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Necessary and Sufficient Conditions for Existence of Decoupling Controllers

Ching-An Lin

***Abstract*—It is well known that if a linear time-invariant plant is free from coincidences of poles and zeros in the right half-plane, then it can be decoupled with internal stability under unity-feedback configuration. We consider plants for which such coincidences do occur and give necessary and sufficient conditions under which stabilizing decoupling controllers exist. The conditions derived, based on transfer matrices and residues, are simple and straightforward.**

***Index Terms*—Decoupling controllers, multivariable systems.**

I. Introduction

Necessary and sufficient conditions for the existence of decoupling controllers, under unity-feedback configuration, have been studied in [7] and, for the two-input/two-output case, in [5]. The approach in [5] and [7] is to find conditions under which there exist open-loop precompensators which decouple the plant while maintaining stabilizability. Existence of such precompensators is equivalent to the existence of stabilizing decoupling controllers.

It is well known that if the plant has no coincidence of pole and zero in the right half-plane, then there exist controllers that stabilize and decouple the system [6]. When a plant cannot be decoupled without sacrificing closed-loop stability, it is precisely due to the coincidences of unstable poles and zeros. Our approach is to look carefully on such coincidences and see how their presence interferes with stability and decoupling requirements. The conditions and derivations are simple and straightforward.

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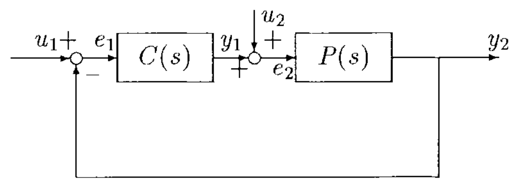

Fig. 1. Unity-feedback system $S(P, C)$.

The paper is organized as follows. Section II describes the system under consideration. Section III derives the necessary and sufficient conditions. Plants with simple pole-zero coincidences are considered first, and proof of the general result, Theorem 3.2, is given in the Appendix. Section IV is a brief conclusion.

II. Notations and Definitions

$\mathbb{C}$:= the field of complex numbers. $\mathbb{C}_- := \{s \in \mathbb{C} | \operatorname{Re}(s) < 0\}$; $\mathbb{C}_+ := \{s \in \mathbb{C} | \operatorname{Re}(s) \geq 0\}$. $\mathbb{R}[s]$:= the ring of polynomials in s with real coefficients; $\mathbb{R}(s)$:= the field of rational functions in s with real coefficients; $\mathbb{R}_p(s)(\mathbb{R}_{po}(s))$:= the set of proper (strictly proper, respectively) rational functions in s with real coefficients. For $H(s) \in \mathbb{R}(s)^{n\times n}$, $\mathcal{Z}[H]$:= the set of all zeros of H in $\mathbb{C}$, $\mathcal{P}[H]$:= the set of all poles of H in $\mathbb{C}$, $\mathcal{Z}_+[H] := \mathcal{Z}[H] \cap \mathbb{C}_+$, and $\mathcal{P}_+[H] := \mathcal{P}[H] \cap \mathbb{C}_+$. A proper transfer matrix $H(s) \in \mathbb{R}_p(s)^{n\times n}$ is stable if and only if $\mathcal{P}[H] \subset \mathbb{C}_-$. For $f, g \in \mathbb{R}[s]$, $\deg(f)$:= degree of f, the relative degree of $f/g \in \mathbb{R}(s)$ is defined as $\deg(f) - \deg(g)$ and $f|g$ means f divides g, or equivalently, $g = fh$ for some $h \in \mathbb{R}[s]$. The relative degree of $v(s) = [v_1(s) \cdots v_n(s)]^T \in \mathbb{R}(s)^n$ is defined as the largest relative degree of $v_i(s)$, $1 \leq i \leq n$. Finally, we use diag $[h_i]$ to denote the $n \times n$ matrix with h_i as its ith diagonal element.

III. Preliminaries

Consider the unity-feedback system $S(P, C)$, shown in Fig. 1, where $P \in \mathbb{R}_{po}(s)^{n\times n}$ is the plant, $C \in \mathbb{R}_p(s)^{n\times n}$ is the controller, (u_1, u_2) is the input, and (y_1, y_2) is the output. We assume that P is nonsingular so that the inverse $P^{-1} \in \mathbb{R}(s)^{n\times n}$ exists. Let $u := [u_1^T \ u_2^T]^T$ and $y := [y_1^T \ y_2^T]^T$.

