the Theory of Nonlinear Oscillations. New York: Gordon Breach, 1961.
[10] S. Lee and S. M. Meerkov, "Vibrational feedback control in the problem of absolute stability," College of Engineering, Univ. Michigan, Ann Arbor, MI, Control Group Rep. CGR-17, Oct. 1989.
[11] S. M. Meerkov, "Averaging of trajectories of slow dynamic systems," *J. Differential Equations*, vol. 9, pp. 1239–1245, 1973.
[12] D. D. Siljak, "New algebraic criteria for positive realness," *J. Franklin Inst.*, vol. 291, no. 2, 1971.

Decoupling Controller Design for Linear Multivariable Plants

Ching-An Lin and Tung-Fu Hsieh

Abstract—**We study decoupling controller design for linear time-invariant square MIMO plants under the unity-feedback configuration. For plants with no coincidences of *unstable* poles and zeros, we give a simplified necessary and sufficient condition for closed-loop stability. The simplified condition leads to a simple parametrization of the set of all achievable decoupled I/O maps and an algorithm which allows the design of decoupling controllers to achieve preassigned closed-loop poles.**

I. Introduction

The design of a decoupling controller for linear MIMO systems has been studied by many researches. Given stable plants, Desoer and Chen [4] provided an algorithm to obtain strictly proper controllers such that the resulting I/O map is decoupled. Safonov and Chen [8] studied the design of a controller to achieve optimal stability margin subject to decoupling and asymptotic tracking constraints. Desoer and Gündes [5] described the set of all diagonal I/O maps and D/O maps achievable by *two-input one-output* controllers. Hammer and Khargonekar [7] gave necessary and sufficient conditions for a plant to be decouplable by a controller placed in the *feedback path*. Vardulakis [9] showed that if a plant has no coincidences of unstable poles and zeros, it can be decoupled by a controller under the unity-feedback configuration.

In this note, we study decoupling controller design for linear time-invariant MIMO square plants under the unity-feedback configuration. For plants with no coincidences of unstable poles and zeros, we give a simplified necessary and sufficient condition for closed-loop stability. The simplified condition leads to a simple parametrization of the set of *all achievable diagonal I/O maps* and the set of *all decoupling controllers*. The parametrization is simple in that it involves only *scalar polynomials satisfying certain interpolation conditions*. Finally, we develop an algorithm for the design of decoupling controllers to achieve preassigned closed-loop poles. The algorithm does not require the computations of coprime factorization, Smith–McMillan form, or structure matrix [9]. In addition to inverting a rational matrix, the only computation required is solving linear algebraic equations.

This paper is organized as follows: Section II introduces the decoupling design problem. In Section III, we prove the simplified stability condition and develop the parametrization results. A design algorithm and an illustrative example are given in Section IV. Section V is the conclusion.

Manuscript received September 29, 1989. The work of C.-A. Lin was supported by the National Science Council under Grant NSC-78-0404-E009-07.
C.-A. Lin is with the Department of Control Engineering, National Chiao-Tung University, Hsinchu, Taiwan 30039, Republic of China.
T.-F. Hsieh is with the Institute of Electronics, National Chiao-Tung University, Hsinchu, Taiwan 30039, Republic of China.
IEEE Log Number 9041207.

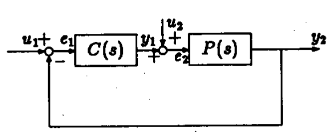

Fig. 1. Unity-feedback system $S(P, C)$.

Notations

$a := b$ means a denotes b. $\mathbb{R}$:= the field of real numbers; C := the field of complex numbers. $C_- := \{s \in C \mid \mathrm{Re}(s) < 0\}$; $C_+ := \{s \in C \mid \mathrm{Re}(s) \geq 0\}$. $\mathbb{R}[s]$:= the ring of polynomials in s with real coefficients; $\mathbb{R}(s)$:= the field of rational functions in s with real coefficients; $\mathbb{R}_p(s)(\mathbb{R}_{po}(s))$:= the set of proper (strictly proper) rational functions in s with real coefficients. For $H(s) \in \mathbb{R}(s)^{n\times n}$, $\mathscr{Z}[H]$:= the set of all zeros of H in C, $\mathscr{P}[H]$:= the set of all poles of H in C [2, p. 75], $\mathscr{Z}_+[H] := \mathscr{Z}[H] \cap C_+$, and $\mathscr{P}_+[H] := \mathscr{P}[H] \cap C_+$. A proper transfer matrix $H(s) \in \mathbb{R}_p(s)^{n\times n}$ is stable if and only if $\mathscr{P}[H] \subset C_-$. For $f, g \in \mathbb{R}[s]$, $\deg f$:= degree of f and $f \mid g$ means f divides g, or equivalently, $g = fh$ for some $h \in \mathbb{R}[s]$.

