國 立 交 通 大 學

機械工程學系

碩士論文

新 Mathieu-Duffing 系統的渾沌現象與其應用

適應逆步控制及部分區域穩定理論之

實用混合投影渾沌同步及辛渾沌同步

Chaos and Pragmatical Hybrid Projective and Symplectic Chaos Synchronization of a New Mathieu-Duffing System by Adaptive Backstepping Control and by Partial Region Stability Theory

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中華民國九十七年六月

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所提論文(中文) 新 Mathieu-Duffing 系統的渾沌現象與其應用適應逆步控制及部分區域穩定理論之實用混合投影渾沌同步及辛渾沌同步

(英文)Chaos and Pragmatical Hybrid Projective and Symplectic Chaos Synchronization of a New Mathieu-Duffing System by Adaptive Backstepping Control and by Partial Region Stability Theory

合於碩士資格水準、業經本委員會評審認可。

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摘要

本論文由三部分構成:(1)以相圖、龐卡萊映射圖、分岐圖、功率譜及 Lyapunov 指數圖等數值方法研究 Mathieu – Duffing 系統的渾沌行為。(2)用適應逆步控制在不同的初始條件下的兩個 Mathieu – Duffing 雙系統對不同的渾沌系統 Duffing – van der Pol 系統與 Lü 系統的實用混合投影渾沌同步及辛渾沌同步。(3)應用部分區域穩定性理論研究廣義渾沌同步、渾沌控制及實用混合投影渾沌同步。

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Student：Yan-Sian Li **Advisor：Zheng-Ming Ge**

ABSTRACT

This thesis consists of three parts: (1) the chaotic behaviors of Mathieu - Duffing system are studied numerically by phase portraits, Poincaré maps, bifurcation diagrams, power spectrum and Lyapunov exponent diagrams. (2) Mathieu - Duffing system is studied for pragmatical hybrid projective hyperchaotic generalized synchronization (PHPHGS) and pragmatical hybrid projective and symplectic synchronization (PHPSS) with different kinds of different chaotic systems, Duffing-van der Pol system and Lü system, by adaptive backstepping control. (3) chaotic generalized synchronization, chaos control and pragmatical hybrid projective generalized synchronization is studied by partial region stability theory.

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CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

Chaotic system features that it has complex dynamical behaviors and sensitive behavior dependence initial conditions. Since Pecora and Carroll [1] introduced a method to synchronize two identical systems with different initial conditions, chaos synchronization has attracted a great deal of attention from various fields during the last two decades. Recently, many valuable control methods and techniques have been developed to synchronize chaotic systems, such as backstepping design method [2], impulsive control method [3], invariant manifold method [4], adaptive control method [5], linear and nonlinear feedback control method [6], and active control approach [7], PC method [8], etc. Most of them are based on the exact knowledge of the system structure and all parameters. But in practice, some or all of the system parameters are uncertain. Additionally, these parameters change at every time. A lot of researchers have studied to solve this problem by adaptive synchronization [12-15].

This thesis is organized as follows. In Chapter 2, the chaotic behaviors of a new Mathieu – Duffing system is studied numerically by phase portraits, Poincaré maps and Lyapunov exponent diagrams. That can be explained that chaos exists in the new Mathieu - Duffing system.

In the current scheme of adaptive synchronization [12-15], the traditional Lyapunov stability theorem and Babalat lemma are used to prove that the error vector approaches zero, as time approaches infinity. But the question of that why the estimated parameters also approach uncertain parameters remains unanswered. In Chapter 3, by the pragmatical asymptotical stability theorem, the question can be answered strictly. That the error vector tends to zero and the estimated parameters approach uncertain values is guaranteed by the pragmatical asymptotical stability theorem [17, 18] and adaptive backstepping control.

In Chapter 4, a new kind of synchronization and a new control Lyapunov function are proposed. The symplectic synchronization

$$y = H(x, y, z, t) \tag{1.1}$$

is studied, where x and y are the state vectors of the "master" and of the "slave" respectively and z is a given function vector of time, which may take various forms, either a regular or a chaotic function of time [22-24]..

A new control Lyapunov function

$$V(e) = \exp(ke^T e) - 1 \tag{1.2}$$

is proposed for backstepping control where e is error dynamics. Using the new control Lyapunov function, the error tolerance can be decreased marvelously to 10^{-17} of that using traditional control Lyapnov function

$$V(e) = e^T e \tag{1.3}$$

In Chapter 5, a new scheme to achieve chaos control by partial region stability theory is proposed [33, 34]. By using the GYC partial region stability theory, Lyapunov function becomes a simple linear homogeneous function of error states and controllers are simpler and introduce less simulation error. Similarly, in Chapter 6, a new chaos generalized synchronization strategy by partial region stability theory is proposed [33, 34].

In Chapter 7, the chaotic behaviors of a new Mathieu – Duffing systems with Bessel function parameters is studied numerically by phase portraits, Poincaré maps, bifurcation and Lyapunov exponent diagrams. If can be discovered that chaos also exists in the new Mathieu – Duffing systems with Bessel function parameters.

In Chapter 8, the error vector tends to zero and that the estimated parameters approach uncertain values is guaranteed by the pragmatical asymptotical stability theorem [17, 18] and GYC partial region stability theory. A pragmatical hybrid projective generalized synchronization (PHPGS) strategy is proposed.

In Chapter 9, conclusions are drawn.

Chapter 2

Chaos in a New Mathieu – Duffing System

2.1 Preliminaries

Abundant chaotic behaviors in a new Mathieu - Duffing system are studied numerically by phase portraits, Poincaré maps and Lyapunov exponent diagram.

2.2 A new Mathieu – Duffing system

Mathieu system and Duffing system are two typical nonlinear nonautonomous systems:

$$\begin{cases} \dfrac{d}{dt}x_1 = x_2 \\ \dfrac{d}{dt}x_2 = -(a+b\sin\omega t)x_1 - (a+b\sin\omega t)x_1^3 - cx_2 + d\sin\omega t \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{d}{dt}x_3 = x_4 \\ \dfrac{d}{dt}x_4 = -x_3 - x_3^3 - ex_4 + f\sin\omega t \end{cases} \tag{2.2}$$

Exchanging $\sin\omega t$ term in Eq.(2.1) with x_3 and $\sin\omega t$ in Eq.(2.2) with x_1, we obtain a new autonomous Mathieu – Duffing system:

$$\begin{cases} \frac{d}{dt}x_1 = x_2 \\ \frac{d}{dt}x_2 = -(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3 \\ \frac{d}{dt}x_3 = x_4 \\ \frac{d}{dt}x_4 = -x_3 - x_3^3 - ex_4 + fx_1 \end{cases} \quad (2.3)$$

where x_1, x_2, x_3, x_4 are state variables, and a, b, c, d, e, f are parameters. We analyses and presents simulation results of the chaotic dynamics produced from a new Mathieu – Duffing system in the state equations of Eq. (2.3). When a=20.30, b=0.5970, c=0.005, d=-24.441, e=0.002, f=14.63, abundant chaotic behaviors are shown by phase portraits, Poincaré maps, power spectrum and Lyapunov exponent diagram in Figs 2.1-2.6.

2.3 Simulation results

The parameters used are: a=20.30, b=0.5970, c=0.005, d=-24.441, e=0.002, and f=14.63. The numerical simulations are carried out by MATLAB with using the fractional operator in the Simulink environment.

The time history of four states, phase portraits, Poincaré maps and power spectrum of a new Mathieu – Duffing system with parameters given are shown in Fig.2.1~Fig.2.5. Chaos exists for all cases.

We vary the system parameter *d*, with other system fixed as: a=20.30, b=0.5970, c=0.005, e=0.002, f=14.63, the chaotic Lyapunov exponents are obtained as shown in Fig.2.6.

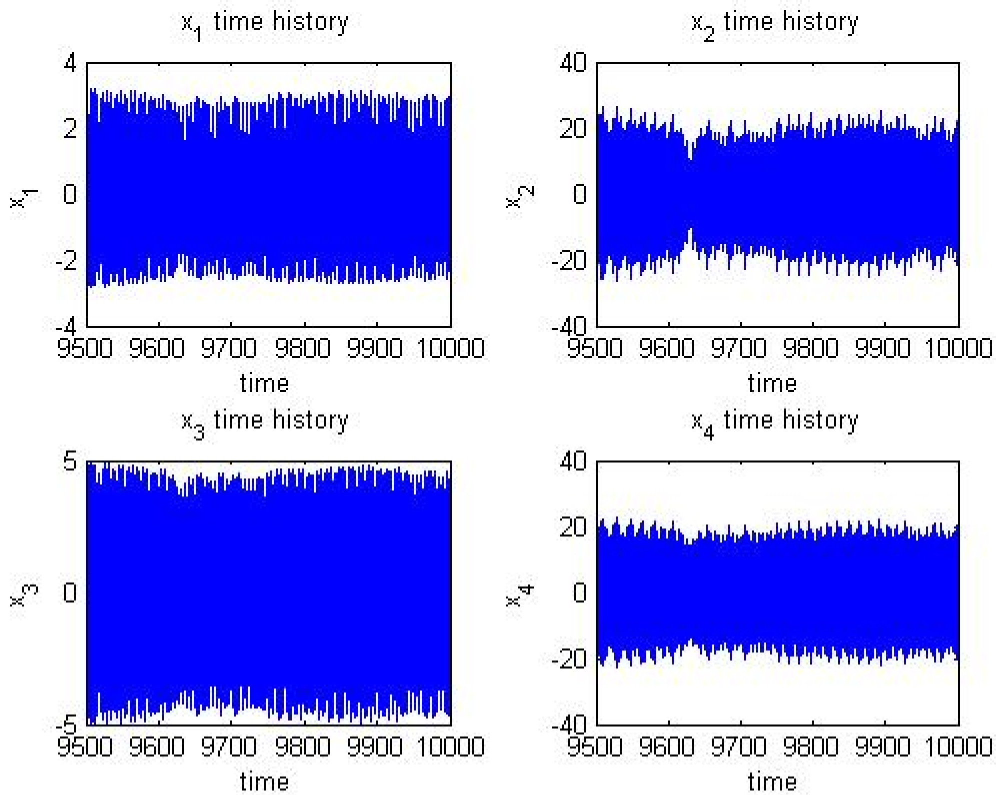

Fig. 2.1 The time history of the four states.

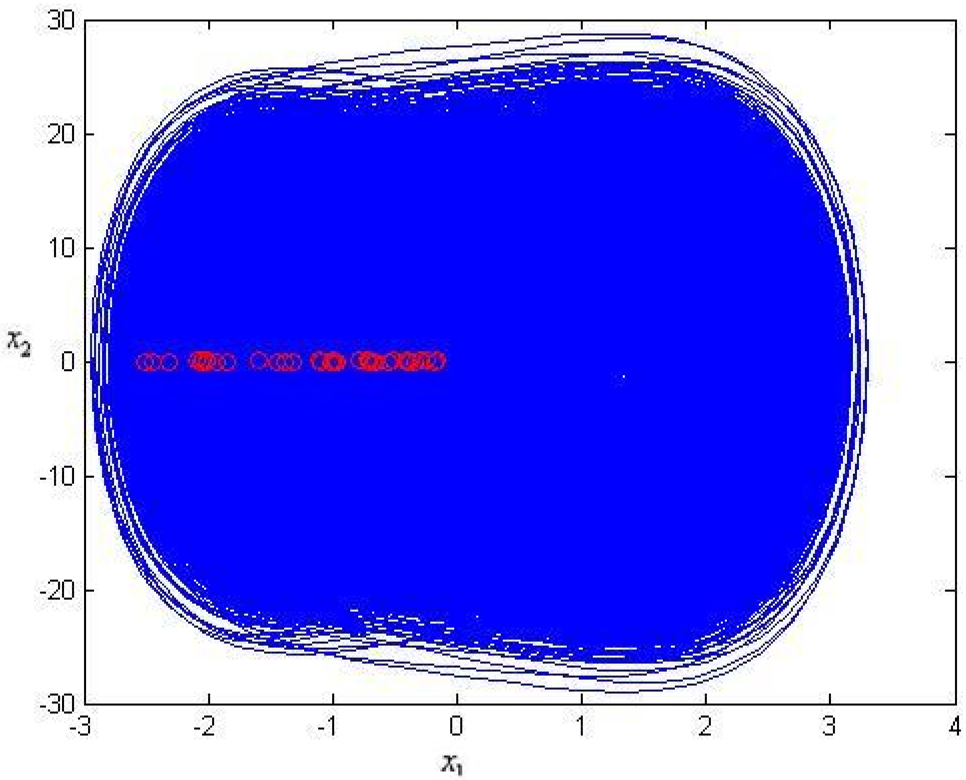

Fig. 2.2 The phase portraits and Poincaré maps of x_1, x_2 dimensions.