The closed-loop transfer matrix is $H_{yu} \in \mathbb{R}_p(s)^{2n\times 2n}$ and is given by

$$
\begin{aligned}
H_{yu} &= \begin{bmatrix} H_{y_1u_1} & H_{y_1u_2} \\ H_{y_2u_1} & H_{y_2u_2} \end{bmatrix} \\
&= \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix}. \qquad (1)
\end{aligned}
$$

We say that the system $S(P, C)$ is (internally) stable and C is a stabilizing controller for P if H_{yu} is stable; the system is decoupled and C is a decoupling controller for P if C stabilizes P and the input–output (I/O) map.[1] $H_{y_2u_1}$ is nonsingular and diagonal.

Since P is strictly proper there is a one-to-one correspondence between the controller C and the transfer matrix $H_{y_1u_1} =: Q$. More precisely, $Q = C(I+PC)^{-1} \in \mathbb{R}_p(s)^{n\times n}$ if and only if $C = Q(I-PQ)^{-1} \in \mathbb{R}_p(s)^{n\times n}$ [1]. In terms of Q, the closed-loop transfer matrix in (1) becomes

$$
H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & (I-PQ)P \end{bmatrix} \qquad (2)
$$

and, in particular, the I/O map $H_{y_2u_1} = PQ$.

[1] For convenience, we call the transfer matrix $H_{y_2u_1}$ the I/O map of the feedback system.

Stability of $S(P, C)$ requires the stability of the four block entries of (2). The following result says that if the (block) diagonal entries of (2) are stable, then the only unstable poles that may appear in the off-diagonal entries are those that are both poles and (transmission) zeros of P.

Lemma 2.1: For the system $S(P, C)$ with H_{yu} given in (2), if Q and $(I-PQ)P$ are stable, then $\mathcal{P}_+[PQ] \subset (\mathcal{P}_+[P] \cap \mathcal{Z}_+[P])$ and $\mathcal{P}_+[QP] \subset (\mathcal{P}_+[P] \cap \mathcal{Z}_+[P])$. □

Proof: The assertion follows easily by noting that the poles of P^{-1} are the zeros of P [1] and that $PQ = I - [(I-PQ)P]P^{-1}$ and $QP = I - P^{-1}[(I-PQ)P]$. □

Comment: If there is no coincidence of poles and zeros in $\mathbb{C}_+$, that is, if $\mathcal{P}_+[P] \cap \mathcal{Z}_+[P] = \emptyset$, then it suffices to check Q and $(I-PQ)P$ for the stability of $S(P, C)$ [2].

IV. Necessary and Sufficient Conditions

A sufficient condition for the existence of a decoupling controller for the plant P is that P has no coincidences of poles and zeros in $\mathbb{C}_+$ [6]; however, the sufficient condition is not necessary [5], [7]. To find a necessary and sufficient condition we only need to consider the cases where coincidences of $\mathbb{C}_+$ pole–zero do occur. To simplify derivations we consider first the case where the $\mathbb{C}_+$-coincidences are all simple.

A. Simple Coincidences

Given the plant $P \in \mathbb{R}_{po}(s)^{n\times n}$ with $P^{-1} \in \mathbb{R}(s)^{n\times n}$, write

$$P(s) = \sum_{j=1}^{M} \frac{R^j}{s-\lambda_j} + U(s)$$

and

$$P(s)^{-1} = \sum_{l=1}^{M} \frac{T^l}{s-\lambda_l} + V(s) \tag{3}$$

where $\lambda_j \in \mathbb{C}_+$ are distinct, $R^j, T^l \in \mathbb{C}^{n\times n}$, $U(s) \in \mathbb{R}_{po}(s)^{n\times n}$, and $V(s) \in \mathbb{R}(s)^{n\times n}$ are analytic at $\{\lambda_j\}_{j=1}^M$, and $\mathcal{P}_+[U] \cap \mathcal{P}_+[V] = \emptyset$. The plant has M simple $\mathbb{C}_+$-coincidences at $\{\lambda_j\}_{j=1}^M$.

