Information Processing Letters 37 (1991) 241–245
North-Holland

14 March 1991

Topological properties of supercube

Shyan-Ming Yuan

Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan, ROC 30050

Communicated by K. Ikeda
Received 31 August 1990
Revised 20 November 1990

Abstract

Yuan, S.-M., Topological properties of supercube, Information Processing Letters 37 (1991) 241–245.

The N-node Supercube is a new generalized version of Hypercube topology. Unlike the Binary Hypercube, the Supercube can be constructed for any number of nodes N. In addition, it maintains the connectivity and diameter properties of the corresponding hypercube. In this paper, we examine some topological properties of the Supercube from the graph-theory point of view.

Keywords: Fault-tolerant networks, parallel processing, graph theory, connectivity, diameter, node-disjoint path

1. Introduction

Hypercube topology has been used to develop several parallel machines, see [4] for references. The major advantage for using hypercube topology to design parallel computers is that it offers high data bandwidth and low message latency. However, there exists a significant drawback which is that the number of nodes in a hypercube topology must be a power of 2. Therefore, it cannot be constructed for any number of nodes. In the literature, several generalized versions of the hypercube have been proposed. In [1], a topology called Generalized Hypercube which can be constructed for any number of nodes N, was proposed, where N can be represented as a product of m_i's, $m_i > 1$ for $1 \leqslant i \leqslant r$ and $N = \prod_{i=1}^{r} m_i$. Although this network can be constructed for any number of nodes, it has two major drawbacks. When the number of nodes N is a prime, it reduces to a completely-connected graph. Also, significant changes have to be made for adding a new node. In [2], a topology called Incomplete Hypercube was proposed, which does not have the drawbacks of the Generalized Hypercube but has serious limitations in the connectivity. In some extreme situations, removing a single node may disconnect the whole system graph. In [5], Sen proposed a new modified hypercube called Supercube which does not have the drawbacks of either the Generalized Hypercube or the Incomplete Hypercube and has the same connectivity and diameter of the corresponding hypercube. In fact, an N-node Supercube is either a supergraph of a $(m-1)$-dimensional hypercube when $2^{m-1} < N < 2^m$ or an m-dimensional hypercube when $N = 2^m$. Thus, the network is called Supercube.

The purpose of this paper is to study the topological properties of the Supercube network topology. The remainder of this paper is organized in three sections. In the next section, we describe the Supercube topology and some of its basic properties. Section 3 derives some more topological properties of the Supercube. The last section contains concluding remarks.

2. The Supercube graph and its basic properties

The following formal definition of the Supercube graph is from [5]. Let $G=(V, E)$ be a graph, where V is the set of vertices and E is the set of edges in the graph. Assume that V contains N vertices, which are numbered from 0 to $N-1$. Then, each vertex X [1] in V can be expressed as a k-bit sequence $x_1 x_2 \ldots x_k$, where $k=\lceil \log_2 N \rceil$, $\forall i$, $1 \leqslant i \leqslant k$, $x_i = 0, 1$, and $X=\sum_{i=1}^{k} x_i 2^{k-i}$. The vertex set V is partitioned into three subsets V_1, V_2 and V_3, where

$$V_3 = \{ X \mid X \in V,\ X = 1u, \text{ where } u \text{ is a } (k-1)\text{-bit sequence} \},$$

$$V_2 = \{ X \mid X \in V,\ X = 0u,\ 1u \notin V, \text{ where } u \text{ is a } (k-1)\text{-bit sequence} \},$$

$$V_1 = \{ X \mid X \in V,\ X = 0u,\ 1u \in V, \text{ where } u \text{ is a } (k-1)\text{-bit sequence} \}.$$

Before we define the edge set E, let us define a term called Hamming distance.

Definition 2.1. The *Hamming distance* between two binary sequences u and v, denoted as $HD(u, v)$, is the number of positions where the bit values of u and v differ. In other words, $HD(u, v)$ is the bitwise XOR [2] of u and v.