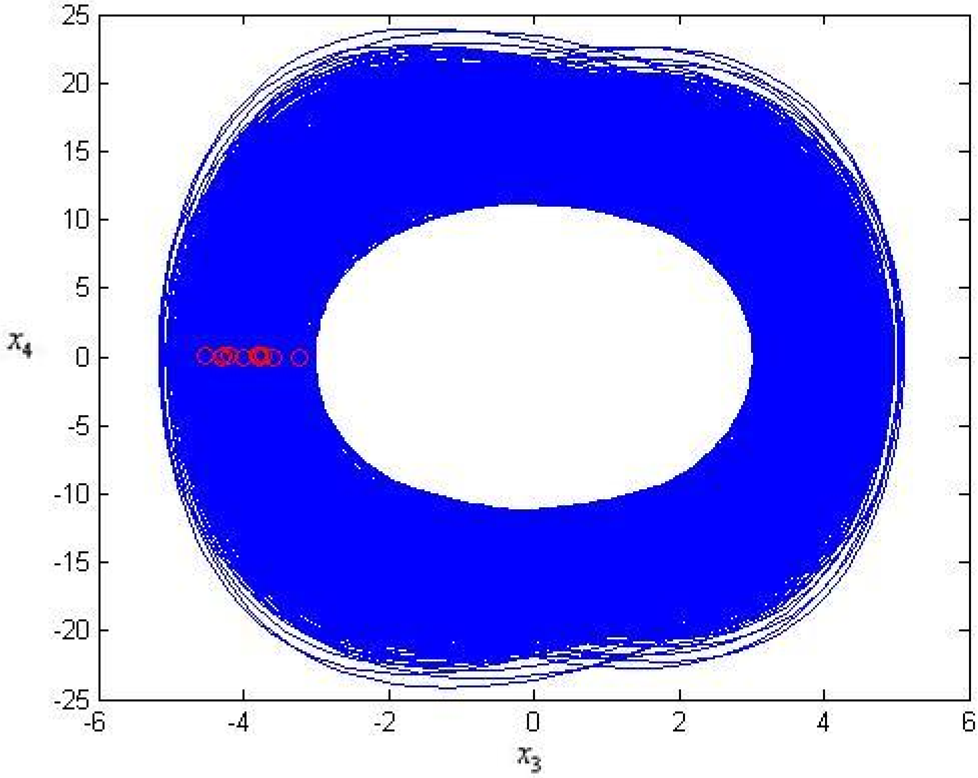

Fig. 2.3 The phase portraits and Poincaré maps of x_3, x_4 dimensions.

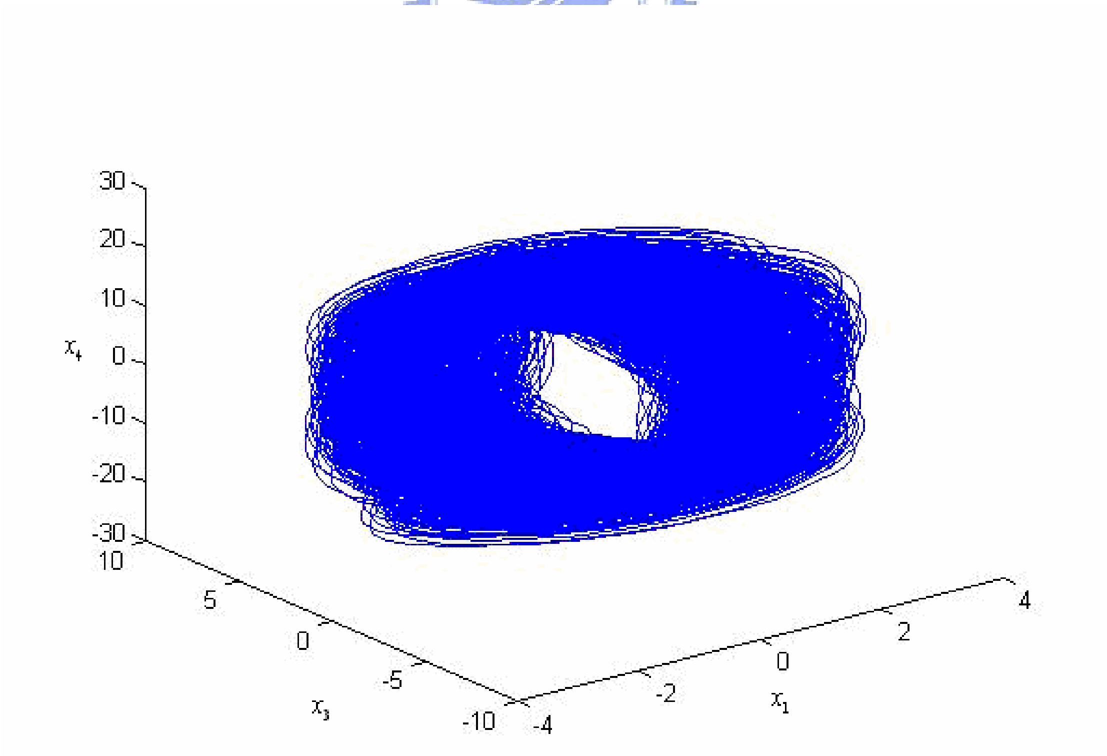

Fig. 2.4 Chaotic phase protrait of a new Mathieu-Duffing system in three dimensions.

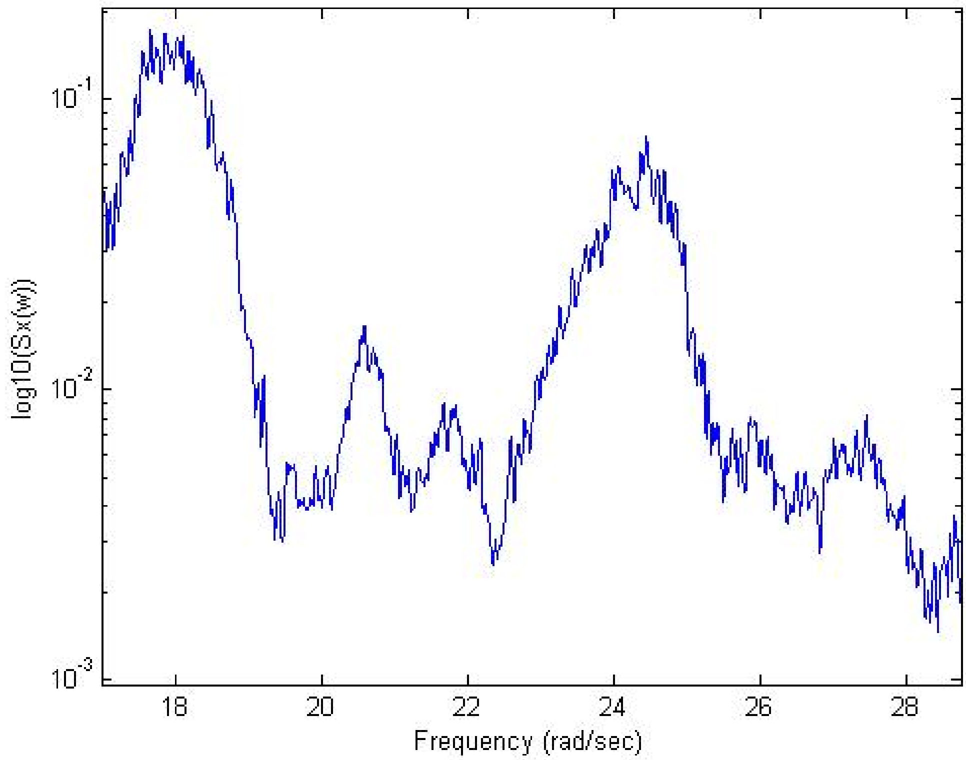

Fig. 2.5 The chaotic power spectrum of x_1 for a new Mathieu-Duffing system.

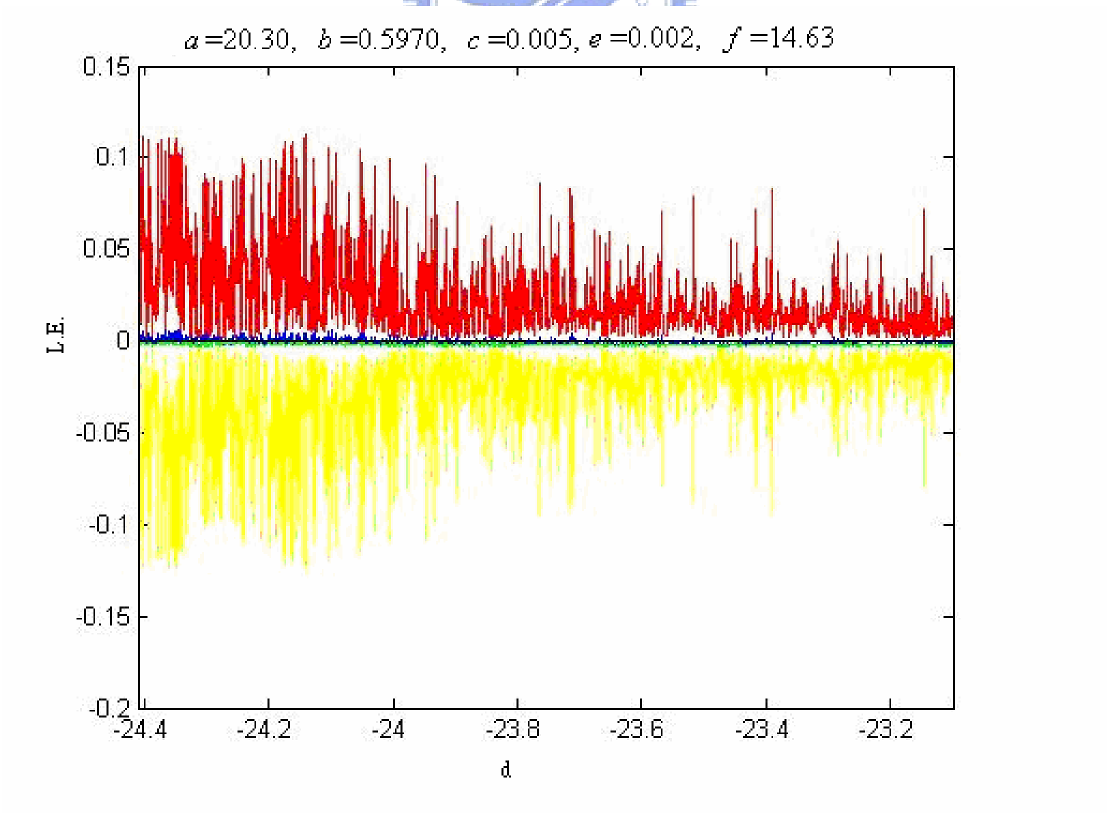

Fig. 2.6 Lyapunov exponents of a chaotic new Mathieu-Duffing system (+,0,0,-).

Chapter 3

Pragmatical Hybrid Projective Hyperchaotic Generalized Synchronization of Hyperchaotic Systems by Adaptive Backstepping Control

3.1Preliminaries

In the current scheme of adaptive synchronization [11-15], the traditional Lyapunov stability theorem and Babalat lemma are used to prove that the error vector approaches zero, as time approaches infinity. But the question of that why the estimated parameters also approach uncertain parameters remains unanswered. By the pragmatical asymptotical stability theorem, the question can be answered strictly. In this Chapter, that the error vector tends to zero and the estimated parameters approach uncertain values is guaranteed by the pragmatical asymptotical stability theorem [16, 17] and adaptive backstepping control.

3.2 Synchronization scheme

Among many kinds of synchronizations [2-6], the generalized synchronization is investigated [7-8]. This means that we can give a functional relationship between the states of the master and slave $y = G(x)$. In this Chapter, a special case of hybrid projective hyperchaotic generalized synchronization

$$y = G(x) = g\,x(t)z(t) \qquad (3.1)$$

is studied, where x and y are state variable vectors of the master and slave, respectively. z is state vector of a third hyperchaotic system, called functional system, since it is a constituent of function G.. While the entries of constant vector g can be either positive or negative, hybrid projective synchronization is named. When z is chaotic, chaotic generalized synchronization is named. Pragmatical asymptotical stability theorem is used, pragmatical synchronization is named. As a whole, pragmatical hybrid projective hyper-chaotic generalized synchronization (PHPHGS) is named.

The master system is

$$\dot{x} = f(x) \qquad (3.2)$$

where $x = [x_1, x_2, ..., x_n]^T \in \Re^n$ is a state vector and all parameters of Eq.(3.2) are uncertain. The slave system is

$$\dot{y} = f(y) + u \qquad (3.3)$$

$y = [y_1, y_2, ..., y_n]^T \in \Re^n$ is also a state vector, where u is a control vector. The functional system is

$$\dot{z} = k(z) \qquad (3.4)$$

where $z = [z_1, z_2, ..., z_n]^T \in \Re^n$ is a hyperchaotic state vector and all parameters of Eq.(3.4) are known.

The control task is to force the slave state vector to track an n-dimensional desired vector

$$h(t) = [h_1(t), h_2(t), ..., h_n(t)] = [g_1 x_1(t) z_1(t), g_2 x_2(t) z_2(t), ..., g_n x_n(t) z_n(t)] \qquad (3.5)$$

where $g = [g_1, g_2, ..., g_n]$ are constant vector with positive and negative entries.