Consider the system $S(P, C)$ shown in Fig. 1. Suppose for some stabilizing controller C the resulting I/O map $H_{y_2u_1} =: H$ is diagonal, that is, C is a decoupling controller for P. Write $H = \text{diag}[h_i]$ where $h_i \in \mathbb{R}_p(s)$ is stable. With $Q = C(I+PC)^{-1}$ we have $H = PQ$. Internal stability of $S(P, C)$ implies that Q, $(I-PQ)P$ and QP are all stable, in particular, they are all analytic at $\{\lambda_j\}_{j=1}^M$.

Let us examine the consequences of these requirements. Since $Q = P^{-1}H$, Q is analytic at $\{\lambda_j\}_{j=1}^M$ if and only if

$$P^{-1}H = \left[\sum_{l=1}^{M} \frac{T^l}{s-\lambda_l} + V(s)\right] H(s)$$

is analytic at $\{\lambda_l\}_{l=1}^M$. Since V and H are analytic at $\{\lambda_l\}_{l=1}^M$, Q is analytic at $\{\lambda_l\}_{l=1}^M$ if and only if

$$T^l H(\lambda_l) = 0, \qquad l = 1, \cdots, M. \tag{4}$$

Let T_i^l be the *ith column* of T^l. Since H is diagonal, (4) is equivalent to

$$T_i^l h_i(\lambda_l) = 0, \qquad l = 1, \cdots, M, \qquad i = 1, \cdots, n. \tag{5}$$

Similarly, $(I-PQ)P$ is analytic at $\{\lambda_j\}_{j=1}^M$ if and only if

$$H(\lambda_j)R^j = R^j, \qquad j = 1, \cdots, M. \tag{6}$$

Let R_i^j be the *ith row* of R^j, and (6) becomes

$$h_i(\lambda_j)R_i^j = R_i^j, \qquad j = 1, \cdots, M, \qquad i = 1, \cdots, n. \tag{7}$$

Conditions (5) and (7) together imply that

$$T_i^l R_i^l = 0_{n\times n}, \qquad l = 1, \cdots, M, \qquad i = 1, \cdots, n. \tag{8}$$

Thus, for each l and each i, either T_i^l is a *zero column* or R_i^l is a *zero row*. Assume that both Q and $(I-PQ)P$ are analytic at $\{\lambda_j\}_{j=1}^M$ and write

$$QP = \left[\sum_{l=1}^{M} \frac{T^l}{s-\lambda_l} + V(s)\right] H(s) \left[\sum_{j=1}^{M} \frac{R^j}{s-\lambda_j} + U(s)\right].$$

Since Q is analytic at $\{\lambda_j\}_{j=1}^M$, QP is analytic at $\{\lambda_j\}_{j=1}^M$ if and only if the associated residues are zero. The residue associated with the pole λ_j is

$$\left.\left[\sum_{l=1,\, l\neq j}^{M} \frac{T^l}{s-\lambda_l} + V(s)\right] H(s)R^j\right|_{s=\lambda_j}.$$

Thus QP is analytic at $\{\lambda_j\}_{j=1}^M$ if and only if

$$\left[\sum_{l=1,\, l\neq j}^{M} \frac{T^l}{\lambda_j-\lambda_l} + V(\lambda_j)\right] R^j = 0 \qquad j = 1, \cdots, M \tag{9}$$

where we have used (6).

We now show that the *necessary conditions* (8) and (9) together are also sufficient to guarantee the existence of a decoupling controller for P. We do this by showing that if (8) and (9) hold, then it is possible to choose a *proper stable diagonal* I/O map H for which the matrices $Q := P^{-1}H$, $(I-PQ)P$, and QP are all proper and stable, and thus the controller $C = P^{-1}H(I-H)^{-1}$ is the decoupling controller achieving the I/O map H.

