30^2\ (ft/s^2)^2, 100^2\ ft^2, 10, 20)$

a Position error *b* Velocity error *c* Acceleration error

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11 Appendixes

11.1 Derivation of eqns. 15–20

Let λ, $\bar{\lambda}$, $\tilde{v}_k$ and $\bar{r}(\bar{r}')$ denote the true noise correlation, the preset noise correlation, the residue measurement noise and the preset variance of the measurement noise after (or before) decorrelation, respectively. Then, from eqns. 7–9:

$$y_k = z_k - \bar{\lambda} z_{k-1} \quad (38)$$

$$= \bar{H} X_k + \tilde{v}_k \quad (39)$$

$$\bar{H} = H - \bar{\lambda} H \phi^{-1} \quad (40)$$

$$B = H - \bar{H} \quad (41)$$

$$\bar{r} = B\bar{Q}B^T + (1 - \bar{\lambda}^2)\bar{r}' \quad (42)$$

$$\tilde{v}_k = v_k - \bar{\lambda} v_{k-1} + BW_{k-1} \quad (43)$$

$$v_k = \lambda v_{k-1} + \nu_k \quad (44)$$

The preset matrices $\bar{r}$ and $\bar{H}$ will be used in eqns. 10–12 to compute the Kalman gain K_k while the true noises $\tilde{v}_k$, v_k and the matrix $\bar{H}$ will be used in error analysis.

Next, let $\tilde{X}_{k|k}$ and $\tilde{X}_{k|k-1}$ denote the estimated state error and the predicted state error, respectively. Then,

$$\begin{aligned}\tilde{X}_{k|k} &= X_k - \hat{X}_{k|k} \\ &= X_k - [\hat{X}_{k|k-1} + K_k(y_k - \bar{H}\hat{X}_{k|k-1})] \\ &= (X_k - \hat{X}_{k|k-1}) \\ &\quad - K_k[(\bar{H}X_k + \tilde{v}_k) - \bar{H}\hat{X}_{k|k-1}] \\ &= (I - K_k\bar{H})\tilde{X}_{k|k-1} - K_k\tilde{v}_k \end{aligned} \quad (45)$$

$$\begin{aligned}\tilde{X}_{k|k-1} &= X_k - \hat{X}_{k|k-1} \\ &= (\phi X_{k-1} + W_{k-1}) - \phi\hat{X}_{k-1|k-1} \\ &= \phi\tilde{X}_{k-1|k-1} + W_{k-1} \\ &= \phi(I - K_{k-1}\bar{H})\tilde{X}_{k-1|k-2} \\ &\quad - \phi K_{k-1}\tilde{v}_{k-1} + W_{k-1}\end{aligned} \quad (46)$$

In the steady state, $K_k = K_\infty = K$. Let $\Psi = \phi(I - K\bar{H})$, then

$$\begin{aligned}\tilde{X}_{k|k-1} &= \Psi\tilde{X}_{k-1|k-2} + (W_{k-1} - \phi K\tilde{v}_{k-1}) \\ &= \Psi^j\tilde{X}_{k-j|k-j-1} + \sum_{q=k-j+1}^{k} \Psi^{k-q} \\ &\quad \times (W_{q-1} - \phi K\tilde{v}_{q-1}) \quad j = 1, 2, \ldots\end{aligned} \quad (47)$$

From eqns. 43–47 and the facts that (i) the process $\{W_k\}$ and $\{\nu_k\}$ are white and are uncorrelated to each other, and that (ii) a white and posterior noise will not affect the prior state error, the following results can be obtained in steady state ($k = \infty$, see the proof in Appendix 11.2).