Define the error vector $e(t) = [e_1, e_2, ..., e_n]^T \in \Re^n$:

$$e(t) = y(t) - h(t) = y - gxz \qquad (3.6)$$

The controlling goal is that

$$\lim_{t\to\infty} e = 0 \tag{3.7}$$

can be accomplished on the base of pragmatical asymptotical stability theorem by adaptive backstepping control.

3.3. Numerical results of PHPHGS by adaptive backstepping control

Take the Mathieu-Duffing system (2.3) as master system, where a, b, c, d, e, f are uncertain parameters and following Mathieu-Duffing system as slave system:

$$\begin{cases} \frac{d}{dt}y_1 = y_2 \\ \frac{d}{dt}y_2 = -(\hat{a}+\hat{b}y_3)y_1 - (\hat{a}+\hat{b}y_3)y_1^3 - \hat{c}y_2 + \hat{d}y_3 \\ \frac{d}{dt}y_3 = y_4 \\ \frac{d}{dt}y_4 = -y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 \end{cases} \tag{3.8}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}$ *and* $\hat{f}$ are estimated parameters. Finally, take a new Duffing – van der Pol system [32]:

$$\begin{cases} \frac{d}{dt}z_1 = z_2 \\ \frac{d}{dt}z_2 = -z_1 - z_1^3 - a_1 z_2 + d_1 z_3 \\ \frac{d}{dt}z_3 = z_4 \\ \frac{d}{dt}z_4 = -b_1 z_3 + c_1(1-z_3^2)z_4 + f_1 z_1 \end{cases} \tag{3.9}$$

as functional system, where z_1, z_2, z_3, z_4, are state variables, and a_1, b_1, c_1, d_1, f_1, are constants. When a_1=0.01, b_1=1.00063, c_1=5, d_1=0.66635, f_1=0.05, chaos occurs as shown in Fing 3.1.

By definition, error states $e_i (i = 1, 2, 3, 4)$ are

$$\begin{cases} e_1 = y_1 - g_1 x_1 z_1 \\ e_2 = y_2 - g_2 x_2 z_2 \\ e_3 = y_3 - g_3 x_3 z_3 \\ e_4 = y_4 - g_4 x_4 z_4 \end{cases} \tag{3.10}$$

where $g_1, g_2, g_3, and\ g_4$ are partly positive and partly negative constants, to form hybrid projective synchronization.

In order to lead (y_1, y_2, y_3, y_4) to $(g_1 x_1 z_1, g_2 x_2 z_2, g_3 x_3 z_3, g_4 x_4 z_4)$, u_1, u_2, u_3 and u_4 are added to each equation of Eq.(3.8), respectively:

$$\begin{cases} \frac{d}{dt} y_1 = y_2 + u_1 \\ \frac{d}{dt} y_2 = -(\hat{a} + \hat{b} y_3) y_1 - (\hat{a} + \hat{b} y_3) y_1^3 - \hat{c} y_2 + \hat{d} y_3 + u_2 \\ \frac{d}{dt} y_3 = y_4 + u_3 \\ \frac{d}{dt} y_4 = -y_3 - y_3^3 - \hat{e} y_4 + \hat{f} y_1 + u_4 \end{cases} \tag{3.11}$$

The error dynamics becomes:

$$\begin{cases} \dot{e}_1 = e_2 + g_2 x_2 z_2 - g_1 x_2 z_1 - g_1 x_1 z_2 + u_1 \\ \dot{e}_2 = -\hat{a} e_1 - \hat{a} g_1 x_1 z_1 - \hat{b} e_1 e_3 - \hat{b} g_3 x_3 z_3 e_1 - \hat{b} g_1 x_1 z_1 e_3 - \hat{b} g_1 x_1 z_1 g_3 x_3 z_3 - \hat{a} e_1^3 \\ \quad - 3\hat{a} g_1 x_1 z_1 e_1^2 - 3\hat{a} g_1^2 x_1^2 z_1^2 e_1 - \hat{a} g_1^3 x_1^3 z_1^3 - \hat{b} e_1^3 e_3 - 3\hat{b} g_1 x_1 z_1 e_1^2 e_3 - 3\hat{b} g_1^2 x_1^2 z_1^2 e_1 e_3 \\ \quad - \hat{b} g_1^3 x_1^3 z_1^3 e_3 - \hat{b} g_3 x_3 z_3 e_1^3 - 3\hat{b} g_1 x_1 z_1 g_3 x_3 z_3 e_1^2 - 3\hat{b} g_1^2 x_1^2 z_1^2 g_3 x_3 z_3 e_1 \\ \quad - \hat{b} g_1^3 x_1^3 z_1^3 g_3 x_3 z_3 - \hat{c} e_2 - \hat{c} g_2 x_2 z_2 + \hat{d} e_3 + \hat{d} g_3 x_3 z_3 + a g_2 x_1 z_2 + b g_2 x_3 x_1 z_2 \\ \quad + a g_2 x_1^3 z_2 + b g_2 x_3 x_1^3 z_2 + c g_2 x_2 z_2 - d g_2 x_3 z_2 + g_2 x_2 z_1 + g_2 x_2 z_1^3 + a_1 g_2 x_2 z_2 \\ \quad - d_1 g_2 x_2 z_3 + u_2 \\ \dot{e}_3 = e_4 + g_4 x_4 z_4 - g_3 x_4 z_3 - g_3 x_3 z_4 + u_3 \\ \dot{e}_4 = -e_3 + g_3 x_3 z_3 - e_3^3 - 3 e_3^2 g_3 x_3 z_3 - 3 e_3 g_3^2 x_3^2 z_3^2 - g_3^3 x_3^3 z_3^3 - \hat{e} e_4 - \hat{e} g_4 x_4 z_4 \\ \quad + \hat{f} e_1 + \hat{f} g_1 x_1 z_1 + g_4 x_3 z_4 + g_4 x_3^3 z_4 + e g_4 x_4 z_4 - f g_4 x_1 z_4 + b_1 g_4 x_4 z_3 \\ \quad - c_1 g_4 x_4 z_4 + c_1 g_4 x_4 z_3^2 z_4 - f_1 g_4 x_4 z_1 + u_4 \end{cases} \tag{3.12}$$

where

$$\dot{e}_i = \dot{y}_i - g_i \dot{x}_i z_i - g_i x_i \dot{z}_i, \quad (i = 1,2,3,4) \tag{3.12}$$

Step1. We consider the asymptotical stability of $e_1 = 0$:

$$\dot{e}_1 = e_2 + g_2 x_2 z_2 - g_1 x_2 z_1 - g_1 x_1 z_2 + u_1 \tag{3.13}$$

where e_2 is regarded as a controller. Choose a control Lyapunov function (CLF) of the form

$$v_1 = \tfrac{1}{2} e_1^2 \tag{3.14}$$

Its time derivative along the solution of Eq.(3.13) is

$$\dot{v}_1 = e_1 (e_2 + g_2 x_2 z_2 - g_1 x_2 z_1 - g_1 x_1 z_2 + u_1) \tag{3.15}$$

Assume virtual controller $e_2 = \alpha_1(e_1) = -e_1$. Eq.(3.15) can be rewritten as

$$\dot{v}_1 = e_1 (-e_1 + g_2 x_2 z_2 - g_1 x_2 z_1 - g_1 x_1 z_2 + u_1) \tag{3.16}$$

Choose

$$u_1 = -g_2 x_2 z_2 + g_1 x_2 z_1 + g_1 x_1 z_2 \tag{3.17}$$

Eq.(3.16) becomes

$$\dot{v}_1 = -e_1^2 < 0 \tag{3.18}$$

This means that $e_1 = 0$ is asymptotically stable.

Step2. When e_2 is considered as a controller, $\alpha_1(e_1)$ is an estimative function.

Define

$$w_2 = e_2 - \alpha_1(e_1) = e_1 + e_2 \tag{3.19}$$

Study the (e_2, w_2) system. We have

$$\dot{w}_2 = \dot{e}_1 + \dot{e}_2 \tag{3.20}$$

Choose a control Lyapunov function of the form

$$v_2 = v_1 + \tfrac{1}{2}(w_2^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2) > 0\,, \tag{3.21}$$

where $\tilde{a} = a - \hat{a}\,, \tilde{b} = b - \hat{b}\,, \tilde{c} = c - \hat{c}\,, \tilde{d} = d - \hat{d}$ and $\hat{a}\,, \hat{b}\,, \hat{c}\,, \hat{d}$ are estimated values of the unknown parameters a, b, c, d, respectively. We have

$$\begin{aligned}
\dot{v}_2 &= \dot{v}_1 + w_2(\dot{e}_1 + \dot{e}_2) + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} \\
&= \dot{v}_1 + w_2(e_2 - \hat{a}e_1 - \hat{a}g_1x_1z_1 - \hat{b}g_3x_3z_3e_1 - \hat{b}g_1x_1z_1g_3x_3z_3 - \hat{a}e_1^3 \\
&\quad - 3\hat{a}g_1x_1z_1e_1^2 - 3\hat{a}g_1^2x_1^2z_1^2e_1 - \hat{a}g_1^3x_1^3z_1^3 - \hat{b}g_3x_3z_3e_1^3 - 3\hat{b}g_1x_1z_1g_3x_3z_3e_1^2 \\
&\quad - 3\hat{b}g_1^2x_1^2z_1^2g_3x_3z_3e_1 - \hat{b}g_1^3x_1^3z_1^3g_3x_3z_3 - \hat{c}e_2 - \hat{c}g_2x_2z_2 + \hat{d}g_3x_3z_3 + ag_2x_1z_2 \\
&\quad + bg_2x_3x_1z_2 + ag_2x_1^3z_2 + bg_2x_3x_1^3z_2 + cg_2x_2z_2 - dg_2x_3z_2 + g_2x_2z_1 + g_2x_2z_1^3 \\
&\quad + a_1g_2x_2z_2 - d_1g_2x_2z_3 + u_2) + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} \\
&\quad + w_2e_3(-\hat{b}e_1 - \hat{b}g_1x_1z_1 - \hat{b}e_1^3 - 3\hat{b}g_1x_1z_1e_1^2 - 3\hat{b}g_1^2x_1^2z_1^2e_1 - \hat{b}g_1^3x_1^3z_1^3 + \hat{d})
\end{aligned} \tag{3.22}$$

Choose controller $e_3 = \alpha_2(e_1) = 0$, and choose

$$\begin{cases}
\dot{\tilde{a}} = -\dot{\hat{a}} = -g_2x_1z_2w_2 - g_2x_1^3z_2w_2 \\
\dot{\tilde{b}} = -\dot{\hat{b}} = -g_2x_1x_3z_2w_2 - g_2x_1^3x_3z_2w_2 \\
\dot{\tilde{c}} = -\dot{\hat{c}} = -g_2x_2z_2w_2 \\
\dot{\tilde{d}} = -\dot{\hat{d}} = g_2x_3z_2w_2 \\
u_2 = -e_2 + \hat{a}e_1 + \hat{a}g_1x_1z_1 + \hat{b}g_3x_3z_3e_1 + \hat{b}g_1x_1z_1g_3x_3z_3 + \hat{a}e_1^3 + 3\hat{a}g_1x_1z_1e_1^2 \\
\quad + 3\hat{a}g_1^2x_1^2z_1^2e_1 + \hat{a}g_1^3x_1^3z_1^3 + \hat{b}g_3x_3z_3e_1^3 + 3\hat{b}g_1x_1z_1g_3x_3z_3e_1^2 \\
\quad + 3\hat{b}g_1^2x_1^2z_1^2g_3x_3z_3e_1 + \hat{b}g_1^3x_1^3z_1^3g_3x_3z_3 + \hat{c}e_2 + \hat{c}g_2x_2z_2 - \hat{d}g_3x_3z_3 - \hat{a}g_2x_1z_2 \\
\quad - \hat{b}g_2x_3x_1z_2 - \hat{a}g_2x_1^3z_2 - \hat{b}g_2x_3x_1^3z_2 - \hat{c}g_2x_2z_2 + \hat{d}g_2x_3z_2 - g_2x_2z_1 - g_2x_2z_1^3 \\
\quad - a_1g_2x_2z_2 + d_1g_2x_2z_3 - w_2
\end{cases} \tag{3.23}$$

Eq.(3.22) becomes

$$\dot{v}_2 = \dot{v}_1 - w_2^2 < 0 \tag{3.24}$$

This means that $e_2 = 0$ is asymptotically stable.

Step3. When e_3 is considered as a controller, $\alpha_2(e_1)$ is estimative function.

Define

$$w_3 = e_3 - \alpha_2(e_1) = e_3 \tag{3.25}$$

Its time derivative is

$$\dot{w}_3 = \dot{e}_3 \tag{3.26}$$

Choose a control Lyapunov function of the form

$$v_3 = v_2 + \tfrac{1}{2} w_3^2 > 0 \tag{3.27}$$

Its time derivative through the third equation of Eq.(3.11) is

$$\dot{v}_3 = \dot{v}_2 + w_3(e_4 + g_4 x_4 z_4 - g_3 x_4 z_3 - g_3 x_3 z_4 + u_3) \tag{3.28}$$

where e_4 is a virtual controller. Take

$$e_4 = \alpha_3(e_1) = -w_3 = -e_3 \tag{3.29}$$

and choose

$$u_3 = -g_4 x_4 z_4 + g_3 x_4 z_3 + g_3 x_3 z_4 \tag{3.30}$$

Eq.(3.28) becomes

$$\dot{v}_3 = \dot{v}_3 - w_3^2 \le 0 \tag{3.31}$$

This means that $e_3 = 0$ is asymptotically stable.