Let $h_i(s) = \tilde{\beta}_i(s)/\alpha_i(s)$ and $H(s) = \text{diag}[h_i(s)]$, where $\tilde{\beta}_i(s), \alpha_i(s) \in \mathbb{R}[s]$ and $\alpha_i(s)$ is Hurwitz, $i = 1, \cdots, n$. Write [2]

$$P = \left[\frac{Z_{ij}}{P_{ij-}P_{ij+}}\right] \tag{10}$$

where $Z_{ij}, P_{ij-}, P_{ij+} \in \mathbb{R}[s]$ are mutually coprime, P_{ij+} is monic, $\mathcal{Z}[P_{ij+}] \subset \mathbb{C}_+$, and $\mathcal{Z}[P_{ij-}] \subset \mathbb{C}_-$; write

$$P^{-1} = \left[\frac{N_{ij}}{D_{ij-}D_{ij+}}\right] \tag{11}$$

where $N_{ij}, D_{ij-}, D_{ij+} \in \mathbb{R}[s]$ are mutually coprime, D_{ij+} is monic, $\mathcal{Z}[D_{ij+}] \subset \mathbb{C}_+$, and $\mathcal{Z}[D_{ij-}] \subset \mathbb{C}_-$.

Let

$$P_{i+} = \text{the monic least common multiple of } \{P_{ij+}\}_{j=1}^n \tag{12}$$

and

$$D_{j+} = \text{the monic least common multiple of } \{D_{ij+}\}_{i=1}^n \tag{13}$$

and γ_j be the relative degree of the jth column of P^{-1}. Since $P \in \mathbb{R}_{po}(s)^{n\times n}$, $\gamma_j > 0$. Note that (8) implies that, for $i = 1, \cdots, n$, the polynomials D_{i+} and P_{i+} are *coprime*. Since $Q = P^{-1}H = P^{-1}\,\text{diag}[\tilde{\beta}_j/\alpha_j]$, Q is proper and stable if and only if

$$D_{j+}|\tilde{\beta}_j \quad \text{and} \quad \deg(\alpha_j) - \deg(\tilde{\beta}_j) \geq \gamma_j, \qquad j = 1, \cdots, n.$$

Or equivalently

$$\tilde{\beta}_j = D_{j+}\beta_j \qquad \text{for some } \beta_j \in \mathbb{R}[s], \qquad j = 1, \cdots, n \tag{14}$$

and

$$\deg(\alpha_j) - \deg(\beta_j) \geq \gamma_j + \deg(D_{j+}), \qquad j = 1, \cdots, n. \tag{15}$$

With (14) and (15) satisfied, $(I-PQ)P$ is stable if and only if

$$P_{i+}|(\alpha_i - D_{i+}\beta_i), \qquad i = 1, \cdots, n. \tag{16}$$

Since D_{i+} and P_{i+} are coprime, there are $\alpha_i(s) \in \mathbb{R}[s]$ which are Hurwitz and $\beta_i(s) \in \mathbb{R}[s]$ such that (15) and (16) are satisfied [2]. We thus have shown that (8) implies the coprimeness of P_{i+} and D_{i+}, which in turn makes possible the choice of a stable proper diagonal I/O map H so that the corresponding Q and $(I - PQ)P$ are proper and stable. In fact, the Hurwitz polynomials α_i are subject only to (15) and can otherwise be arbitrarily chosen. It remains to show that if (9) is also satisfied, then the matrix QP is stable. By Lemma 2.1, with Q and $(I - PQ)P$ stable, the $\mathbb{C}_+$-poles of QP form a subset of $\{\lambda_1, \cdots, \lambda_M\}$. However, (9) and the stability of Q and $(I - PQ)P$ together ensure that QP is analytic at $\{\lambda_j\}_{j=1}^M$. Thus QP is stable. We have thus established the following necessary and sufficient conditions for the existence of a decoupling controller.

Theorem 3.1: For the plant $P(s)$, together with its inverse $P(s)^{-1}$ given in (3), there exists a decoupling controller if and only if (8) and (9) hold. □

Example 1: Consider the plant [7]

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} \end{bmatrix} = \frac{\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}}{s-1} + \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{-1}{2(s+1)} & \frac{2}{3(s+2)} \end{bmatrix} =: \frac{R}{s-1} + U(s)$$

$$P(s)^{-1} = \frac{\begin{bmatrix} 2 & 0 \\ -3 & 0 \end{bmatrix}}{s-1} + \begin{bmatrix} s+2 & -(s+1) \\ -1 & s+2 \end{bmatrix} =: \frac{T}{s-1} + V(s).$$

The plant has a $\mathbb{C}_+$-coincidence at $s = 1$. Condition (8) is satisfied, but

$$V(1)R = \begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -1 & \frac{-2}{3} \\ \frac{3}{2} & 1 \end{bmatrix} \neq 0_{2\times 2}.$$

Thus the plant cannot be decoupled, and any controller that makes $H_{y_2u_1}$ diagonal will result in $H_{y_1u_2}$ containing a pole at $s = 1$.