$$(a)\quad E\{\tilde{v}_k\tilde{v}_k\} = \lambda_a r + (BQ_1B^T)s \quad (48)$$

$$(b)\quad E\{\tilde{v}_k\tilde{v}_{k-j}\} = \lambda^{j-1}\lambda_b r \quad j = 1, 2, \ldots \quad (49)$$

$$(c)\quad E\{\tilde{X}_{k|k-1}\tilde{v}_k\} = P_a r + (Q_1B^T)s \quad (50)$$

$$\begin{aligned}(d)\quad & E\{\tilde{X}_{k|k-1}\tilde{v}_{k-j}\} \\ &= \Psi^j[P_a r + (Q_1B^T)s] \\ &\quad - (\lambda^{j-2}I + \lambda^{j-3}\Psi + \cdots + \Psi^{j-2})\phi K(\lambda_b r) \\ &\quad - \Psi^{j-1}\phi K[\lambda_a r + (BQ_1B^T)s] \\ &\qquad j = 1, 2, \ldots\end{aligned} \quad (51)$$

$$(e)\quad E\{\tilde{v}_k\tilde{X}^T_{k|k-1}\} = P_b r + (BQ_1)s \quad (52)$$

$$(f)\quad E\{\tilde{v}_k\tilde{X}^T_{k-j|k-j-1}\} = \lambda^j P_b r \quad j = 1, 2, \ldots \quad (53)$$

$$(g)\quad E\{\tilde{X}_{k|k-1}\tilde{X}^T_{k|k-1}\} = P' = \alpha_0 s + \beta_0 r \quad (54)$$

where α_0 and β_0 are matrices, and can be solved from

$$\begin{aligned}P' &= \Psi P'\Psi + (\phi KBQ_1B^TK^T\phi^T \\ &\quad + Q_1 - \Psi Q_1B^TK^T\phi^T - \phi KBQ_1\Psi^T)s \\ &\quad + (\phi K\lambda_a K^T\phi^T - \Psi P_a K^T\phi^T - \phi KP_b\Psi^T)r\end{aligned} \quad (55)$$

$$\begin{aligned}(h)\quad & E\{\tilde{X}_{k|k}\tilde{X}^T_{k|k}\} \\ &= P_{k|k} \\ &= (I - K\bar{H})P'(I - K\bar{H})^T \\ &\quad + K[\lambda_a r + (BQ_1B^T)s]K^T \\ &\quad - (I - K\bar{H})[P_a r + Q_1B^Ts]K^T \\ &\quad - K[P_b r + BQ_1 s](I - K\bar{H})^T\end{aligned} \quad (56)$$

where Q_1, λ_a, λ_b, P_a, P_b are defined in eqn. 21 and eqns. 25–38, respectively.

Since

$$\varepsilon_k = y_k - \bar{H}\hat{X}_{k|k-1} = \bar{H}\hat{X}_{k|k-1} + \tilde{v}_k \quad (57)$$

the autocorrelations of the innovation can be derived from eqns. 43, 44 and 47–55 as follows (note: the subscript '$k = \infty$' is omitted in the following equations for convenience):

$$\begin{aligned}\rho_0 &= E\{\varepsilon_k\varepsilon_k\} \\ &= (\alpha 1) + (\alpha 2) + (\alpha 3) + (\alpha 4)\end{aligned} \quad (58)$$

$$(\alpha 1) = \bar{H}E\{\tilde{X}_{k|k-1}\tilde{X}^T_{k|k-1}\}\bar{H}^T$$
$$= (\bar{H}\alpha_0\bar{H}^T)s + (\bar{H}\beta_0\bar{H}^T)r \quad (59)$$
$$(\alpha 2) = E\{\tilde{v}_k\tilde{v}_k\}$$
$$= (BQ_1B^T)s + \lambda_a r \quad (60)$$
$$(\alpha 3) = \bar{H}E\{\tilde{X}_{k|k-1}\tilde{v}_k\}$$
$$= (\bar{H}Q_1B^T)s + (\bar{H}P_a)r \quad (61)$$
$$(\alpha 4) = E\{\tilde{v}_k\tilde{X}^T_{k|k-1}\}\bar{H}^T$$
$$= (BQ_1\bar{H}^T)s + (P_b\bar{H}^T)r \quad (62)$$