II. Problem Statement

Consider the unity-feedback system $S(P, C)$ shown in Fig. 1, where $P(s) \in \mathbb{R}_{po}(s)^{n\times n}$ is the given plant, $C(s) \in \mathbb{R}_p(s)^{n\times n}$ the controller to be designed, (u_1, u_2) the input, and (y_1, y_2) the output. It is assumed that the dynamical systems described by $P(s)$ and $C(s)$ contain no unstable hidden modes. Let $u := [u_1^T u_2^T]^T$ and $y := [y_1^T y_2^T]^T$. Since $P(s)$ is strictly proper, $S(P, C)$ is well posed.
The closed-loop transfer matrix $H_{yu} \in \mathbb{R}_p(s)^{2n\times 2n}$ and is given by

$$H_{yu} = \begin{bmatrix} H_{y_1u_1} & H_{y_1u_2} \\ H_{y_2u_1} & H_{y_2u_2} \end{bmatrix} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix}. \tag{2.1}$$

We say that the system $S(P, C)$ is (internally) stable and C is a stabilizing controller for P if and only if H_{yu} is stable.

The problems studied in this note are the following. Given that $P(s)$ satisfies the following assumptions:
P1: $P(s) \in \mathbb{R}_{po}(s)^{n\times n}$ is nonsingular, that is, $\det P \not\equiv 0$,
P2: $\mathscr{Z}_+[P] \cap \mathscr{P}_+[P] = \varnothing$,
1) describe the set of all controller $C(s) \in \mathbb{R}_p(s)^{n\times n}$ for which the system $S(P, C)$ is stable and the input–output transfer matrix $H_{y_2u_1}$ is nonsingular and diagonal, and 2) develop an algorithm for the design of decoupling controllers. Note that the assumptions are quite weak: P1 means that there are no trivial inputs or outputs and P2 is generically satisfied [9]. It has been shown [9] that P1 and P2 are sufficient to guarantee the existence of a stabilizing and decoupling controller.

III. The Main Results

In this section, we prove a simplified necessary and sufficient condition for closed-loop stability which leads to simple characterizations of the set of *all stabilizing and decoupling controllers* and the set of *all achievable input–output diagonal transfer matrices*. We start by noting that with the definition

$$Q := C(I+PC)^{-1} \tag{3.1}$$

the closed-loop transfer matrix in (2.1) becomes

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & P(I-QP) \end{bmatrix} \tag{3.2}$$

and in particular, the input–output transfer matrix $H_{y_2u_1} = PQ$. We shall need the following lemma which gives the relation between poles and zeros of a square rational matrix.

Lemma 1 [2, p. 75, Fact 8]: Let $H(s) \in \mathbb{R}(s)^{n\times n}$ be nonsingular. Then

1) $\mathscr{Z}[H] = \mathscr{P}[H^{-1}]$;
2) $\mathscr{P}[H] = \mathscr{Z}[H^{-1}]$; and in particular,
3) $\mathscr{Z}_+[H] = \mathscr{P}_+[H^{-1}]$.

Theorem 1: Consider the system $S(P, C)$ shown in Fig. 1. Suppose P1 and P2 are satisfied, and let Q be as defined in (3.1). Under these conditions, $S(P, C)$ is stable if and only if Q and $P(I - QP)$ are both stable.

Comment: 1) By definition, to check the stability of $S(P, C)$, we have to check the stability of the four submatrices in (2.1). The theorem says that if the plant has the same number of inputs and outputs, and has no coincidences of poles and zeros in C_+, only the stability of two submatrices has to be checked. 2) It is easy to see from (3.2) that if P is stable, then Q is stable implies that $S(P, C)$ is stable.

Proof:

($\Rightarrow$) Follows from definition.

($\Leftarrow$) We shall prove that QP and PQ are both stable.

Let $G := P(I - QP)$ and write

$$(I - QP) = P^{-1}G \tag{3.3}$$

where $P^{-1} \in \mathbb{R}(s)^{n\times n}$ by P1.