Step4. When e_4 is a virtual controller, $\alpha_3(e_1)$ is estimative function.

Define

$$w_4 = e_3 - \alpha_3(e_1) = e_3 + e_4 \tag{3.32}$$

then

$$\dot{w}_4 = \dot{e}_3 + \dot{e}_4 \tag{3.33}$$

Choose a Lyapunov function of the form

$$V = v_4 = v_3 + \tfrac{1}{2}(w_4^2 + \tilde{e}^2 + \tilde{f}^2) > 0 \tag{3.34}$$

Its time derivative is

$$\begin{aligned} \dot{V} = \dot{v}_4 = \dot{v}_3 + w_4(&e_4 - e_3 - g_3 x_3 z_3 - e_3^3 - 3e_3^2 g_3 x_3 z_3 - 3e_3 g_3^2 x_3^2 z_3^2 - g_3^3 x_3^3 z_3^3 - \hat{e} e_4 \\ &- \hat{e} g_4 x_4 z_4 + \hat{f} e_1 + \hat{f} g_1 x_1 z_1 + g_4 x_3 z_4 + g_4 x_3^3 z_4 + e g_4 x_4 z_4 - f g_4 x_1 z_4 \\ &+ b_1 g_4 x_4 z_3 - c_1 g_4 x_4 z_4 + c_1 g_4 x_4 z_3^2 z_4 - f_1 g_4 x_4 z_1 + u_4) + \tilde{e}\dot{\tilde{e}} + \tilde{f}\dot{\tilde{f}} \end{aligned} \tag{3.35}$$

Choose

$$
\begin{cases}
\dot{\tilde{e}} = -\dot{\hat{e}} = -g_4 x_4 z_4 w_4 \\
\dot{\tilde{f}} = -\dot{\hat{f}} = g_4 x_1 z_4 w_4 \\
u_4 = -e_4 + e_3 + g_3 x_3 z_3 + e_3^3 + 3e_3^2 g_3 x_3 z_3 + 3e_3 g_3^2 x_3^2 z_3^2 + g_3^3 x_3^3 z_3^3 + \hat{e} e_4 \\
\qquad + \hat{e} g_4 x_4 z_4 - \hat{f} e_1 - \hat{f} g_1 x_1 z_1 - g_4 x_3 z_4 - g_4 x_3^3 z_4 - \hat{e} g_4 x_4 z_4 + \hat{f} g_4 x_1 z_4 \\
\qquad - b_1 g_4 x_4 z_3 + c_1 g_4 x_4 z_4 - c_1 g_4 x_4 z_3^2 z_4 + f_1 g_4 x_4 z_1 - w_4
\end{cases}
\tag{3.36}
$$

Eq.(3.35) becomes

$$
\begin{aligned}
\dot{V} &= \dot{v}_4 = \dot{v}_3 - w_4^2 \\
&= -e_1^2 - w_2^2 - w_3^2 - w_4^2 < 0
\end{aligned}
\tag{3.37}
$$

This means that $e_4 = 0$ is asymptotically stable. Rewrite Eq.(3.37)as

$$
\dot{V} = \dot{v}_4 = -2e_1^2 - e_2^2 - 2e_3^2 - e_4^2 < 0 \tag{3.38}
$$

which is a negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, and $\tilde{f}$,while from Eqs(3.14),(3.21),(3.27),(3.34)

$$
\begin{aligned}
V = v_4 &= \tfrac{1}{2}e_1^2 + \tfrac{1}{2}(e_1 + e_2)^2 + \tfrac{1}{2}e_3^2 + \tfrac{1}{2}(e_3 + e_4)^2 \\
&\quad + \tfrac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{e}^2 + \tilde{f}^2)
\end{aligned}
\tag{3.39}
$$

is a positive definite function of e_1, e_2, e_3, e_4, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, and $\tilde{f}$.

The Lyapunov asymptotical stability theorem can not be satisfied in this case. The common origin of error dynamics and parameter update dynamics cannot be determined to be asymptotically stable. By pragmatical asymptotical theorem (see Appendix) D is a 10 -manifold, n = 10 and the number of error state variables p = 4.

When e_1 = e_2 = e_3 = e_4= 0 and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ take arbitrary values , $\dot{V} = 0$, so X is a 6-dimational space, m = n – p = 10 – 4 = 6. m + 1≤ n are satisfied. By the pragmatical asymptotical stability theorem, not only error vector e tends to zero but also all estimated parameters approach their uncertain parameters. PHPHCGS of chaotic systems by adaptive backstepping control is accomplished. The equilibrium point e_1 =

$e_2 = e_3 = e_4 = \tilde{a} = \tilde{b} = \tilde{c} = \tilde{d} = \tilde{e} = \tilde{f} = 0$ is pragmatically asymptotically stable. The generalized synchronization is accomplished after 2000s with $g_1 = -0.1$, $g_2 = 4$, $g_3 = -0.1$, and $g_4 = 0.2$ while $e_1(0) = -2.4001$, $e_2(0) = -85.9999$, $e_3(0) = -3.0001$, and $e_4(0) = -1.9999$. The estimated parameters approach the uncertain parameters of the chaotic system as shown in Figs 3.2-3.8. The initial values of estimated parameters are $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{d}(0) = \hat{e}(0) = 0$.

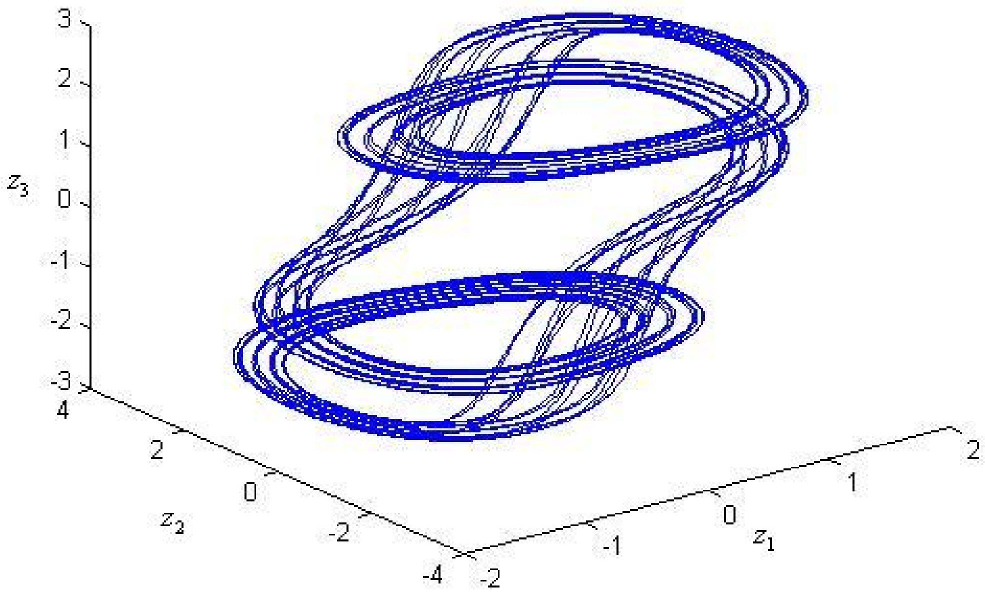

Fig. 3.1 Chaotic phase portrait of new Duffing – Van der Pol system.

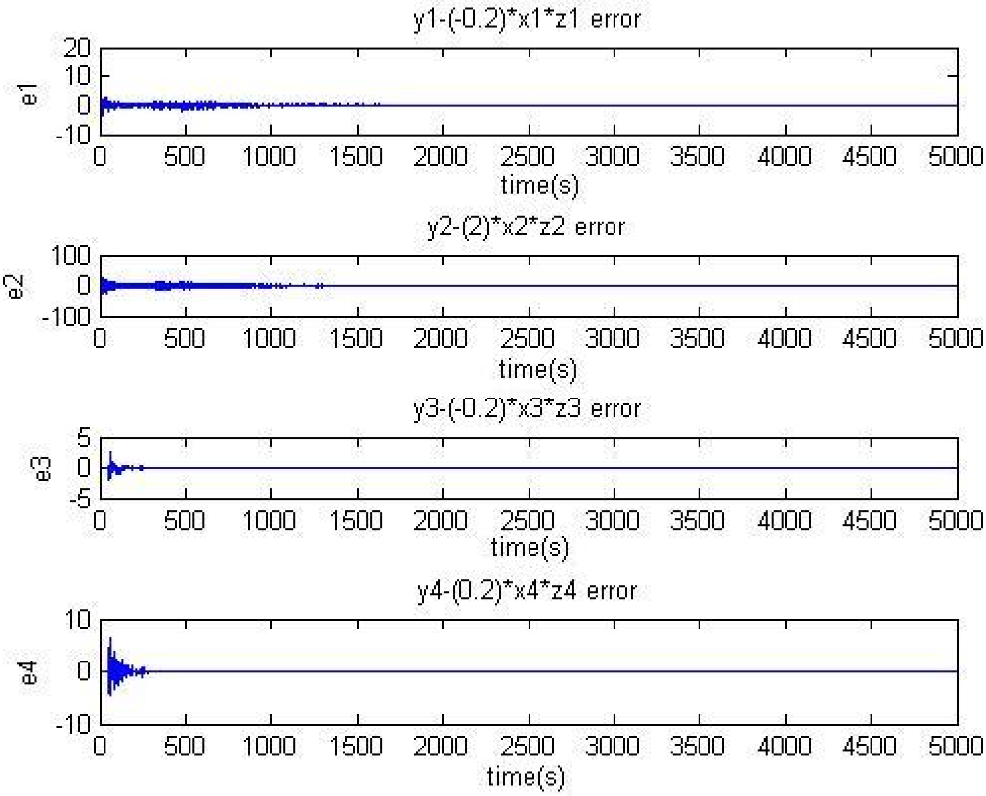

Fig. 3.2 The time histories of errors (e_1, e_2, e_3, e_4).

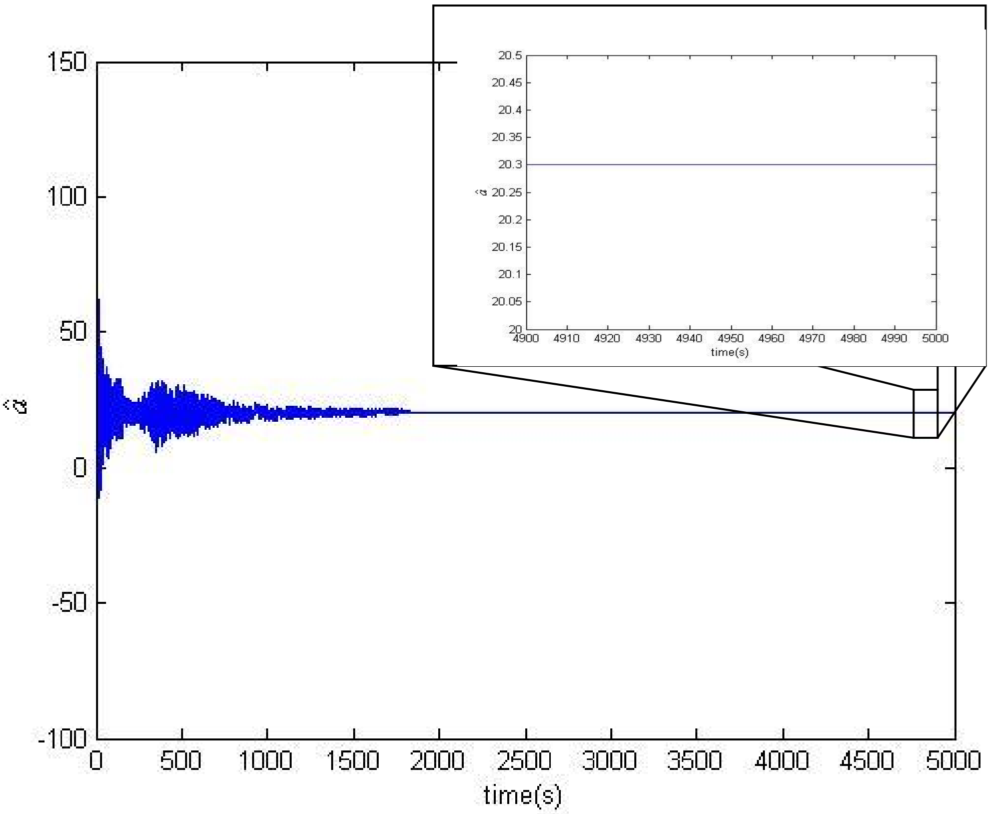

Fig. 3.3 The time histories of estimated parameter $\hat{a}$.