Example 2: Consider the plant [4]

$$P(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ \frac{s-1}{(s+1)^2} & \frac{2(s-1)}{(s+1)^2} \end{bmatrix} = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}{s-1} + \begin{bmatrix} 0 & 0 \\ \frac{s-1}{(s+1)^2} & \frac{2(s-1)}{(s+1)^2} \end{bmatrix} =: \frac{R}{s-1} + U(s)$$

$$P(s)^{-1} = \frac{\begin{bmatrix} 0 & -4 \\ 0 & 4 \end{bmatrix}}{s-1} + \begin{bmatrix} 2(s-1) & -(s+3) \\ -(s-1) & s+3 \end{bmatrix} =: \frac{T}{s-1} + V(s).$$

Again the plant has a $\mathbb{C}_+$-coincidence at $s = 1$. Condition (8) is satisfied, and

$$V(1)R = \begin{bmatrix} 0 & -4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, for this plant a decoupling controller exists.

B. General Case

We now consider the general case. Let

$$P(s) = \sum_{j=1}^{M} \sum_{k=1}^{K_j} \frac{R^{jk}}{(s-\lambda_j)^k} + U(s)$$

and

$$P(s)^{-1} = \sum_{m=1}^{M} \sum_{l=1}^{L_m} \frac{T^{ml}}{(s-\lambda_m)^l} + V(s) \tag{17}$$

where $\lambda_j \in \mathbb{C}_+$, $R^{jk}, T^{ml} \in \mathbb{C}^{n\times n}$, $K_j \geq 1$, $L_m \geq 1$, $U \in \mathbb{R}_{po}(s)^{n\times n}$, and $V \in \mathbb{R}(s)^{n\times n}$ are analytic at $\{\lambda_j\}_{j=1}^M$ and $\mathcal{P}_+[U] \cap \mathcal{P}_+[V] = \emptyset$. Let R_i^{jk} be the ith row of R^{jk} and T_i^{ml} be the ith column of T^{ml}, and let, for $j = 1, \cdots, M$

$$W_j(s) = \sum_{m=1,\, m\neq j}^{M} \sum_{l=1}^{L_m} \frac{T^{ml}}{(s-\lambda_m)^l} + V(s). \tag{18}$$

The sufficient and necessary condition for the existence of a decoupling controller is the following Theorem whose proof is given in the Appendix.

Theorem 3.2: For the plant P, together with its inverse P^{-1} given in (17), there exists a decoupling controller if and only if, for $j = 1, \cdots, M$

$$T_i^{jl} R_i^{jk} = 0, \qquad i = 1, \cdots, n, \qquad l = 1, \cdots, L_j, \qquad k = 1, \cdots, K_j \tag{19}$$

and

$$\sum_{k=0}^{n_j} W_j^{(k)}(\lambda_j) R^{j(K_j+k-n_j)} \cdot \frac{1}{k!} = 0, \qquad n_j = 0, 1, \cdots, K_j - 1. \tag{20}$$

□

Comments:

1) Conditions (19) and (20) reduce to (8) and (9) if all the coincidences are simple.
2) Condition (19) ensures that P_{i+} and D_{i+} defined in (12) and (13), respectively, are coprime; (20) ensures that the stability of $H_{y_1u_1}$ and $H_{y_2u_2}$ imply stability of $H_{y_1u_2}$.
3) The condition is simple in that no computations of either coprime factorizations or Smith–Mcmillan form is required.
4) We note that since a transfer matrix generically does not have any pole-zero coincidence, the conditions hold generically. This result, however, allows a quantitative discussion of the relation of the cost of decoupling when the conditions are “nearly violated” [3].

V. Conclusions

We derive necessary and sufficient conditions for the existence of decoupling controllers. The conditions and derivations based on transfer matrices and residues are simple and straightforward. The necessary and sufficient conditions easily can be extended to the block decoupling [4].