Since, by assumption, Q and G are both stable

$$\mathscr{P}_+[(I - QP)] \subset \mathscr{P}_+[P] \tag{3.4}$$

and

$$\mathscr{P}_+[(I - QP)] = \mathscr{P}_+[P^{-1}G] \subset \mathscr{P}_+[P^{-1}] = \mathscr{Z}_+[P] \tag{3.5}$$

where we have used Lemma 1 in the last equality.

From (3.4), (3.5), and P2 we have

$$\mathscr{P}_+[QP] = \mathscr{P}_+[I - QP] \subset \{\mathscr{P}_+[P] \cap \mathscr{Z}_+[P]\} = \varnothing \tag{3.6}$$

and thus QP is stable.

Write $(I - PQ) = GP^{-1}$ and similar arguments as above show that $\mathscr{P}_+[PQ] = \varnothing$. □

Since P is strictly proper, $Q = C(I + PC)^{-1} \in \mathbb{R}_p(s)^{n\times n}$ if and only if $C = Q(I - PQ)^{-1} \in \mathbb{R}_p(s)^{n\times n}$ [2]. From Theorem 1, we see that, by $C = Q(I - PQ)^{-1}$, every stabilizing controller $C \in \mathbb{R}_p(s)^{n\times n}$ is defined by a stable $Q \in \mathbb{R}_p(s)^{n\times n}$ which satisfies that $P(I - QP)$ is stable. The converse is also true: every stable $Q \in \mathbb{R}_p(s)^{n\times n}$ satisfying that $P(I - QP)$ is stable defines a stabilizing controller for P. More precisely is the following corollary:

Corollary 1: Suppose P satisfies P1 and P2. Let $Q \in \mathbb{R}_p(s)^{n\times n}$ be stable and be such that $P(I - QP)$ is stable. Under these conditions, with $C := Q(I - PQ)^{-1}$, the system $S(P, C)$ is stable.

Proof: It is easy to check that the closed-loop transfer matrix of $S(P, Q(I - PQ)^{-1})$ is

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & P(I - QP) \end{bmatrix}. \tag{3.7}$$

From the proof of Theorem 1, we see that Q and $P(I - QP)$ both stable implies that PQ and $-QP$ are both stable, and thus the assertion follows. □

Consequently, for P satisfying P1 and P2, the set of *all stabilizing controllers* is given by

$$\{Q(I - PQ)^{-1} \mid Q \in \mathbb{R}_p(s)^{n\times n} \text{ is stable and is such that } P(I - QP) \text{ is stable}\}. \tag{3.8}$$

We now turn to the problem of decoupling. Since the design objective is to make the system decoupled, every achievable diagonal I/O transfer matrix from u_1 to y_2 has the form

$$H_{y_2u_1} = \begin{bmatrix} \dfrac{\tilde\beta_1(s)}{\alpha_1(s)} & & 0 \\ & \ddots & \\ 0 & & \dfrac{\tilde\beta_n(s)}{\alpha_n(s)} \end{bmatrix} =: M \tag{3.9}$$

where $\alpha_j, \tilde\beta_j \in \mathbb{R}[s]$, with α_j Hurwitz, for $j = 1, \cdots, n$.

In the following, we shall derive necessary and sufficient conditions on α_j and $\tilde\beta_j$ so that M can be achieved by a stabilizing controller, that is, by an element in the set described by (3.8). Write $P(s)$ as

$$P(s) = \left[\frac{Z_{ij}(s)}{P_{ij-}(s)P_{ij+}(s)}\right] \tag{3.10}$$

where $Z_{ij}(s), P_{ij-}(s), P_{ij+}(s) \in \mathbb{R}[s]$ are mutually coprime, $P_{ij+}(s)$ is monic with $\mathscr{Z}[P_{ij+}] \subset C_+$, and $\mathscr{Z}[P_{ij-}] \subset C_-$. Write $P^{-1}(s) \in \mathbb{R}(s)^{n\times n}$ as

$$P^{-1}(s) = \left[\frac{N_{ij}(s)}{D_{ij-}(s)D_{ij+}(s)}\right] \tag{3.11}$$

where $N_{ij}(s), D_{ij-}(s), D_{ij+}(s) \in \mathbb{R}[s]$ are mutually coprime, $D_{ij+}(s)$ is monic with $\mathscr{Z}[D_{ij+}] \subset C_+$, and $\mathscr{Z}[D_{ij-}] \subset C_-$.