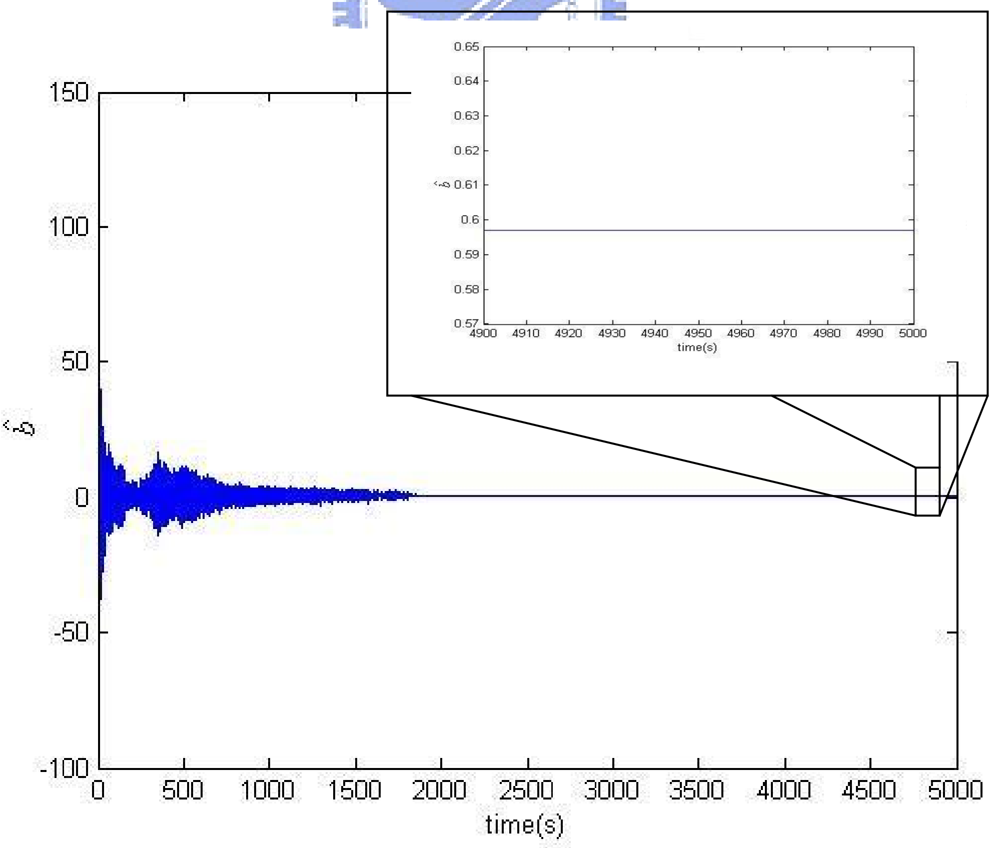

Fig. 3.4 The time histories of estimated parameter $\hat{b}$.

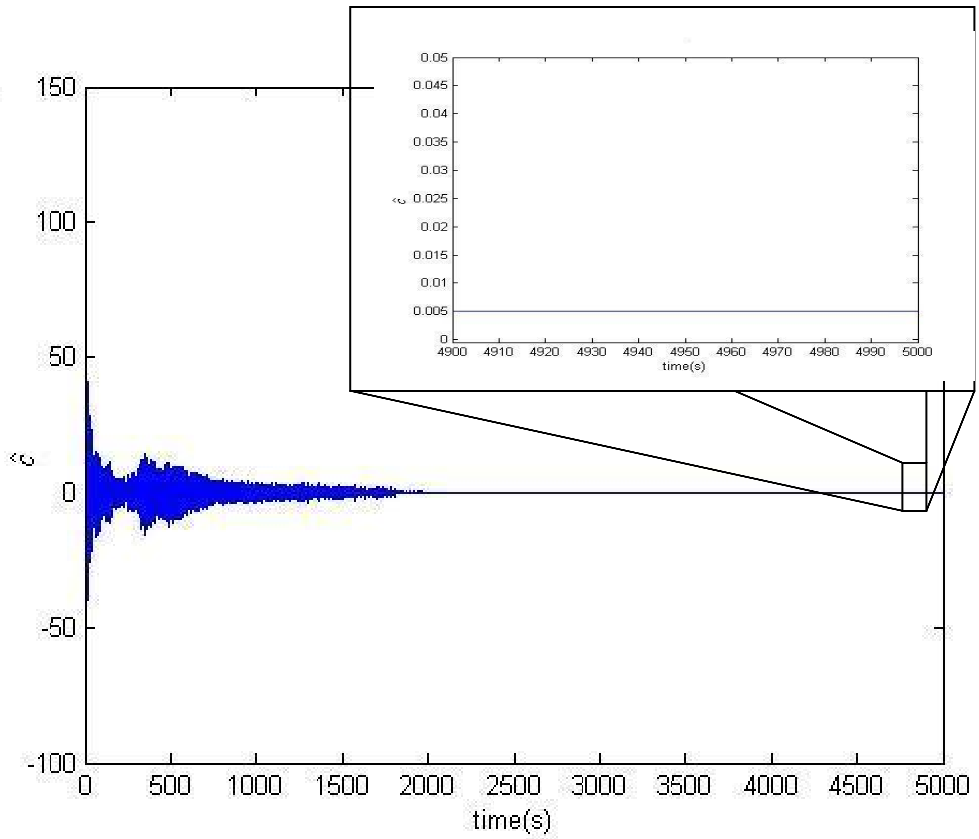

Fig. 3.5 The time histories of estimated parameter $\hat{c}$.

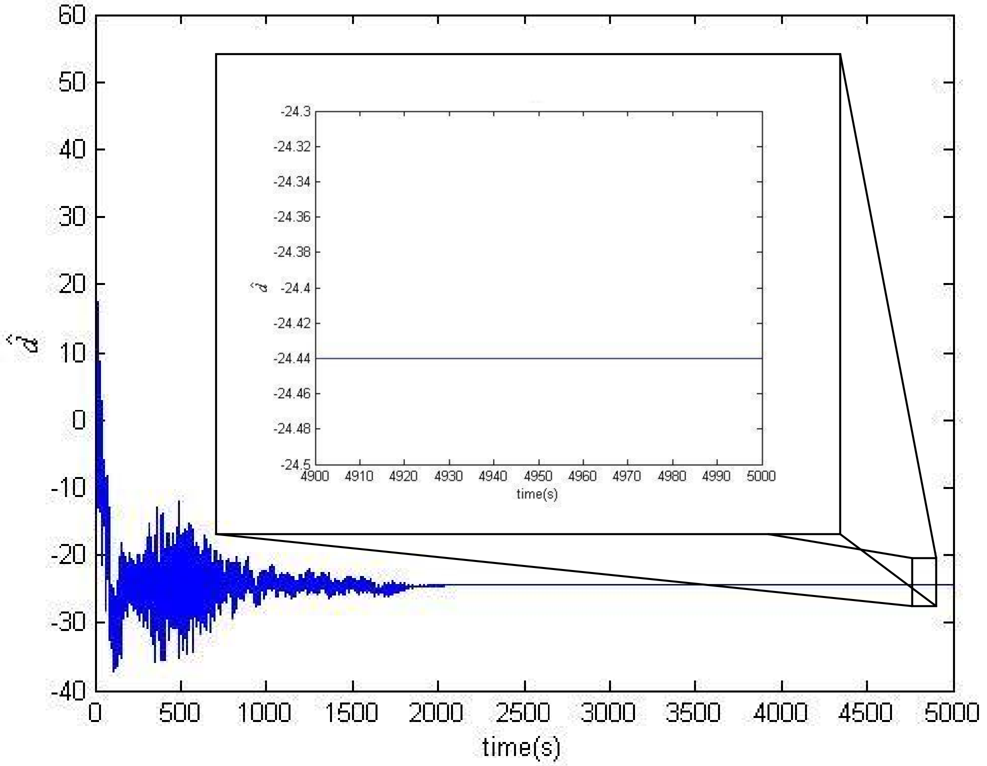

Fig. 3.6 The time histories of estimated parameter $\hat{d}$.

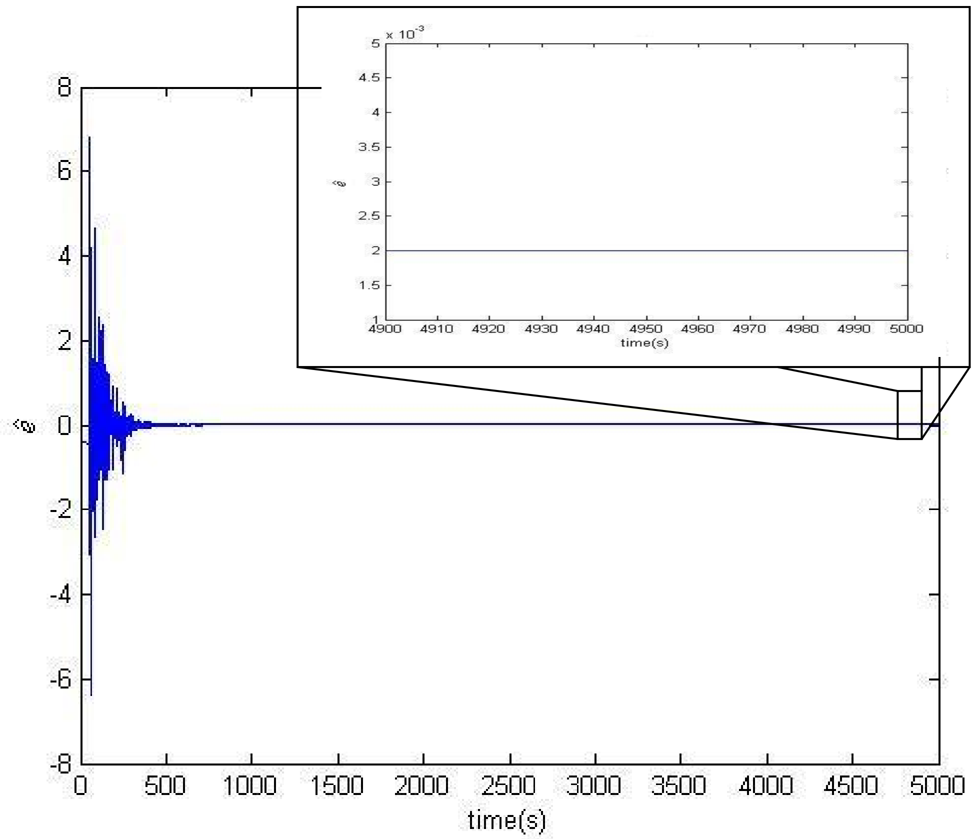

Fig. 3.7 The time histories of estimated parameter $\hat{e}$.

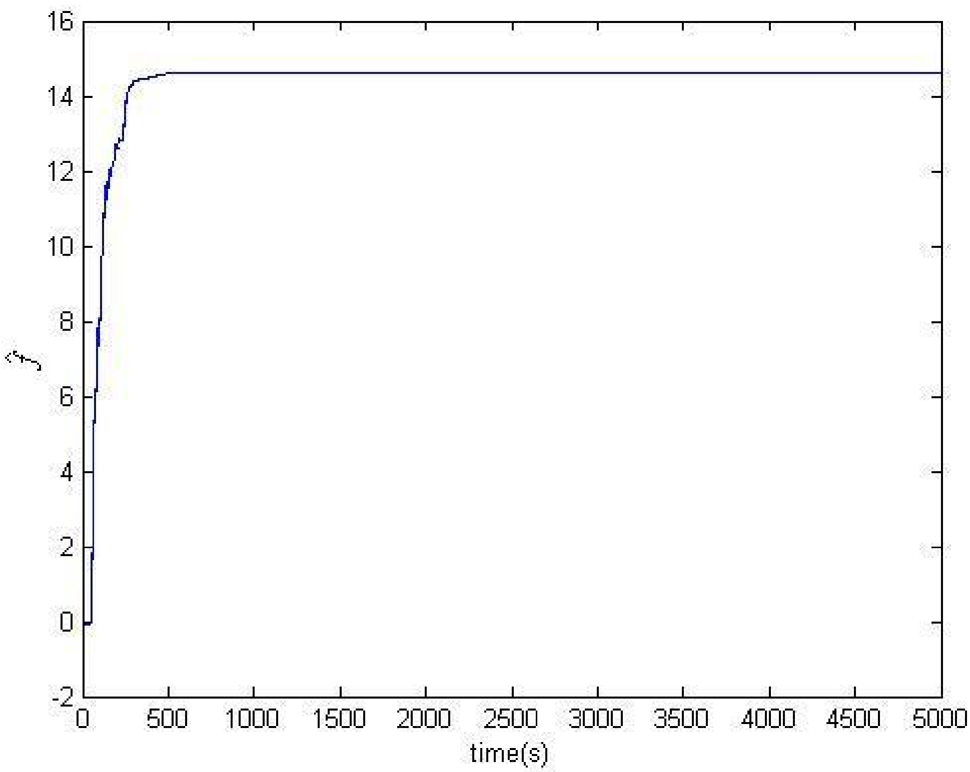

Fig. 3.8 The time histories of estimated parameter $\hat{f}$.