and

$$\rho_j = E\{\varepsilon_k\varepsilon_{k-j}\}$$
$$= (\beta 1) + (\beta 2) + (\beta 3) + (\beta 4) \quad j = 1, 2, \ldots \quad (63)$$
$$(\beta 1) = \bar{H}E\{\tilde{X}_{k|k-1}\tilde{X}^T_{k-j|k-j-1}\}\bar{H}^T$$
$$= \bar{H}E\left\{\left[\Psi^j\tilde{X}_{k-j|k-j-1} + \sum_{q=k-j+1}^{k}\Psi^{k-q}(W_{q-1} - \phi K\tilde{v}_{q-1})\right] \times \tilde{X}^T_{k-j|k-j-1}\right\}\bar{H}^T$$
$$= \bar{H}\Psi^j(\alpha_0 s + \beta_0 r)\bar{H}^T - \bar{H}(\lambda^{j-1}I + \lambda^{j-2}\Psi + \cdots + \Psi^{j-1})\phi K(P_b r)\bar{H}^T - \bar{H}\Psi^{j-1}\phi K(BQ_1 s)\bar{H}^T \quad (64)$$
$$(\beta 2) = E\{\tilde{v}_k\tilde{v}_{k-j}\} = \lambda^{j-1}\lambda_b r \quad (65)$$
$$(\beta 3) = \bar{H}E\{\tilde{X}_{k|k-1}\tilde{v}_{k-j}\}$$
$$= \bar{H}\Psi^j[P_a r + (Q_1B^T)s] - \bar{H}(\lambda^{j-2}I + \lambda^{j-3}\Psi + \cdots + \Psi^{j-2})\phi K(\lambda_b r) - \bar{H}\Psi^{j-1}\phi K[\lambda_a r + (BQ_1B^T)s] \quad (66)$$
$$(\beta 4) = E\{\tilde{v}_k\tilde{X}^T_{k-j|k-j-1}\}\bar{H}^T$$
$$= (\lambda^j P_b r)\bar{H}^T \quad (67)$$

then, eqns. 15–17 and 18–20 can be obtained from eqns. 58–62 and 63–67, respectively.

11.2 Proof of eqns. 48–56 in Appendix 11.1

Eqns. 48 and 49 can be proved easily from eqns. 43, 44, 25 and 26. To prove eqns. 50, 52 and 54–56, let

$$P_{\alpha,k} = E\{\tilde{X}_{k|k-1}\tilde{v}_k\}$$

and

$$P_{\beta,k} = E\{\tilde{v}_k\tilde{X}^T_{k|k-1}\}$$

From eqns. 45, 46 and 48, and the fact that the white and posterior noise W_{k-1} will not affect the prior state error $\tilde{X}_{k-1|k-2}$, then

$$P_{k|k-1} = E\{\tilde{X}_{k|k-1}\tilde{X}^T_{k|k-1}\}$$
$$= \phi(I - K_{k-1}\bar{H})P_{k-1|k-2}(I - K_{k-1}\bar{H})^T\phi^T + \phi K_{k-1}[\lambda_a r + (BQ_1B^T)s]K^T_{k-1}\phi^T + Q_1 s - \phi(I - K_{k-1}\bar{H})P_{\alpha,k-1}K^T_{k-1}\phi^T - \phi K_{k-1}P_{\beta,k-1}(I - K_{k-1}\bar{H})^T\phi^T \quad (68)$$

$$P_{k|k} = E\{\tilde{X}_{k|k}\tilde{X}^T_{k|k}\}$$
$$= (I - K_k\bar{H})P_{k|k-1}(I - K_k\bar{H})^T + K_k[\lambda_a r + (BQ_1B^T)s]K^T_k - (I - K_k\bar{H})P_{\alpha,k}K^T_k - K_kP_{\beta,k}(I - K_k\bar{H})^T \quad (69)$$

To solve $P_{\alpha,k}$ and $P_{\beta,k}$, let $P_{\alpha 1,k} = E\{\tilde{X}_{k|k-1}v_{k-1}\}$ and $P_{\beta 1,k} = E\{v_{k-1}\tilde{X}^T_{k|k-1}\}$. Then, from eqns. 43–46 and the fact that the white and posterior noise v_k will not affect the prior state error $\tilde{X}_{k|k-1}$,

$$P_{\alpha,k} = (\lambda - \bar{\lambda})P_{\alpha 1,k} + (Q_1B^T)s \quad (70)$$
$$P_{\beta,k} = (\lambda - \bar{\lambda})P_{\beta 1,k} + (BQ_1)s \quad (71)$$
$$P_{\alpha 1,k} = \lambda\phi(I - K_{k-1}\bar{H})P_{\alpha 1,k-1} - (1 - \lambda\bar{\lambda})\phi K_{k-1}r \quad (72)$$
$$P_{\beta 1,k} = \lambda P_{\beta 1,k-1}(I - K_{k-1}\bar{H})^T\phi^T - (1 - \lambda\bar{\lambda})K^T_{k-1}\phi^T r \quad (73)$$