From (3.7) and (3.9), with $C = Q(I - PQ)^{-1}$, we have $M = PQ$, thus

$$Q = P^{-1}M = \left[\frac{N_{ij}}{D_{ij-}D_{ij+}}\right] \begin{bmatrix} \dfrac{\tilde\beta_1}{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \dfrac{\tilde\beta_n}{\alpha_n} \end{bmatrix} = \left[\frac{N_{ij}\tilde\beta_j}{D_{ij-}D_{ij+}\alpha_j}\right]. \tag{3.12}$$

Since N_{ij} and D_{ij+} are coprime and D_{ij-}, α_j are Hurwitz, Q is stable if and only if

$$D_{ij+} \mid \tilde\beta_j, \quad \text{for } i = 1, \cdots, n,\ j = 1, \cdots, n. \tag{3.13}$$

Let $D_{j+}(s)$ be the monic least common multiple of $\{D_{ij+}(s), i = 1, \cdots, n\}$, then (3.13) holds if and only if for some $\beta_j \in \mathbb{R}[s]$

$$\tilde\beta_j = D_{j+}\beta_j, \qquad \text{for } j = 1, \cdots, n. \tag{3.14}$$

Also from (3.12) and (3.14), Q is proper if and only if

$$\deg(D_{ij-}D_{ij+}\alpha_j) \ge \deg(N_{ij}D_{j+}\beta_j), \quad \text{for } i = 1, \cdots, n,\ j = 1, \cdots, n. \tag{3.15}$$

Note that (3.15) holds if and only if

$$\deg(\alpha_j) - \deg(\beta_j) \ge \max_i [\deg(N_{ij}) - \deg(D_{ij-}) - \deg(D_{ij+})] + \deg(D_{j+}), \qquad \text{for } j = 1, \cdots, n. \tag{3.16}$$

Thus, Q is proper and stable if and only if α_j and $\tilde\beta_j$ satisfy (3.16) and (3.14).

Suppose M is chosen so that (3.14) and (3.16) are satisfied, then

$$P(I-QP)=(I-PQ)P=(I-M)P=\begin{bmatrix}\frac{\alpha_1-D_{1+}\beta_1}{\alpha_1} & & 0\\ & \ddots & \\ 0 & & \frac{\alpha_n-D_{n+}\beta_n}{\alpha_n}\end{bmatrix}\cdot\left[\frac{Z_{ij}}{P_{ij-}P_{ij+}}\right]=\left[\frac{(\alpha_i-D_{i+}\beta_i)Z_{ij}}{\alpha_iP_{ij-}P_{ij+}}\right]. \quad (3.17)$$

Since Z_{ij} and P_{ij+} are coprime and α_i, P_{ij-} are Hurwitz, $P(I-QP)$ is stable if and only if

$$P_{ij+}\,|\,(\alpha_i-D_{i+}\beta_i),\qquad \text{for } i=1,\cdots,n,\ j=1,\cdots,n. \quad (3.18)$$

Let $P_{i+}(s)$ be the monic least common multiple of $\{P_{ij+}(s), j=1,\cdots,n\}$, then (3.18) holds if and only if

$$P_{i+}\,|\,(\alpha_i-D_{i+}\beta_i),\qquad \text{for } i=1,\cdots,n. \quad (3.19)$$

Let $\lambda_{ik}\in C_+$, $k=1,\cdots,t_i$, be the zeros of $P_{i+}(s)$ with multiplicity n_{ik} and write $P_{i+}(s)$ as

$$P_{i+}(s)=(s-\lambda_{i1})^{n_{i1}}(s-\lambda_{i2})^{n_{i2}}\cdots(s-\lambda_{it_i})^{n_{it_i}}. \quad (3.20)$$

Then (3.19) is satisfied if and only if

$$(\alpha_i-D_{i+}\beta_i)^{(l-1)}(s)\,|_{s=\lambda_{ik}}=0,\quad \text{for } i=1,\cdots,n,\ k=1,\cdots,t_i,\ l=1,\cdots,n_{ik}. \quad (3.21)$$

Since D_{i+} and P_{i+} are coprime by assumption P2 and Lemma 1, we have $D_{i+}(\lambda_{ik})\neq 0$, for $i=1,\cdots,n$, $k=1,\cdots,t_i$. It can be checked that (3.21) holds if and only if

$$\left(\frac{\alpha_i-D_{i+}\beta_i}{D_{i+}}\right)^{(l-1)}(s)\,|_{s=\lambda_{ik}}=0\quad \text{for } i=1,\cdots,n,\ k=1,\cdots,t_i,\ l=1,\cdots,n_{ik} \quad (3.22)$$

that is

$$\beta_i^{(l-1)}(s)\,|_{s=\lambda_{ik}}=\left(\frac{\alpha_i}{D_{i+}}\right)^{(l-1)}(s)\,|_{s=\lambda_{ik}},\quad \text{for } i=1,\cdots,n,\ k=1,\cdots,t_i,\ l=1,\cdots,n_{ik}. \quad (3.23)$$

We summarize the aforementioned analysis in the following.