Chapter 4

Pragmatical Hybrid Projective and Symplectic Synchronization of Different Order Systems with New Control Lyapunov Function by Adaptive Backstepping Control

4.1Preliminaries

In this Chapter, a new kind of synchronization and a new control Lyapunov function for backstepping are proposed. The symplectic* synchronization

$$y = H(x, y, z, t) \tag{4.1}$$

is studied, where x and y are the state vectors of the "master" and of the "slave" respectively, and z is a given function vector of time, which may take various forms, either a regular or a chaotic function of time [28]. When

$$y = H(x, y, z, t) = H(x, z, t) \tag{4.2}$$

it reduces to the generalized synchronization. Therefore symplectic [72] symchronization is an extension of generalized synchronization.

In Eq. (4.1), the final desired state y of the "slave" system not only depends upon the "master" system state x, but also depends upon the "slave" system state y itself. Therefore the "slave" system is not a traditional pure slave completely obeying the

*The term **"symplectic"** comes from the Greek for "interwined". H. Weyl first introduced the term in 1939 in his book "The Classical Groups" (P. 165 in both the first edition, 1939, and second edition, 1946, Princeton University Press) “master” system but plays a role to determine the final desired state of the “slave” system. In other words, it plays a “interwined” role, so we call this kind of synchronization “symplectic synchronization”, and call the “master” system partner A, the “slave” system partner B.

The application of symplectic synchronization has great potential. For instance, if the symplectically synchronized chaotic signal is used as the signal carrier from a transmitter, secure communication is more difficult to be deciphered.

In this Chapter, a new control Lyapunov function

$$V(e) = \exp(ke^T e) - 1 \tag{4.3}$$

is proposed for backstepping control where e is error dynamics. Using the new control Lyapunov function, error tolerance can be decreased marvelously as comparing to that obtained by traditional control Lyapnov function

$$V(e) = e^T e \tag{4.4}$$

This Chapter is organized as follows. In Section 2, by the GYC pragmatical asymptotical stability theorem with new control Lyapunov functions, a pragmatical hybrid projective and symplectic synchronization scheme is achieved. In Section 3, chaos in the Mathieu - Duffing system and Lü system [36] with four dimensions are given. In Section 4, using the new control Lyapunov functions numerical results of PHPSS by adaptive backstepping control is achieved.

4.2. Symplectic synchronization scheme

Assume that there are two different nonlinear chaotic dynamical systems and that the partner A controls the partner B partially. The partner A is given by

$$\dot{x} = f(x) \tag{4.5}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ denotes the state vector and f a vector function.

The partner B is given by

$$\dot{y} = g(y) \tag{4.6}$$

where $y = (y_1, y_2, \cdots, y_n)^T \in R^n$ denotes the state vector, g is another vector-function.

The controlling partner B becomes

$$\dot{y}_u = g(y) + u(t) \tag{4.6b}$$

where $u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T \in R^n$ is a control input vector.

$$\dot{z} = h(z) \tag{4.7}$$

is a functional system, where $z = (z_1, z_2, ..., z_n)^T \in \Re^n$ is a state vector which is either chaotic or regular function vector of time, h is a given vector function.

Our goal is to design the controller $u(t)$ so that the state vector y of the partner B asymptotically approaches $H(x, y, z, t)$ and finally to accomplish the synchronization in the sense that the limit of the error vector $e(t) = (e_1, e_2, \cdots, e_n)^T$ approaches zero:

$$\lim_{t \to \infty} e = 0 \tag{4.8}$$

where

$$e = y_u - H(x, y, z, t) \tag{4.9}$$

From Eq. (4.9) , it is obtained that

$$\dot{e} = \dot{y}_u - \frac{\partial H}{\partial x}\dot{x} - \frac{\partial H}{\partial y}\dot{y} - \frac{\partial H}{\partial z}\dot{z} - \frac{\partial H}{\partial t} \tag{4.10}$$

By Eq. (4.5), Eq. (4.6), Eq. (4.6b), Eq. (4.7), and Eq. (4.9) becomes

$$\dot{e} = g(y) + u(t) - \frac{\partial H}{\partial x} f(x) - \frac{\partial H}{\partial y} g(y) - \frac{\partial H}{\partial y} h(z) - \frac{\partial H}{\partial t} \tag{4.11}$$

A new control Lyapnuov function $V(e)$ is chosen as a positive definite function of e:

$$V(e) = \exp(ke^T e) - 1 \tag{4.12}$$

where k is a positive constant. Its time derivative along any solution of Eq.(4.11)becomes

$$\dot{V}(e) = 2ke^T \{g(y) + u(t) - \frac{\partial H}{\partial x} f(x) - \frac{\partial H}{\partial y} g(y) - \frac{\partial H}{\partial z} h(z) - \frac{\partial H}{\partial t}\} \exp(ke^T e) \tag{4.13}$$

When $u(t)$ is chosen so that $\dot{V} = 2ke^T C_{n\times n} e \exp(ke^T e)$ where $C_{n\times n}$ is a diagonal negative definite constant matrix, then $\dot{V}$ is a negative definite function of e. By Lyapunov theorem of asymptotical stability

$$\lim_{t\to\infty} e = 0 \tag{4.14}$$

The symplectic synchronization is obtained [22-24].

4.3. Lü system

Lü system can be described as follows [36]:

$$\begin{cases} \frac{d}{dt} z_1 = a_1(z_2 - z_1) \\ \frac{d}{dt} z_2 = -z_1 z_3 + c_1 z_1 \\ \frac{d}{dt} z_3 = z_1 z_2 - b_1 z_3 \end{cases} \tag{4.15}$$

where z_1, z_2, z_3 are state variables, and a_1, b_1, c_1 are parameters. When the parameter $a_1 = 36,\ b_1 = 3,$ and $c_1 = 15$, the system dynamics is chaotic. We choose Lü system of three states as the different order system for four states new Mathieu – Duffing system. The additional fourth equation is chosen as

$$\dot{z}_4 = \dot{z}_1 + \dot{z}_2 + \dot{z}_3 \tag{4.16}$$

From Eq. (4.16) , it is obtained that

$$\frac{d}{dt}z_4 = a(z_2 - z_1) - z_1 z_3 + c z_1 + z_1 z_2 - b z_3 \qquad (4.17)$$

Augmented Lü system with four states becomes:

$$\begin{cases} \frac{d}{dt}z_1 = a_1(z_2 - z_1) \\ \frac{d}{dt}z_2 = -z_1 z_3 + c_1 z_1 \\ \frac{d}{dt}z_3 = z_1 z_2 - b_1 z_3 \\ \frac{d}{dt}z_4 = a_1(z_2 - z_1) - z_1 z_3 + c_1 z_1 + z_1 z_2 - b_1 z_3 \end{cases} \qquad (4.18)$$

where z_1 , z_2 , z_3 , z_4 are state variables, and a_1, b_1, c_1 are parameters. When $a_1 = 36$, $b_1 = 3$, and $c_1 = 15$, the system dynamics is chaotic as shown in Fig.4.1

4.4. Numerical results of CHPS by adaptive backstepping control

Take Mathieu-Duffing system (2.3) as master system, where a, b, c, d, e, f are uncertain parameters and following Mathieu-Duffing system as slave system:

$$\begin{cases} \frac{d}{dt}y_1 = y_2 \\ \frac{d}{dt}y_2 = -(\hat{a} + \hat{b}y_3)y_1 - (\hat{a} + \hat{b}y_3)y_1^3 - \hat{c}y_2 + \hat{d}y_3 \\ \frac{d}{dt}y_3 = y_4 \\ \frac{d}{dt}y_4 = -y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 \end{cases} \qquad (4.19)$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}$ *and* $\hat{f}$ are estimated parameters. Finally, take Eq.(4.19) as functional system.

Let

$$\begin{cases} H_1(x,y,z,t) = g_1x_1z_1y_1 \\ H_2(x,y,z,t) = g_2x_2z_2y_2 \\ H_3(x,y,z,t) = g_3x_3z_3y_3 \\ H_4(x,y,z,t) = g_4x_4z_4y_4 \end{cases} \tag{4.20}$$

Eq.(4.9) becomes

$$\begin{cases} e_1 = y_1 - g_1x_1z_1y_1 \\ e_2 = y_2 - g_2x_2z_2y_2 \\ e_3 = y_3 - g_3x_3z_3y_3 \\ e_4 = y_4 - g_4x_4z_4y_4 \end{cases} \tag{4.21}$$

where $g_1, g_2, g_3, and\ g_4$ are partly positive and partly negative constants in order to form hybrid projective synchronization.

In order to lead (y_1, y_2, y_3, y_4) to $(g_1x_1z_1y_1, g_2x_2z_2y_1, g_3x_3z_3y_3, g_4x_4z_4y_4)$, u_1, u_2, u_3 and u_4 are added to Eq.(4.19) to form the partner B in Eq.(4.6b), respectively:

$$\begin{cases} \frac{d}{dt}y_1 = y_2 + u_1 \\ \frac{d}{dt}y_2 = -(\hat{a} + \hat{b}y_3)y_1 - (\hat{a} + \hat{b}y_3)y_1^3 - \hat{c}y_2 + \hat{d}y_3 + u_2 \\ \frac{d}{dt}y_3 = y_4 + u_3 \\ \frac{d}{dt}y_4 = -y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 + u_4 \end{cases} \tag{4.22}$$

By Eq.(4.10)

$$\dot{e}_i = \dot{y}_{ui} - g_i\dot{x}_iz_iy_i - g_ix_i\dot{z}_iy_i - g_ix_iz_i\dot{y}_i, (i = 1,2,3,4) \tag{4.23}$$

The error dynamics is obtained:

$$
\begin{cases}
\dot{e}_1 = e_2 + g_2x_2z_2y_2 - g_1x_2z_1y_1 - g_1a_1x_1z_2y_1 + g_1a_1x_1z_1y_1 - g_1x_2z_1y_2 + u_1 \\
\dot{e}_2 = -\hat{a}y_1 - \hat{a}y_1{}^3 - \hat{c}y_2 + \hat{d}g_3x_3z_3y_3 - \hat{b}g_3x_3z_3y_3y_1 - \hat{b}g_3x_3z_3y_3y_1^3 \\
\qquad + cg_2x_2z_2y_2 + (\hat{d} - \hat{b}y_1 - \hat{b}y_1^3)e_3 + ag_2x_1z_2y_2 + bg_2x_1x_3z_2y_2 + ag_2x_1^3z_2y_2 \\
\qquad + bg_2x_1^3x_3z_2y_2 - dg_2x_3z_2y_2 + g_2x_2z_1z_3y_2 - b_1g_2x_2z_2y_2 + \hat{a}g_2x_2z_2y_1 \\
\qquad + \hat{b}g_2x_2z_2y_1y_3 + \hat{a}g_2x_2z_2y_1^3 + \hat{b}g_2x_2z_2y_1^3y_3 + \hat{c}g_2x_2z_2y_2 - \hat{d}g_2x_2z_2y_3 + u_2 \\
\dot{e}_3 = e_4 + g_4x_4z_4y_4 - g_3x_4z_3y_3 - g_3x_3z_1z_2y_3 + g_3c_1x_3z_3y_3 - g_3c_1x_3z_3y_4 + u_3 \\
\dot{e}_4 = -y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 + g_4x_3z_4y_4 + g_4x_3^3z_4y_4 + eg_4x_4z_4y_4 - fg_4x_1z_4y_4 \\
\qquad - a_1g_4x_4z_2y_4 + a_1g_4x_4z_1y_4 + g_4x_4z_1z_3y_4 - b_1g_4x_4z_2y_4 - g_4x_4z_1z_2y_4 \\
\qquad + c_1g_4x_4z_3y_4 + g_4x_4z_4y_3^3 + \hat{e}g_4x_4z_4y_4 - \hat{f}g_4x_4z_4y_1 + u_4
\end{cases} \tag{4.24}
$$

Step1. We consider the asymptotical stability of $e_1 = 0$:

$$
\dot{e}_1 = e_2 + g_2x_2z_2y_2 - g_1x_2z_1y_1 - g_1a_1x_1z_2y_1 + g_1a_1x_1z_1y_1 - g_1x_2z_1y_2 + u_1 \tag{4.25}
$$

where e_2 is regarded as a controller. Choose a control Lyapunov function of the form

$$
v_1 = \exp(k_1e_1^2) - 1 > 0 \tag{4.26}
$$

Its time derivative along the solution of Eq.(4.25) is

$$
\begin{aligned}
\dot{v}_1 = {} & 2k_1e_1(e_2 + g_2x_2z_2y_2 - g_1x_2z_1y_1 - g_1a_1x_1z_2y_1 + g_1a_1x_1z_1y_1 \\
& - g_1x_2z_1y_2 + u_1)\exp(k_1e_1^2)
\end{aligned} \tag{4.27}
$$

Assume virtual controller $e_2 = \alpha_1(e_1) = -e_1$. Eq.(4.27) can be rewritten as

$$
\begin{aligned}
\dot{v}_1 = {} & 2k_1e_1(-e_1 + g_2x_2z_2y_2 - g_1x_2z_1y_1 - g_1a_1x_1z_2y_1 + g_1a_1x_1z_1y_1 \\
& - g_1x_2z_1y_2 + u_1)\exp(k_1e_1^2)
\end{aligned} \tag{4.28}
$$

Choose

$$
u_1 = -g_2x_2z_2y_2 + g_1x_2z_1y_1 + g_1a_1x_1z_2y_1 - g_1a_1x_1z_1y_1 + g_1x_2z_1y_2 \tag{4.29}
$$

Eq.(4.27) becomes

$$
\dot{v}_1 = -2k_1e_1^2\exp(k_1e_1^2) < 0 \tag{4.30}
$$

This means that $e_1 = 0$ is asymptotically stable.