The edge set E is the union of E_1, E_2, E_3 and E_4, where

$$E_1 = \{ (X, Y) \mid X, Y \in V,\ X = 0u,\ Y = 0v, \text{ where } u, v \text{ are } (k-1)\text{-bit sequences and } HD(u, v) = 1 \},$$

$$E_2 = \{ (X, Y) \mid X, Y \in V_3,\ X = 1u,\ Y = 1v, \text{ where } u, v \text{ are } (k-1)\text{-bit sequences and } HD(u, v) = 1 \},$$

$$E_3 = \{ (X, Y) \mid X \in V_3,\ Y \in V_2,\ X = 1u,\ Y = 0v, \text{ where } u, v \text{ are } (k-1)\text{-bit sequences and } HD(u, v) = 1 \},$$

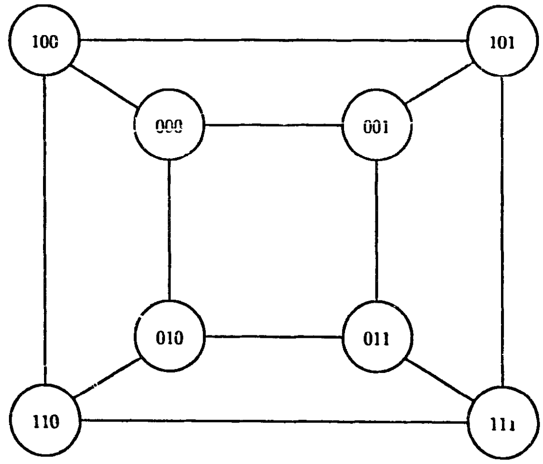


Fig. 1. A 3-dimensional hypercube.

$$E_4 = \{ (X, Y) \mid X \in V_3,\ Y \in V_1,\ X = 1u,\ Y = 0u, \text{ where } u \text{ is a } (k-1)\text{-bit sequence} \}.$$

From the above formal definition, we find that an N-node Supercube graph can be constructed from an m-dimensional hypercube, where $2^{m-1} < N \leqslant 2^m$. Assume that nodes in an m-dimensional hypercube are labeled from 0 to $2^m - 1$. For each node u, $N \leqslant u \leqslant 2^m - 1$, merging nodes u and $u - 2^{m-1}$ in the m-dimensional hypercube into a single node labeled as $u - 2^{m-1}$ and leaving other nodes in the m-dimensional hypercube unchanged, an N-node Supercube is obtained. Figures 1, 2 and 3 demonstrate how to construct a

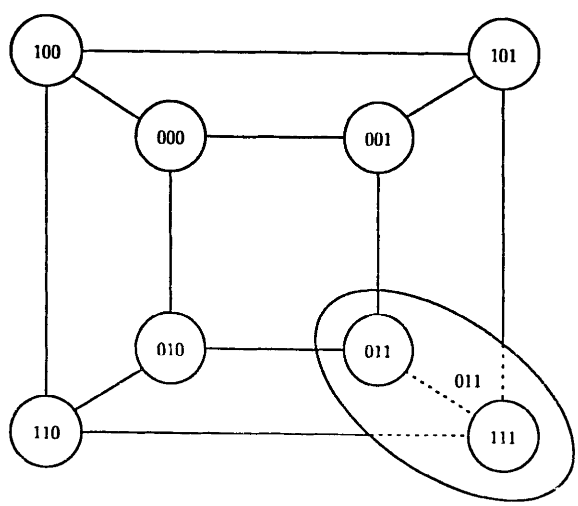
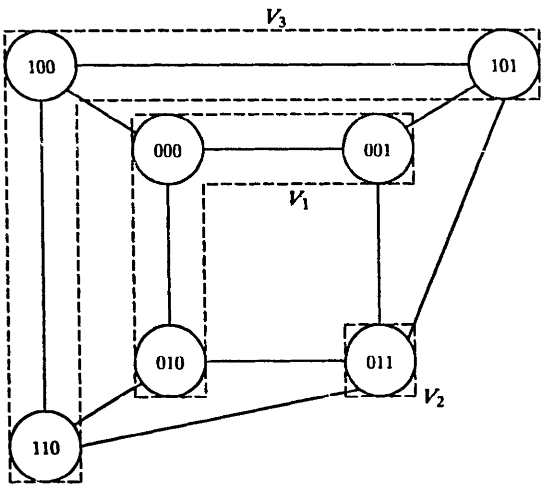


Fig. 2. Merge nodes 011 and 111 into the new 011.

[1] X is an integer between 0 and $N-1$.
[2] XOR is the exclusive or.