Theorem 2: Assume that P satisfies P1 and P2. Then an input–output transfer matrix $H_{y_2u_1}$ of the form in (3.9) is achievable by a stabilizing controller under the unity-feedback configuration shown in Fig. 1 if and only if

M1: α_i is Hurwitz and $\tilde{\beta}_i=D_{i+}\beta_i$ for some $\beta_i\in\mathbb{R}[s]$, for $i=1,\cdots,n$,

M2: $\deg(\alpha_i)-\deg(\beta_i)\geq \max_j[\deg(N_{ji})-\deg(D_{ji-})-\deg(D_{ji+})]+\deg(D_{i+})$, for $i=1,\cdots,n$,

M3: $\beta_i^{(l-1)}(s)\,|_{s=\lambda_{ik}}=(\alpha_i/D_{i+})^{(l-1)}(s)\,|_{s=\lambda_{ik}}$, for $i=1,\cdots,n$, $k=1,\cdots,t_i$, $l=1,\cdots,n_{ik}$, where α_i, β_i, etc., are defined in (3.9)–(3.14).

It follows from Theorem 2 that the set of *all achievable diagonal I/O maps* is given by

$$\left\{\operatorname{diag}\left[\frac{D_{1+}\beta_1}{\alpha_1}\cdots\frac{D_{n+}\beta_n}{\alpha_n}\right]\,\middle|\,\alpha_i \text{ is Hurwitz},\ \alpha_i,\beta_i \text{ satisfy M2 and M3}\right\} \quad (3.24)$$

and the set of *all decoupling controllers* is given by

$$\left\{P^{-1}\operatorname{diag}\left[\frac{D_{1+}\beta_1}{\alpha_1-D_{1+}\beta_1}\cdots\frac{D_{n+}\beta_n}{\alpha_n-D_{n+}\beta_n}\right]\,\middle|\,\alpha_i \text{ is Hurwitz},\ \alpha_i,\beta_i \text{ satisfy M2 and M3}\right\}. \quad (3.25)$$

Note that the conditions M1 and M2 are the same as those obtained in [4] for stable plants; M3 are the interpolation conditions due to the unstable poles of the plant. The conditions are specified "channel by channel" and are in a form suitable for computations.

IV. Computing a Decoupling Controller

In this section, we give an algorithm for finding decoupling controllers which achieve preassigned closed-loop poles.

Let k_i be the total number of zeros (counting multiplicities) of $P_{i+}(s)$, that is

$$k_i:=\sum_{k=1}^{t_i}n_{ik},\qquad \text{for } i=1,\cdots,n \quad (4.1)$$

where n_{ik} and t_i are defined in (3.20). For $i=1,\cdots,n$, let $\beta_i(s)$ be a polynomial of degree k_i-1, i.e.,

$$\beta_i(s):=\beta_{i1}s^{k_i-1}+\beta_{i2}s^{k_i-2}+\cdots+\beta_{ik_i} \quad (4.2)$$

and let $\alpha_i(s)$ be any Hurwitz polynomial. Then condition M3 becomes a set of linear equations in the coefficients of $\beta_i(s)$. More precisely, let