Step2. When e_2 is considered as a controller, $\alpha_1(e_1)$ is an estimative function.

Define

$$w_2 = e_2 - \alpha_1(e_1) = e_1 + e_2 \tag{4.31}$$

Study the (e_2, w_2) system. We have

$$\dot{w}_2 = \dot{e}_1 + \dot{e}_2 \tag{4.32}$$

Choose a control Lyapunov function of the form

$$v_2 = v_1 + \exp(k_2 w_2^2) + \tfrac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2) - 1 > 0, \tag{4.33}$$

where $\tilde{a} = a - \hat{a}$, $\tilde{b} = b - \hat{b}$, $\tilde{c} = c - \hat{c}$, $\tilde{d} = d - \hat{d}$ and $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ are estimated values of the unknown parameters a, b, c, d, respectively. We have

$$\begin{aligned}
\dot{v}_2 &= \dot{v}_1 + 2k_2 w_2(\dot{e}_1 + \dot{e}_2)\exp(k_2 w_2^2) + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} \\
&= \dot{v}_1 + 2k_2 w_2(-\hat{a}y_1 - \hat{a}y_1^{\ 3} - \hat{c}y_2 + \hat{d}g_3x_3z_3y_3 - \hat{b}g_3x_3z_3y_3y_1 - \hat{b}g_3x_3z_3y_3y_1^3 \\
&\quad + cg_2x_2z_2y_2 + ag_2x_1z_2y_2 + bg_2x_1x_3z_2y_2 + ag_2x_1^3z_2y_2 + bg_2x_1^3x_3z_2y_2 \\
&\quad - dg_2x_3z_2y_2 + g_2x_2z_1z_3y_2 - b_1g_2x_2z_2y_2 + \hat{a}g_2x_2z_2y_1 + \hat{b}g_2x_2z_2y_1y_3 \\
&\quad + \hat{a}g_2x_2z_2y_1^3 + \hat{b}g_2x_2z_2y_1^3y_3 + \hat{c}g_2x_2z_2y_2 - \hat{d}g_2x_2z_2y_3 + e_2 + u_2)\exp(k_2 w_2^2) \\
&\quad + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + 2k_2 w_2 e_3(\hat{d} - \hat{b}y_1 - \hat{b}y_1^3)\exp(k_2 w_2^2)
\end{aligned} \tag{4.34}$$

Choose controller $e_3 = \alpha_2(e_1) = 0$, and choose

$$\begin{cases}
\dot{\tilde{a}} = -\dot{\hat{a}} = 2k_2 w_2 \exp(k_2 w_2^2)(-g_2x_1z_2y_2 - g_2x_1^3z_2y_2) + \tilde{a} \\
\dot{\tilde{b}} = -\dot{\hat{b}} = 2k_2 w_2 \exp(k_2 w_2^2)(-g_2x_1x_3z_2y_2 - g_2x_3x_1^3z_2y_2) + \tilde{b} \\
\dot{\tilde{c}} = -\dot{\hat{c}} = 2k_2 w_2 \exp(k_2 w_2^2)(-g_2x_2z_2y_2) + \tilde{c} \\
\dot{\tilde{d}} = -\dot{\hat{d}} = 2k_2 w_2 \exp(k_2 w_2^2)(g_2x_3z_2y_2) + \tilde{d} \\
u_2 = \hat{a}y_1 + \hat{a}y_1^{\ 3} + \hat{c}y_2 - \hat{d}g_3x_3z_3y_3 + \hat{b}g_3x_3z_3y_3y_1 + \hat{b}g_3x_3z_3y_3y_1^3 \\
\qquad - \hat{c}g_2x_2z_2y_2 - \hat{a}g_2x_1z_2y_2 - \hat{b}g_2x_1x_3z_2y_2 - \hat{a}g_2x_1^3z_2y_2 + \hat{b}g_2x_1^3x_3z_2y_2 \\
\qquad + \hat{d}g_2x_3z_2y_2 - g_2x_2z_1z_3y_2 + b_1g_2x_2z_2y_2 - \hat{a}g_2x_2z_2y_1 - \hat{b}g_2x_2z_2y_1y_3 \\
\qquad - \hat{a}g_2x_2z_2y_1^3 - \hat{b}g_2x_2z_2y_1^3y_3 - \hat{c}g_2x_2z_2y_2 + \hat{d}g_2x_2z_2y_3 - e_2 \\
\qquad + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 - w_2
\end{cases} \tag{4.35}$$

Eq.(4.33) becomes

$$\dot{v}_2 = \dot{v}_1 - 2k_2 w_2^2 \exp(k_2 w_2^2) < 0 \tag{4.36}$$

This means that $e_2 = 0$ is asymptotically stable.

Step3. When e_3 is considered as a controller, $\alpha_2(e_1)$ is estimative function.

Define

$$w_3 = e_3 - \alpha_2(e_1) = e_3 \tag{4.37}$$

Its time derivative is

$$\dot{w}_3 = \dot{e}_3 \tag{4.38}$$

Choose a control Lyapunov function of the form

$$v_3 = v_2 + \exp(k_3 w_3^2) - 1 > 0 \tag{4.39}$$

Its time derivative through the third equation of Eq.(4.39) is

$$\begin{aligned} \dot{v}_3 = \dot{v}_2 + 2k_3 w_3 (&e_4 + g_4 x_4 z_4 y_4 - g_3 x_4 z_3 y_3 - g_3 x_3 z_1 z_2 y_3 + g_3 a_3 x_3 z_3 y_3 \\ &- g_3 a_3 x_3 z_3 y_4 + u_3) \exp(k_3 w_3^2) \end{aligned} \tag{4.40}$$

where e_4 is a virtual controller. Take

$$e_4 = \alpha_3(e_1) = -w_3 = -e_3 \tag{4.41}$$

and choose

$$u_3 = -g_4 x_4 z_4 y_4 + g_3 x_4 z_3 y_3 + g_3 x_3 z_1 z_2 y_3 - g_3 c_1 x_3 z_3 y_3 + g_3 c_1 x_3 z_3 y_4 \tag{4.42}$$

Eq.(4.40) becomes

$$\dot{v}_3 = \dot{v}_3 - 2k_3 w_3^2 \exp(k_3 w_3^2) \le 0 \tag{4.43}$$

This means that $e_3 = 0$ is asymptotically stable.

Step4. When e_4 is a virtual controller, $\alpha_3(e_1)$ is estimative function.

Define

$$w_4 = e_3 - \alpha_3(e_1) = e_3 + e_4 \tag{4.44}$$

then

$$\dot{w}_4 = \dot{e}_3 + \dot{e}_4 \tag{4.45}$$

Choose a control Lyapunov function of the form

$$V = v_4 = v_3 + \exp(k_4 w_4^2) + \tfrac{1}{2}(\tilde{e}^2 + \tilde{f}^2) - 1 > 0 \tag{4.46}$$

Its time derivative is

$$\begin{aligned}\dot{V} = \dot{v}_4 = \dot{v}_3 + 2k_4 w_4 (&e_4 - y_3 - y_3^3 - \hat{e} y_4 + \hat{f} y_1 + g_4 x_3 z_4 y_4 + g_4 x_3^3 z_4 y_4 \\ &+ e g_4 x_4 z_4 y_4 - f g_4 x_1 z_4 y_4 - a_1 g_4 x_4 z_2 y_4 + a_1 g_4 x_4 z_1 y_4 + g_4 x_4 z_1 z_3 y_4 \\ &- b_1 g_4 x_4 z_2 y_4 - g_4 x_4 z_1 z_2 y_4 + c_1 g_4 x_4 z_3 y_4 + g_4 x_4 z_4 y_3^3 + \hat{e} g_4 x_4 z_4 y_4 \\ &- \hat{f} g_4 x_4 z_4 y_1 + u_4) \exp(k_4 w_4^2) + \tilde{e}\dot{\tilde{e}} + \tilde{f}\dot{\tilde{f}}\end{aligned} \tag{4.47}$$

Choose

$$\begin{cases}\dot{\tilde{e}} = -\dot{\hat{e}} = 2k_4 w_4 \exp(k_4 w_4^2)(-g_4 x_4 z_4 w_4) + \tilde{e} \\ \dot{\tilde{f}} = -\dot{\hat{f}} = 2k_4 w_4 \exp(k_4 w_4^2)(g_4 x_1 z_4 w_4) + \tilde{f} \\ u_4 = -e_4 + y_3 + y_3^3 + \hat{e} y_4 - \hat{f} y_1 - g_4 x_3 z_4 y_4 - g_4 x_3^3 z_4 y_4 \\ \quad - e g_4 x_4 z_4 y_4 + f g_4 x_1 z_4 y_4 + a_1 g_4 x_4 z_2 y_4 - a_1 g_4 x_4 z_1 y_4 - g_4 x_4 z_1 z_3 y_4 \\ \quad + b_1 g_4 x_4 z_2 y_4 + g_4 x_4 z_1 z_2 y_4 - c_1 g_4 x_4 z_3 y_4 - g_4 x_4 z_4 y_3^3 - \hat{e} g_4 x_4 z_4 y_4 \\ \quad + \hat{f} g_4 x_4 z_4 y_1 + \tilde{e}^2 + \tilde{f}^2 - w_4\end{cases} \tag{4.48}$$

Eq.(4.47) becomes

$$\begin{aligned}\dot{V} &= \dot{v}_4 = \dot{v}_3 - 2k_4 w_4^2 \exp(k_4 w_4^2) \\ &= -2k_1 e_1^2 \exp(k_1 e_1^2) - 2k_2 w_2^2 \exp(k_2 w_2^2) \\ &\quad - 2k_3 w_3^2 \exp(k_3 w_3^2) - 2k_4 w_4^2 \exp(k_4 w_4^2) < 0\end{aligned} \tag{4.49}$$

This means that $e_4 = 0$ is asymptotically stable. Rewrite Eq.(4.49) as

$$\begin{aligned}\dot{V} = \dot{v}_4 &= -2k_1 e_1^2 \exp(k_1 e_1^2) - 2k_2 (e_1 + e_2)^2 \exp(k_2 (e_1 + e_2)^2) \\ &\quad - 2k_3 e_3^2 \exp(k_3 e_3^2) - 2k_4 (e_3 + e_4)^2 \exp(k_4 (e_3 + e_4)^2) < 0\end{aligned} \tag{4.50}$$

which is a negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, and $\tilde{f}$, while from Eqs(4.26),(4.33),(4.39),(4.46) .

$$\begin{aligned}V = v_4 &= \exp(k_1 e_1^2) + \exp(k_2 (e_1 + e_2)^2) + \exp(k_3 e_3^2) + \exp(k_4 (e_3 + e_4)^2) \\ &\quad + \tfrac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{e}^2 + \tilde{f}^2) - 4\end{aligned} \tag{4.51}$$

is a positive definite function of e_1, e_2, e_3, e_4, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, and $\tilde{f}$.

The Lyapunov asymptotical stability theorem can not be satisfied in this case. The common origin of error dynamics and parameter update dynamics cannot be determined to be asymptotically stable. By GYC pragmatical asymptotical stability theorem (see Appendix) D is a 10 -manifold, n = 10 and the number of error state variables p = 4. When e_1 = e_2 = e_3 = e_4 = 0 and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ take arbitrary values , $\dot{V}=0$, so X is of 6 dimensions, m = n – p = 10 – 4 = 6. m + 1 ≤ n are satisfied. By the GYC pragmatical asymptotical stability theorem, not only error vector e tends to zero but also all estimated parameters approach their uncertain parameters. PHPSS of chaotic systems by adaptive backstepping control is accomplished. The equilibrium point e_1 = e_2 = e_3 = e_4 = $\tilde{a}$ = $\tilde{b}$ = $\tilde{c}$ = $\tilde{d}$ = $\tilde{e}$ = $\tilde{f}$ = 0 is asymptotically stable.

The numerical results are shown in Figs 4.2-4.6. the numerical data used are: g_1= 0.0001 , g_2= – 0.0001 , g_3 = -0.0001 , g_4= -0.0001 , $k_1 = 0.01$, $k_2 = 0.01$, $k_3 = 0.01$, and $k_4 = 0.01$, $e_1(0) = -2.00014$, $e_2(0) = 9.99999$, $e_3(0) = -2.00014$ and $e_4(0) = 9.99999$. Using the new control Lyapunov functions, the error tolerances are reduced marvelously to 10^{-17} of that using traditional control Lyapnov function $V(e) = e^T e$ as shown in Figs.4.2-4.3. The estimated parameters approach the uncertain parameters of the chaotic system as shown in Figs. 4.4-4.6. The initial values of estimated parameters are $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{d}(0) = \hat{e}(0) = \hat{f}(0) = 0$. This property makes the proposed PHPSS is very effective.