Appendix

A. Proof of Theorem 3.2

For simplicity we prove only the case where $M = 2$. The extension to the case where $M > 2$ is straightforward, though tedious.

Write

$$P(s)=\sum_{k=1}^{K_1}\frac{R^{1k}}{(s-\lambda_1)^k}+\sum_{k=1}^{K_2}\frac{R^{2k}}{(s-\lambda_2)^k}+U(s)$$

and

$$P(s)^{-1}=\sum_{l=1}^{L_1}\frac{T^{1l}}{(s-\lambda_1)^l}+\sum_{l=1}^{L_2}\frac{T^{2l}}{(s-\lambda_2)^l}+V(s)$$

where U and V are analytic at λ_1 and λ_2. Assume that H is diagonal and analytic at λ_1 and λ_2.

We show that (19) and (20) are necessary. Since $Q=P^{-1}H$, by taking the partial fraction expansion, we have

$$\begin{aligned}Q=&\sum_{l=1}^{L_1}\frac{1}{(s-\lambda_1)^l}\sum_{k=0}^{L_1-l}\frac{1}{k!}T^{1(l+k)}H^{(k)}(\lambda_1)\\&+\sum_{l=1}^{L_2}\frac{1}{(s-\lambda_2)^l}\sum_{k=0}^{L_2-l}\frac{1}{k!}T^{2(l+k)}H^{(k)}(\lambda_2)+G(s)\end{aligned}$$

for some $G(s)\in\mathbb{R}(s)^{n\times n}$ analytic at λ_1 and λ_2.

Thus Q is analytic at λ_1 if and only if

$$\sum_{k=0}^{L_1-l}\frac{1}{k!}T^{1(l+k)}H^{(k)}(\lambda_1)=0,\qquad l=1,\cdots,L_1. \tag{21}$$

The last equation (i.e., $l=L_1$) of (21) is

$$T^{1L_1}H(\lambda_1)=0. \tag{22}$$

Since $H=\operatorname{diag}[h_i]$, (22) is equivalent to

$$h_i(\lambda_1)=0\quad\text{if}\quad T_i^{1L_1}\neq 0,\qquad i=1,\cdots,n. \tag{23}$$

The second-to-last equation ($l=L_1-1$) of (21) can be written as

$$T_i^{1(L_1-1)}h_i(\lambda_1)+T_i^{1L_1}h_i'(\lambda_1)=0,\qquad i=1,\cdots,n. \tag{24}$$

It follows from (23) and (24) that

$$T_i^{1(L_1-1)}h_i(\lambda_1)=0$$

and

$$T_i^{1L_1}h_i'(\lambda_1)=0,\qquad i=1,\cdots,n.$$

The second-to-last equation of (21) thus becomes

$$T^{1(L_1-1)}H(\lambda_1)=0\quad\text{and}\quad T^{1L_1}H'(\lambda_1)=0.$$

By examining the equations in (21) from last to first, it follows that each individual term in (21) equals zero, that is

$$T^{1(l+k)}H^{(k)}(\lambda_1)=0,\qquad \text{for all}\quad \begin{aligned}&l=1,\cdots,L_1\\&k=0,\cdots,L_1-l.\end{aligned} \tag{25}$$

In particular

$$T^{1l}H(\lambda_1)=0\quad\text{for all}\quad l=1,\cdots,L_1. \tag{26}$$

Similarly, the requirement that Q is analytic at λ_2 implies that

$$T^{2(l+k)}H^{(k)}(\lambda_2)=0,\qquad \text{for all}\quad \begin{aligned}&l=1,\cdots,L_2\\&k=0,\cdots,L_2-l\end{aligned} \tag{27}$$