$$x_i:=[\beta_{i1}\,\beta_{i2}\cdots\beta_{ik_i}]^T \quad (4.3)$$

then M3 is equivalent to

$$A_ix_i=b_i,\qquad \text{for } i=1,\cdots,n \quad (4.4)$$

where

$$A_i:=\begin{bmatrix}
\lambda_{i1}^{k_i-1} & \lambda_{i1}^{k_i-2} & \cdots & \cdots & \lambda_{i1}^2 & \lambda_{i1} & 1\\
(k_i-1)\lambda_{i1}^{k_i-2} & (k_i-2)\lambda_{i1}^{k_i-3} & \cdots & \cdots & 2\lambda_{i1} & 1 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{(k_i-1)!}{(k_i-n_{i1})!}\lambda_{i1}^{k_i-n_{i1}} & \frac{(k_i-2)!}{(k_i-n_{i1}-1)!}\lambda_{i1}^{k_i-n_{i1}-1} & \cdots & (n_{i1}-1)! & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\lambda_{it_i}^{k_i-1} & \lambda_{it_i}^{k_i-2} & \cdots & \cdots & \lambda_{it_i}^2 & \lambda_{it_i} & 1\\
(k_i-1)\lambda_{it_i}^{k_i-2} & (k_i-2)\lambda_{it_i}^{k_i-3} & \cdots & \cdots & 2\lambda_{it_i} & 1 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{(k_i-1)!}{(k_i-n_{it_i})!}\lambda_{it_i}^{k_i-n_{it_i}} & \frac{(k_i-2)!}{(k_i-n_{it_i}-1)!}\lambda_{it_i}^{k_i-n_{it_i}-1} & \cdots & (n_{it_i}-1)! & 0 & \cdots & 0
\end{bmatrix} \quad (4.5)$$

$$b_i := \begin{bmatrix} \frac{\alpha_i}{D_{i+}}(\lambda_{i1}) \\ \frac{\alpha_i}{D_{i+}}^{(1)}(\lambda_{i1}) \\ \vdots \\ \frac{\alpha_i}{D_{i+}}^{(n_{i1}-1)}(\lambda_{i1}) \\ \vdots \\ \frac{\alpha_i}{D_{i+}}(\lambda_{it_i}) \\ \frac{\alpha_i}{D_{i+}}^{(1)}(\lambda_{it_i}) \\ \vdots \\ \frac{\alpha_i^{(n_{it_i}-1)}}{D_{i+}}(\lambda_{it_i}) \end{bmatrix}. \tag{4.6}$$

Each matrix A_i in (4.5) is a $k_i \times k_i$ *confluent Vandermonde matrix* which is nonsingular [1], and hence, the linear equation $A_i x_i = b_i$ has a unique solution for each i. Thus, for every Hurwitz polynomial α_i which satisfies M2 with deg $\beta_i = k_i - 1$, there is a *unique* polynomial β_i of degree $k_i - 1$ so that M3 is satisfied. In design, for each channel the poles of I/O transfer functions can be chosen as desired and the zeros, with the C_+-zeros of the plant properly kept, computed by solving (4.4).

In general, A_i and b_i are complex. Since the λ_{ik}'s occur in complex conjugate pairs, elementary row operations can be used to convert (4.4) into equivalent linear equations which involve only real elements and thus all solutions $x_i \in \mathbb{R}^{k_i}$.

If the zeros of $P_{i+}(s)$ are simple, that is, if $n_{ik} = 1$ for all i and k, then A_i and b_i in (4.5) and (4.6) are simply

$$A_i := \begin{bmatrix} \lambda_{i1}^{k_i-1} & \lambda_{i1}^{k_i-2} & \cdots & \lambda_{i1} & 1 \\ \lambda_{i2}^{k_i-1} & \lambda_{i2}^{k_i-2} & \cdots & \lambda_{i2} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{it_i}^{k_i-1} & \lambda_{it_i}^{k_i-2} & \cdots & \lambda_{it_i} & 1 \end{bmatrix}$$

$$b_i := \begin{bmatrix} \frac{\alpha_i}{D_{i+}}(\lambda_{i1}) \\ \frac{\alpha_i}{D_{i+}}(\lambda_{i2}) \\ \vdots \\ \frac{\alpha_i}{D_{i+}}(\lambda_{it_i}) \end{bmatrix}$$

and (4.4) becomes standard *Vandermonde* equations.

Algorithm

Data: Given $P(s)$ in the form as defined in (3.10), with assumptions P1 and P2 satisfied.

Step 1: Compute $P^{-1}(s)$ and put it in the form as defined in (3.11).

Step 2: For $i = 1, \cdots, n$, do the following.

a) Calculate k_i defined in (4.1).

b) With deg $\beta_i = k_i - 1$, choose any Hurwitz polynomial $\alpha_i(s)$ for which the condition in (3.16) holds.

c) Solve the linear equation $A_i x_i = b_i$ defined in (4.4).

d) Set $\beta_i(s)$ as that in (4.2).