$E_1 = \{(0,1),(0,2),(1,3),(2,3)\}$
$E_2 = \{(4,5),(4,6)\}$
$E_3 = \{(3,5),(3,6)\}$
$E_4 = \{(0,4),(1,5),(2,6)\}$

Fig. 3. The resulting 7-node Supercube.

7-node Supercube from a 3-dimensional hypercube.

The following basic properties of the Supercubes are established in [5].

Theorem 2.2. *The node connectivity of an N-node Supercube is at least* $\lfloor \log_2 N \rfloor$.

Theorem 2.3. *The diameter of an N-node Supercube is at most* $\lfloor \log_2 N \rfloor$.

Theorem 2.4. *The node degree of an N-node Supercube is between* $k-1$ *and* $2k-2$, *where* $k = \lceil \log_2 N \rceil$.

3. Distances and paths in supercube

One of the major advantages of a k-dimensional hypercube is that there exist a lot of node-disjoint paths between any two nodes [3]. In particular, there are at least $k-1$ node-disjoint paths of length $\leqslant k$ between any two nodes in a k-dimensional hypercube.

Lemma 3.1. *There exist at least* $k-1$ *node-disjoint paths of length* $\leqslant k$ *between any two nodes in a* k*-dimensional hypercube.*

Proof. From Proposition 3.2 of [3], we know that there are i node-disjoint paths of length $= i$ between any two nodes A and B if $HD(A, B) = i$. From Proposition 3.3 of [3], we know that there are k node-disjoint paths of length $\leqslant HD(A, B) + 2$ between any two nodes A and B if $HD(A, B) \leqslant k-1$. Therefore, for any two nodes A and B in a k-dimensional hypercube,

- if $HD(A, B) = k$, there exist k node-disjoint paths of length $= k$,
- if $HD(A, B) = k-1$, there exist $k-1$ node-disjoint paths of length $= k-1$,
- if $HD(A, B) \leqslant k-2$, there exist k node-disjoint paths of length $\leqslant k$.

Thus, there exist at least $k-1$ node-disjoint paths of length $\leqslant k$ between any two nodes in a k-dimensional hypercube. □

As long as no more than $k-2$ nodes or links failed, the distance between any two nodes in a k-dimensional hypercube will be at most k. Here, we will show that the Supercube topology has a similar property.

Theorem 3.2. *There exist at least* $k-1$ *disjoint paths of length* $\leqslant k$ *between any two nodes in an N-node Supercube, where* $k = \lceil \log_2 N \rceil$.

Proof. There are three cases to be considered.

Case 1. Let the source node be $0s$ and the destination node be $0d$, where s and d are binary sequences of $k-1$ bits. All $0x$-nodes in an N-nodes Supercube form a binary hypercube of dimension $k-1$ and from [3],

- if $HD(s, d) = k-1$, there are $k-1$ node-disjoint paths of length $= k-1$,
- if $HD(s, d) \leqslant k-2$, there are $k-1$ node-disjoint paths of length $\leqslant k$ $(= k-2+2)$.

Therefore, there exist $k-1$ node-disjoint paths of length $\leqslant k$ between nodes $0s$ and $0d$ and all interior nodes of these paths are in the form of $0x$.

Since all nodes in the form of $0x$ in a k-dimensional hypercube belong to the N-node Supercube, these $k-1$ node-disjoint paths are legal paths of the N-node Supercube.

Case 2. Let the source node be $1s$ and the destination node be $1d$. Similar to Case 1, there are $k-1$ node-disjoint paths of length $\leqslant k$ between $1s$ and $1d$ and all interior nodes of these paths are in the form of $1x$ in a k-dimensional hypercube. It is known that not all nodes in the form of $1x$ in a k-dimensional hypercube belong to the N-node Supercube. For any path P: $1s = x_0 - x_1 - x_2 - \cdots - x_m = 1d$ contains some nodes $1x$ not in the N-node Supercube, let nodes $1x_{i_1}, 1x_{i_2}, \ldots, 1x_{i_p}$ be the only nodes in P not in the Supercube. By replacing each $1x_{i_j}$ by $0x_{i_j}$, we obtain another path P_s. Since for each $1x$ not in the Supercube, the $0x$ must be in the Supercube. Thus, all nodes in the path P_s are in the Supercube. If all adjacent nodes in P_s have an edge between them in the Supercube, then P_s is a legal path of the Supercube. There are the following cases:

- The adjacent nodes are $0u$ and $1v$. Since $1u$ and $1v$ are adjacent nodes in a k-dimensional hypercube, $HD(u, v) = HD(1u, 1v) = 1$. Because $1u$ is not in the Supercube, the edge $(0u, 1v)$ is in E_3.
- The adjacent nodes are $1u$ and $0v$. Similar to the previous case, the edge $(1u, 0v)$ is in E_3.
- The adjacent nodes are $0u$ and $0v$. Since $1u$ and $1v$ are adjacent nodes in a k-dimensional hypercube, $HD(u, v) = HD(0u, 0v) = 1$. Thus, the edge $(0u, 0v)$ is in E_1.

Since the $k-1$ P paths are node-disjoint in the k-dimensional hypercube and all interior nodes are in the form of $1x$, the $k-1$ paths P_s are constructed by replacing each $1x$ in P_s by either $0x$ or $1x$, the resulting $k-1$ P_s paths are node-disjoint in the Supercube.

Case 3. Let the source node be $1s$ and the destination node be $0d$. We need to consider the following cases: (a) $HD(s, d) \leqslant k-3$, (b) $HD(s, d) = k-2$ and (c) $HD(s, d) = k-1$.

(a) $HD(s, d) \leqslant k-3$. From [3], there are $k-1$ node-disjoint paths of length $\leqslant k-1$ between s and d in a $(k-1)$-dimensional hypercube. Thus, there are $k-1$ paths of the following form between $1s$ and $0d$ and all interior nodes are in the form of $0x$.

$$1s - 0s - 0u - \cdots - 0d$$

The node $0s$ is the only common node in these $k-1$ paths. Now transform these $k-1$ paths so that at most one of them contains node $0s$ without introducing other common nodes. Since all $0x$'s except $0s$ in these paths only appear in at most one path, if the transformation replaces some $0x$ by $1x$, then the $1x$ only appears in at most one path.

If $1s - 0s - 0d$ is one of the paths, it remains unchanged, as the only path that will contain nodes $0s$. For each path of $1s - 0s - 0u - \cdots - 0d$, if $1u$ is in the Supercube, then $1s - 1u - 0u - \cdots - 0d$ is used to replace the original path because $HD(1s, 1u) = HD(s, u) = 1$. If $1u$ is not in the Supercube, then $1s - 0u - \cdots - 0d$ can be used to replace the original path because the edge $(1s, 0u)$ is in E_3. Since the transformation does not increase the length of paths, the path length of the resulting $k-1$ node-disjoint paths is $\leqslant k-1+1 = k$.

(b) $HD(s, d) = k-2$. Since $HD(s, d) = k-2$, $HD(1s, 0d) = k-1$. From Proposition 3.2 of [3], there are $k-1$ node-disjoint paths of length $= k-1$ between $1s$ and $0d$ in a k-dimensional hypercube. Since $HD(s, d) = k-2$, if $s = s_1 s_2 \cdots s_{k-1}$ and $d = d_1 d_2 \cdots d_{k-1}$, then there exists an integer j, $1 \leqslant j \leqslant k-1$ such that $s_j = d_j = y$ and $\forall i$, $1 \leqslant i \leqslant k-1$, $i \neq j$, $s_i = \bar{d}_i$. Let $s_0 = 1$ and $d_0 = 0$, then $1s = s_0 s_1 \cdots s_{k-1}$ and $0d = d_0 d_1 \cdots d_{k-1}$. These $k-1$ node-disjoint paths of length $= k-1$ are as follows.

$\forall i$, $1 \leqslant i \leqslant j$, Path P_i:

$$\text{node } 0 = s_0 s_1 \cdots s_{i-2} s_{i-1} s_i \cdots s_{j-1} y s_{j+1} \cdots s_{k-1} = 1s$$

$$\text{node } 1 = s_0 s_1 \cdots s_{i-2} d_{i-1} s_i \cdots s_{j-1} y s_{j+1} \cdots s_{k-1}$$

$$\text{node } 2 = s_0 s_1 \cdots s_{i-2} d_{i-1} d_i s_{i+1} \cdots s_{j-1} y s_{j+1} \cdots s_{k-1}$$