and in particular

$$T^{2l}H(\lambda_2)=0,\qquad \text{for all}\quad l=1,\cdots,L_2. \tag{28}$$

By taking the partial fraction expansion $(I-PQ)P$ can be written as

$$\begin{aligned}(I-PQ)P=&\sum_{k=1}^{K_1}\frac{1}{(s-\lambda_1)^k}\Bigg[(I-H(\lambda_1))R^{1k})\\&\qquad+\sum_{k_1=1}^{K_1-k}\frac{-1}{k_1!}H^{(k_1)}(\lambda_1)R^{1(k_1+k)}\Bigg]\\&+\sum_{k=1}^{K_2}\frac{1}{(s-\lambda_2)^k}\Bigg[(I-H(\lambda_2))R^{2k})\\&\qquad+\sum_{k_2=1}^{K_2-k}\frac{-1}{k_2!}H^{(k_2)}(\lambda_2)R^{2(k_2+k)}\Bigg]\\&+\tilde{G}(s)\end{aligned}$$

for some $\tilde{G}(s)\in\mathbb{R}(s)^{n\times n}$ analytic at λ_1 and λ_2. From similar arguments as above, it follows that $(I-PQ)P$ is analytic at λ_1 if and only if for $k=1,\cdots,K_1$

$$H(\lambda_1)R^{1k}=R^{1k} \tag{29}$$

and

$$H^{(k_1)}(\lambda_1)R^{1(k_1+k)}=0,\qquad k_1=1,\cdots,K_1-k \tag{30}$$

and $(I-PQ)P$ is analytic at λ_2 if and only if for $k=1,\cdots,K_2$

$$H(\lambda_2)R^{2k}=R^{2k} \tag{31}$$

and

$$H^{(k_2)}(\lambda_2)R^{2(k_2+k)}=0,\qquad k_2=1,\cdots,K_2-k. \tag{32}$$

Since H is diagonal, (26) and (29) imply that

$$T_i^{1l}R_i^{1k}=0,\qquad \text{for}\quad \begin{aligned}&i=1,\cdots,n,\ l=1,\cdots,L_1\\&k=1,\cdots,K_1\end{aligned} \tag{33}$$

and (28) and (31) imply that

$$T_i^{2l}R_i^{2k}=0,\qquad \text{for}\quad \begin{aligned}&i=1,\cdots,n,\ l=1,\cdots,L_2\\&k=1,\cdots,K_2.\end{aligned} \tag{34}$$

Suppose now that (33) and (34) hold and that both Q and $(I-PQ)P$ are analytic at λ_1 and λ_2. Write

$$\begin{aligned}QP=&\sum_{l=1}^{L_1}\frac{T^{1l}}{(s-\lambda_1)^l}H(s)\sum_{k=1}^{K_1}\frac{R^{1k}}{(s-\lambda_1)^k}\\&+\sum_{l=1}^{L_2}\frac{T^{2l}}{(s-\lambda_2)^l}H(s)\sum_{k=1}^{K_2}\frac{R^{2k}}{(s-\lambda_2)^k}\\&+\sum_{l=1}^{L_1}\frac{T^{1l}}{(s-\lambda_1)^l}H(s)\sum_{k=1}^{K_2}\frac{R^{2k}}{(s-\lambda_2)^k}\\&+\sum_{l=1}^{L_2}\frac{T^{2l}}{(s-\lambda_2)^l}H(s)\sum_{k=1}^{K_1}\frac{R^{1k}}{(s-\lambda_1)^k}\\&+\sum_{l=1}^{L_1}\frac{T^{1l}}{(s-\lambda_1)^l}H(s)U(s)\\&+\sum_{l=1}^{L_2}\frac{T^{2l}}{(s-\lambda_2)^l}H(s)U(s)\\&+V(s)H(s)\sum_{k=1}^{K_1}\frac{R^{1k}}{(s-\lambda_1)^k}\\&+V(s)H(s)\sum_{k=1}^{K_2}\frac{R^{2k}}{(s-\lambda_2)^k}\\&+V(s)H(s)U(s).\end{aligned} \tag{35}$$

Since H is diagonal, by (33) and (34) the first two terms in the right-hand side of (35) are zero. Since $Q = P^{-1}H$ is analytic at λ_1 and λ_2, the third term is analytic at λ_1, the fourth term is analytic at λ_2, and the fifth and sixth terms are analytic at λ_1 and λ_2. Thus the matrix QP is analytic at λ_1 if and only if

$$\left[\sum_{l=1}^{L_2} \frac{T^{2l}}{(s-\lambda_2)^l} + V(s)\right] H(s) \sum_{k=1}^{K_1} \frac{R^{1k}}{(s-\lambda_1)^k} \tag{36}$$

is analytic at λ_1; QP is analytic at λ_2 if and only if

$$\left[\sum_{l=1}^{L_1} \frac{T^{1l}}{(s-\lambda_1)^l} + V(s)\right] H(s) \sum_{k=1}^{K_2} \frac{R^{2k}}{(s-\lambda_2)^k} \tag{37}$$

is analytic at λ_2.