Step 3: Compute

$$C = P^{-1} \begin{bmatrix} \frac{D_{1+}\beta_1}{\alpha_1 - D_{1+}\beta_1} & & 0 \\ & \ddots & \\ 0 & & \frac{D_{n+}\beta_n}{\alpha_n - D_{n+}\beta_n} \end{bmatrix}.$$

Comment: The performance of the ith channel is completely determined by the polynomial $\alpha_i(s)$ since the zero locations depend on the choices of $\alpha_i(s)$. In practical design, the α_i's should probably be chosen through some optimization procedure with inequality constraints [3], [6].

Example

Consider the plant

$$P(s) = \begin{bmatrix} \frac{(s+1)}{s(s+2)} & \frac{(s-2)}{s(s+2)} \\ \frac{1}{(s-1)} & \frac{(s-2)}{s(s-1)} \end{bmatrix}.$$

It is easy to check that

$$P_{1+}(s) = s, \quad k_1 = 1; \quad P_{2+}(s) = s(s-1), \quad k_2 = 2.$$

By computation

$$P^{-1}(s) = \begin{bmatrix} s(s+2) & -s(s-1) \\ \frac{-s^2(s+2)}{(s-2)} & \frac{s(s+1)(s-1)}{(s-2)} \end{bmatrix}$$

with

$$D_{1+}(s) = (s-2), \quad D_{2+}(s) = (s-2)$$

and the I/O map can only be chosen to be

$$M(s) = \begin{bmatrix} \frac{(s-2)\beta_1(s)}{\alpha_1(s)} & 0 \\ 0 & \frac{(s-2)\beta_2(s)}{\alpha_2(s)} \end{bmatrix}.$$

For $i = 1$, $x_1 := \beta_{11}$, and choose $\alpha_1(s) = (s+4)^3$. Then by solving the equation

$$\underbrace{1}_{A_1} \cdot \underbrace{\beta_{11}}_{x_1} = \underbrace{\left.\frac{(s+4)^3}{(s-2)}\right|_{s=0}}_{b_1} = -32$$

we get

$$\beta_1(s) = -32.$$

For $i = 2$, $x_2 := [\beta_{21}\ \beta_{22}]^T$, and choose $\alpha_2(s) = (s+2)^4$. Then by solving the equation

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} \beta_{21} \\ \beta_{22} \end{bmatrix}}_{x_2} = \underbrace{\begin{bmatrix} \left.\frac{(s+2)^4}{(s-2)}\right|_{s=0} \\ \left.\frac{(s+2)^4}{(s-2)}\right|_{s=1} \end{bmatrix}}_{b_2} = \begin{bmatrix} -8 \\ -81 \end{bmatrix}$$

we have

$$x_2 = \begin{bmatrix} \beta_{21} \\ \beta_{22} \end{bmatrix} = \begin{bmatrix} -73 \\ -8 \end{bmatrix}.$$

Thus

$$\beta_2(s) = -(73s + 8).$$

It follows that the I/O transfer matrix is

$$M(s) = \begin{bmatrix} \dfrac{-32(s-2)}{(s+4)^3} & 0 \\ 0 & \dfrac{-(s-2)(73s+8)}{(s+2)^4} \end{bmatrix}$$

and the resulting controller is

$$C(s) = \begin{bmatrix} \dfrac{-32(s-2)(s+2)}{(s^2+12s+80)} & \dfrac{(s-2)(73s+8)}{(s^2+9s+106)} \\ \dfrac{32s(s+2)}{(s^2+12s+80)} & \dfrac{-(s+1)(73s+8)}{(s^2+9s+106)} \end{bmatrix}.$$

V. Conclusions

For linear time-invariant MIMO square plant under unity-feedback configuration, we give a simplified necessary and sufficient condition for closed-loop stability. We parametrize the set of all achievable decoupled I/O maps in a way suitable for computation and give an algorithm for computing a decoupling controller which achieves preassigned closed-loop poles in each channel. The computations involved are inverting a rational matrix and solving linear algebraic equations. In practical design, the closed-loop pole locations should probably be determined by an optimization procedure with inequality constraints.

References

[1] Å. Brörck and T. Elfving, "Algorithms for confluent vandermonde systems," *Numerische Mathmatik*, vol. 21, pp. 130–137, 1973.

[2] F. M. Callier and C. A. Desoer, *Multivariable Feedback Systems*. New York: Springer-Verlag, 1982.

[3] S. H. Chen and C. A. Lin, "Computer-aided controller design using Q-parametrization method," in *Proc. 12th Nat. Symp. Automat. Contr.*, Taiwan, Republic of China, 1988, pp. 247–260.