$\cdots$

$$\text{node } j-i+1 = s_0 s_1 \cdots s_{i-2} d_{i-1} \cdots d_{j-1} y s_{j+1} s_{j+2} \cdots s_{k-1}$$

node $j-i+2 = s_0 s_1 \cdots s_{i-2} d_{i-1} \cdots d_{j-1}$
$y d_{j+1} s_{j+2} \cdots s_{k-1}$
...
node $k-i = s_0 s_1 \cdots s_{i-2} d_{i-1} \cdots d_{j-1}$
$y d_{j+1} \cdots d_{k-1}$
node $k+1-i = d_0 s_1 \cdots s_{i-2} d_{i-1} \cdots d_{j-1}$
$y d_{j+1} \cdots d_{k-1}$
...
node $k-2 = d_0 \cdots d_{i-3} s_{i-2} d_{i-1} \cdots d_{j-1}$
$y d_{j+1} \cdots d_{k-1}$
node $k-1 = d_0 \cdots d_{j-1} y d_{j+1} \cdots d_{k-1} = d$ (1)

$\forall i,\ j+1 \leqslant i \leqslant k-1$, Path P_i:
node $0 = s_0 s_1 \cdots s_{j-1} y s_{j+1} \cdots s_{i-1}$
$s_i \cdots s_{k-1} = 1s$
node $1 = s_0 s_1 \cdots s_{j-1} y s_{j+1} \cdots s_{i-1}$
$d_i s_{i+1} \cdots s_{k-1}$
node $2 = s_0 s_1 \cdots s_{j-1} y s_{j+1} \cdots s_{i-1}$
$d_i d_{i+1} s_{i+2} \cdots s_{k-1}$
...
node $k-i = s_0 s_1 \cdots s_{j-1} y s_{j+1}$
$\cdots s_{i-1} d_i \cdots d_{k-1}$
node $k-i+1 = d_0 s_1 \cdots s_{j-1} y s_{j+1}$
$\cdots s_{i-1} d_i \cdots d_{k-1}$
...
node $k-i+j = d_0 d_1 \cdots d_{j-1} y s_{j+1}$
$\cdots s_{i-1} d_i \cdots d_{k-1}$
node $k-i+j+1 = d_0 d_1 \cdots d_{j-1} y d_{j+1} s_{j+2}$
$\cdots s_{i-1} d_i \cdots d_{k-1}$
...
node $k-2 = d_0 d_1 \cdots d_{j-1} y d_{j+1} \cdots d_{i-2}$
$s_{i-1} d_i \cdots d_{k-1}$
node $k-1 = d_0 d_1 \cdots d_{j-1} y d_{j+1} \cdots d_{k-1} = d$ (2)

From the definition of the P_i's, we can observe that if a node $1x$ belongs to a path P_i, then the node $0x$ either belongs to P_i or does not belong to any path. Furthermore, if $1x$ and $0x$ belong to a path P_i, then P_i must be in the form of $1s - \cdots - 1x - 0x - \cdots - 0d$.

Since some of the nodes in these paths do not belong to the Supercube, we now show how to convert these $k-1$ paths so that all interior nodes are in the Supercube without introducing any common nodes. We first replace all nodes $1x$ not in the Supercube by $0x$. We then remove all duplications in the resulting paths. From the same argument as in Case 2, we know that all adjacent nodes in the resulting paths are directly connected in the Supercube.

(c) $HD(s, d) = k-1$. Since $HD(s, d) = k-1$, from [3], there are $k-1$ node-disjoint paths of length $= k-1$ between s and d in a $(k-1)$-dimensional hypercube. Thus, there are $k-1$ paths with length $= k-1$ of the following form between $1s$ and $0d$ and all interior nodes are in the form of $0x$.

$1s - 0s - 0u - \cdots - 0d$

The node $0s$ is the only common node in these $k-1$ paths. From the same argument as in Case 3(a), we can convert these $k-1$ paths into $k-1$ node-disjoint paths of length $\leqslant k$. □

4. Concluding remarks

We have shown some topological properties of Supercube network topology that sheds light upon some of the reasons why the Supercube is attractive. Because the Supercube can be constructed for any number of nodes and have all the nice properties of Hypercube such as $O(\log_2 N)$ node connectivity, $O(\log_2 N)$ diameter, and $\lceil \log_2 N \rceil - 1$ node-disjoint paths of length $\leqslant \lceil \log_2 N \rceil$ between any two nodes, it becomes an ideal network for fault-tolerant and parallel computer designs.

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