Thus with

$$W_1(s) = \sum_{l=1}^{L_2} \frac{T^{2l}}{(s-\lambda_2)^l} + V(s)$$

and

$$W_2(s) = \sum_{l=1}^{L_1} \frac{T^{1l}}{(s-\lambda_1)^l} + V(s)$$

QP is analytic at λ_1 and λ_2 if and only if

$$\sum_{k=0}^{n_1} W_1^{(k)}(\lambda_1) R^{1(K_1+k-n_1)} \cdot \frac{1}{k!} = 0, \qquad n_1 = 0, 1, \cdots, K_1 - 1 \tag{38}$$

and

$$\sum_{k=0}^{n_2} W_2^{(k)}(\lambda_2) R^{2(K_2+k-n_2)} \cdot \frac{1}{k!} = 0, \qquad n_2 = 0, 1, \cdots, K_2 - 1 \tag{39}$$

where we have used (29)–(32) in computing the partial fractions of (36) and (37).

We have shown that the conditions (33), (34), (38), and (39) are necessary. The proof that these conditions together are also sufficient is exactly the same as that for the simple coincidence case and is omitted.

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Improving Stability Margins via Dynamic-State Feedback for Systems with Constant Uncertainty

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***Abstract*—It is well known that if a linear system with time-varying uncertainty in the system matrix and/or the input connection matrix is quadratically stabilizable by *linear dynamic* state feedback, then it is also quadratically stabilizable by *linear static* state feedback. In this paper, we provide an example of a system with unknown constant real uncertainty which is stabilizable by a linear, dynamic-state feedback controller but not by a static-state feedback controller.**

***Index Terms*— Dynamic state feedback control, quadratic stability, robust controls.**

I. Introduction

Consider the uncertain linear system

$$\dot{x}(t) = [A + \Delta A(r)]\, x(t) + [B + \Delta B(r)]\, u(t) \tag{1}$$

where $x(t) \varepsilon R^n$ is the state, $u(t) \varepsilon R^m$ is the control, and $r \varepsilon R^q$ is a vector of uncertain real parameters belonging to a compact set $\Re$. It is assumed that the uncertainty satisfies

$$[\Delta A(r) \quad \Delta B(r)] = DF(r)[E_1 \quad E_2] \tag{2a}$$

$$||F(r)|| \le \overline{r}. \tag{2b}$$

For problems where r is allowed to be time varying, Rotea and Khargonekar [1] have shown that if the system is quadratically stabilizable by dynamic-state feedback, i.e., a controller of the form

$$\dot{z}(t) = A_c z(t) + B_c(t) x(t) \tag{3a}$$

$$u(t) = K_1 z(t) + K_2(t) x(t) \tag{3b}$$

where $z \varepsilon R^{n_c}$, then it is quadratically stabilizable via a static linear-state feedback controller

$$u(t) = Kx(t). \tag{4}$$

We give an example to show that if the vector of uncertain parameters is time invariant, there may not exist a control of the form (4) that stabilizes (1), but there does exist a control of the form (3) which stabilizes (1).

The system equations are

$$\dot{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1.5 + r \\ 1 \end{bmatrix} u(t), \qquad |r| \le \overline{r}. \tag{5}$$

For $\overline{r} > 0.5$, it is easily verified that there does not exist a controller of the form (4) which stabilizes (5). To see this, let $u = k_1 x_1 + k_2 x_2$. The closed-loop system is

$$\dot{x}(t) = \begin{bmatrix} 2 + (1.5 + r)k_1 & -1 + (1.5 + r)k_2 \\ 1 + k_1 & l + k_2 \end{bmatrix} x(t).$$

From the Routh–Hurwitz condition, the system is stable if and only if $-3 - (1.5 + r)k_1 - k_2 > 0$ and $3 + (2.5 + r)k_1 + (0.5 - r)k_2 > 0$.

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