[4] C. A. Desoer and M. J. Chen, "Design of multivariable feedback systems with stable plant," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 2, pp. 408–415, Apr. 1981.

[5] C. A. Desoer and A. N. Gündes, "Decoupling linear multiinput multioutput plants by dynamic output feedback: An algebraic theory," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 8, pp. 744–750, Aug. 1986.

[6] C. L. Gustafson and C. A. Desoer, "Controller design for linear multivariable feedback systems with stable plants, using optimization with inequality constraints," *Int. J. Contr.*, vol. 37, no. 5, pp. 881–907, 1983.

[7] J. Hammer and P. P. Khargonekar, "Decoupling of linear systems by dynamical output feedback," *Math. Syst. Theory*, vol. 17, no. 2, pp. 135–157, 1984.

[8] M. G. Safonov and B. S. Chen, "Multivariable stability-margin optimisation with decoupling and output regulation," *Proc. IEE*, vol. 129, no. 6, pt. D, pp. 276–282, Nov. 1982.

[9] A. I. G. Vardulakis, "Internal stabilization and decoupling in linear multivariable systems by unity output feedback compensation," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 8, pp. 735–739, Aug. 1987.

On Normalized Bezout Fractions of Distributed LTI Systems

S. Q. Zhu

Abstract—**It is shown that if a transfer matrix with entries in the Nevanlinna class has a bezout fraction, then it has a *normalized* one. This means that the full power of the theories developed by using normalized Bezout fractions can be applied to the transfer matrices with entries in the Nevanlinna class.**

I. Introduction

In recent years, the so-called stable Bezout fraction theory has been quite popular in system and control synthesis. The central concepts of this theory are *Bezout fractions* and *normalized Bezout fractions*. Many methods and techniques in stable Bezout fraction theory are based on the use of normalized Bezout fractions, for example: 1) to define a *graph metric* [9]; 2) to provide a necessary and sufficient condition for robust stabilization of feedback systems with additive perturbations to a normalized Bezout fraction [6]; 3) to give lower and upper bounds for the gap metric [12]. However, the existence of normalized bezout fractions has not been shown for distributed LTI systems. A general framework for the study of distributed LTI systems is the *Nevanlinna class N*, which is the quotient field of the integral domain H_∞, i.e., the Hardy space on the right half-plane. The set of transfer function matrices with entries in N includes many cases of interest for system and control theory. For example, it covers the following: 1) the Pritchard–Salamon class, i.e., semigroup systems with unbounded input and output operators [8]; 2) the Callier–Desoer class, i.e., systems with finite unstable poles [1], [2]. Curtain [5] proved that if a system in the Pritchard–Salamon class is stabilizable and detectable, then it is also in the Callier–Desoer class. It was shown that a system in the Pritchard–Salamon class has a normalized Bezout fraction, provided it is stabilizable and detectable [6], [11]. Callier *et al.* [3] proved that each SISO system in the Callier–Desoer class has a normalized Bezout fraction. But, in general, the existence on normalized Bezout fraction is still unknown. Hence, the theories developed by using normalized Bezout fractions cannot be applied to the MIMO systems in the Callier–Desoer class yet, and cannot be applied to some distributed LTI systems yet, such as neutral delay systems and wave equations, etc. whose transfer matrices have entries in the Nevanlinna class. The present note studies this open problem. In this note, it is shown that, if a transfer function matrix with entries in the Nevanlinna class N has a Bezout fraction, then it has a *normalized* one. This means that the full power of the theories developed by using normalized Bezout fractions can be applied to transfer function matrices with entries in N having a Bezout fraction.

This note is organized as follows. Section II is a preparation section in which we introduce the Hardy space H_∞ and Bezout fractions and discuss some properties of transfer function matrices with entries in the Nevanlinna class. In Section III, we introduce shift-invariant subspaces in the Hardy space H_2^k and prove the existence of normalized right Bezout fractions. Finally, in Section IV, we consider the existence of normalized left Bezout fractions and normalized Bezout fractions of discrete LTI systems.

II. Preparation

In this section, we introduce transfer function matrices whose entries belong to the Nevanlinna class and discuss some properties

Manuscript received October 18, 1988; revised March 14, 1989 and January 8, 1990.

The author is with the Faculty of Mathematics and Computing Society, Eindoven University of Technology, Eindoven, The Netherlands.

IEEE Log Number 9041204.