In steady state

$$P_{\alpha 1,\infty} = -(1 - \lambda\bar{\lambda})(I - \lambda\Psi)^{-1}\phi K r \quad (74)$$
$$P_{\beta 1,\infty} = -(1 - \lambda\bar{\lambda})K^T\phi^T(I - \lambda\Psi^T)^{-1}r \quad (75)$$
$$P_{\alpha,\infty} = -(\lambda - \bar{\lambda})(1 + \lambda\bar{\lambda})(I - \lambda\Psi)^{-1}\phi K r + (Q_1B^T)s$$
$$= P_a r + (Q_1B^T)s \quad (76)$$
$$P_{\beta,\infty} = -(\lambda - \bar{\lambda})(1 - \lambda\bar{\lambda})K^T\phi^T(I - \lambda\Psi^T)^{-1}r + (BQ_1)s$$
$$= P_b r + (BQ_1)s \quad (77)$$

Let $P' = P_{k|k-1,k=\infty}$, then from eqns. 68, 76 and 77:

$$P' = \Psi P'\Psi^T + \phi K[\lambda_a r + (BQ_1B^T)s]K^T\phi^T + Q_1 s - \Psi[P_a r + (Q_1B^T)s]K^T\phi^T - \phi K[P_b r + (BQ_1)s]\Psi^T \quad (78)$$

By some simple algebra manipulations eqn. 78 can be rewritten as the following form to obtain α_0 and β_0:

$$P' = \alpha_0 s + \beta_0 r \quad (79)$$

and from eqns. 69, 76 and 77:

$$P_{k|k,k=\infty} = (I - K\bar{H})P'(I - K\bar{H})^T + K[\lambda_a r + (BQ_1B^T)s]K^T - (I - K\bar{H})[P_a r + Q_1B^Ts]K^T - K[P_b r + BQ_1 s](I - K\bar{H})^T \quad (80)$$

Eqns. 53 and 51 can be proved from eqns. 43, 44, 47, 75 and 48–50 as follows:

$$E\{\tilde{v}_k\tilde{X}^T_{k-j|k-j-1}\}_{,k=\infty}$$
$$= E\{(v_k - \bar{\lambda}v_{k-1} + BW_{k-1})\tilde{X}^T_{k-j|k-j-1}\}_{,k=\infty}$$
$$= E\left\{\left[\lambda^j(\lambda - \bar{\lambda})v_{k-j-1} + \sum_{q=0}^{j-1}\lambda^{q-1}(\lambda - \bar{\lambda})v_{k-q} + v_k + BW_{k-1}\right]\tilde{X}^T_{k-j|k-j-1}\right\}_{,k=\infty}$$
$$= \lambda^j(\lambda - \bar{\lambda})P_{\alpha 1,\infty} = \lambda^j P_b r \quad (81)$$
$$E\{\tilde{X}_{k|k-1}\tilde{v}_{k-j}\}_{,k=\infty}$$
$$= E\left\{\left[\Psi^j\tilde{X}_{k-j|k-j-1} + \sum_{q=k-j-1}^{k}\Psi^{k-q}(W_{q-1} - \phi K\tilde{v}_{q-1})\right]\tilde{v}_{k-j}\right\}_{,k=\infty}$$
$$= \Psi^j[P_a r + (Q_1B^T)s] - (\lambda^{j-2}I + \lambda^{j-3}\Psi + \cdots + \Psi^{j-2})\phi K(\lambda_b r) - \Psi^{j-1}\phi K[\lambda_a r + (BQ_1B^T)s] \quad (82)$$