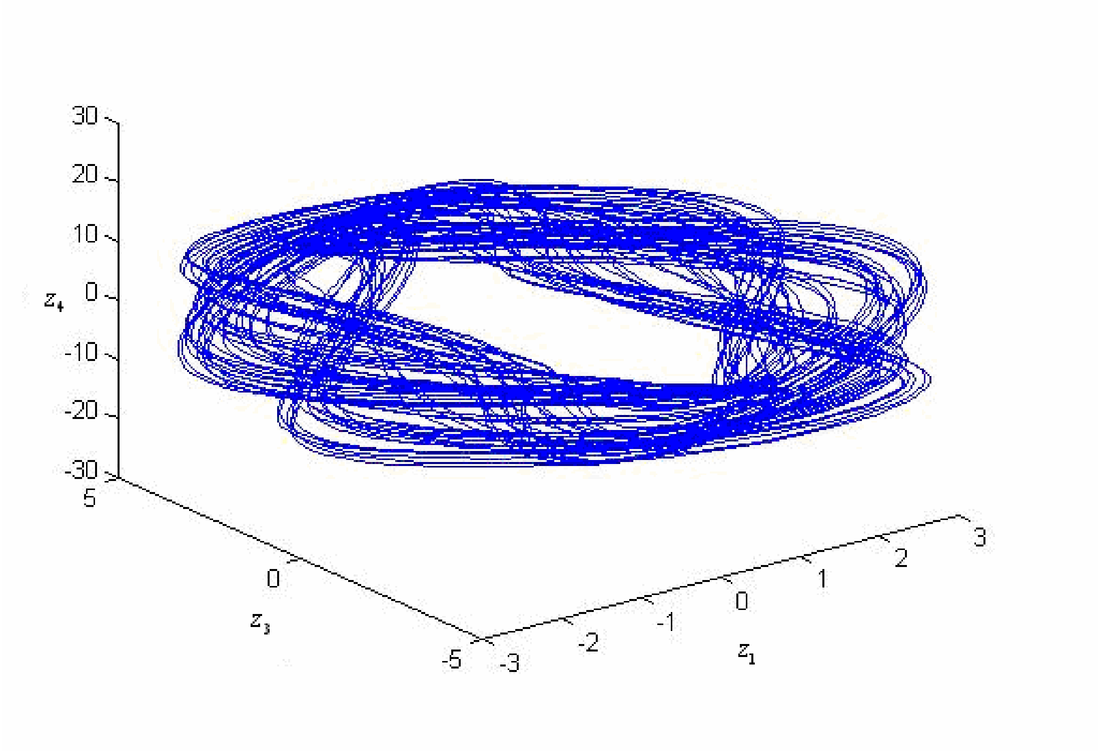

Fig. 4.1 Chaotic phase protract of Lü system.

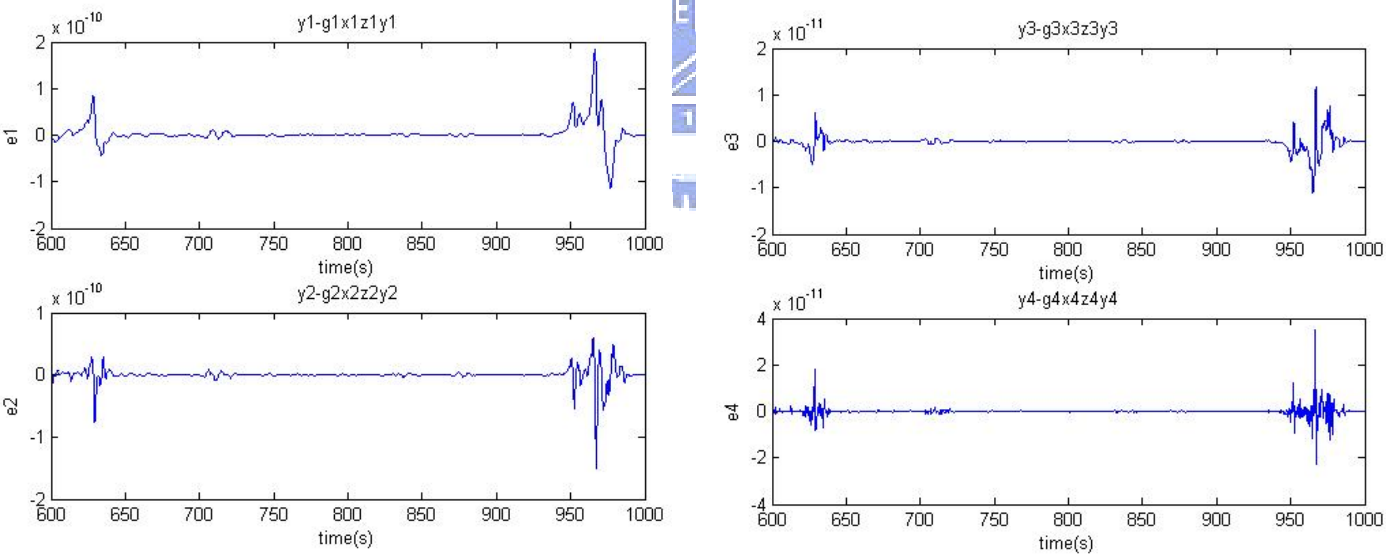

Fig. 4.2 Synchronization error for traditional control Lyapunov function $V(e) = e^T e$ from 600s ~ 1000s.

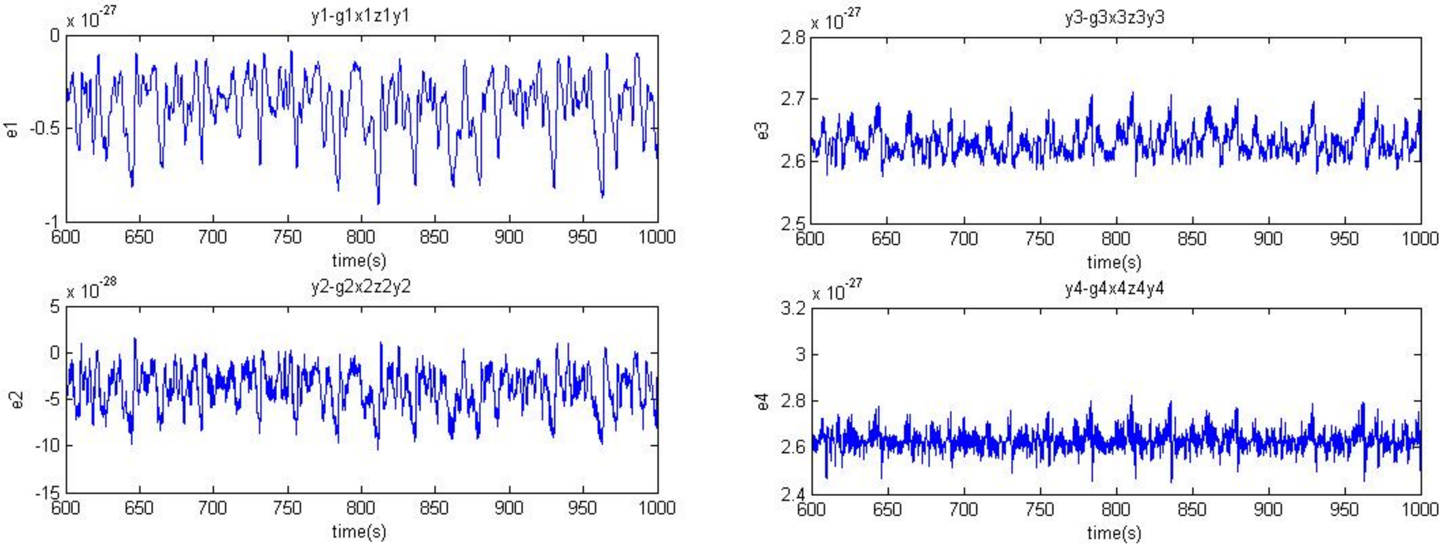

Fig. 4.3 Synchronization error for new control Lyapunov function $V(e) = \exp(ke^T e) - 1$ from 600s ~ 1000s.

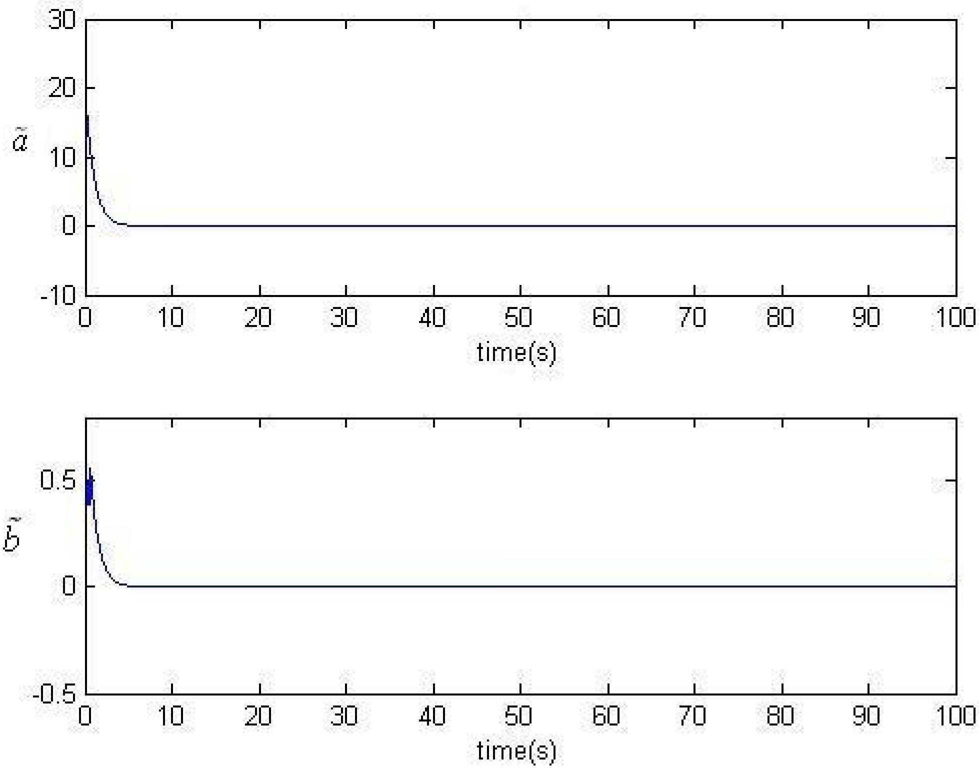

Fig. 4.4 The time histories of estimated parameter $\tilde{a}$ and $\tilde{b}$.

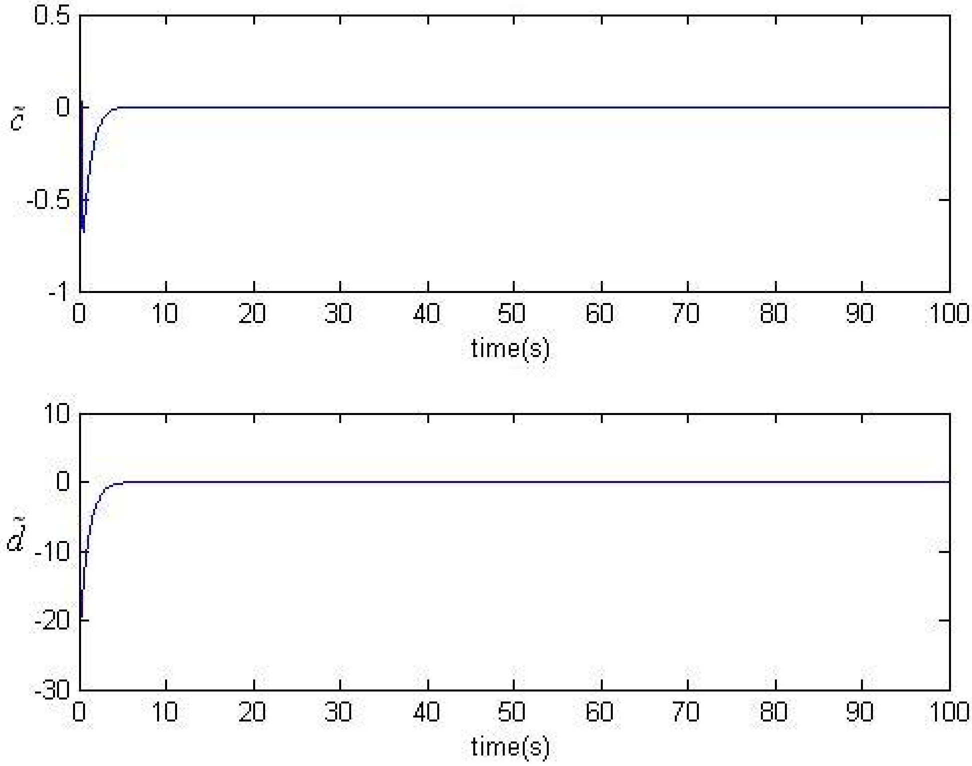

Fig. 4.5 The time histories of estimated parameter $\tilde{c}$ and $\tilde{d}$.

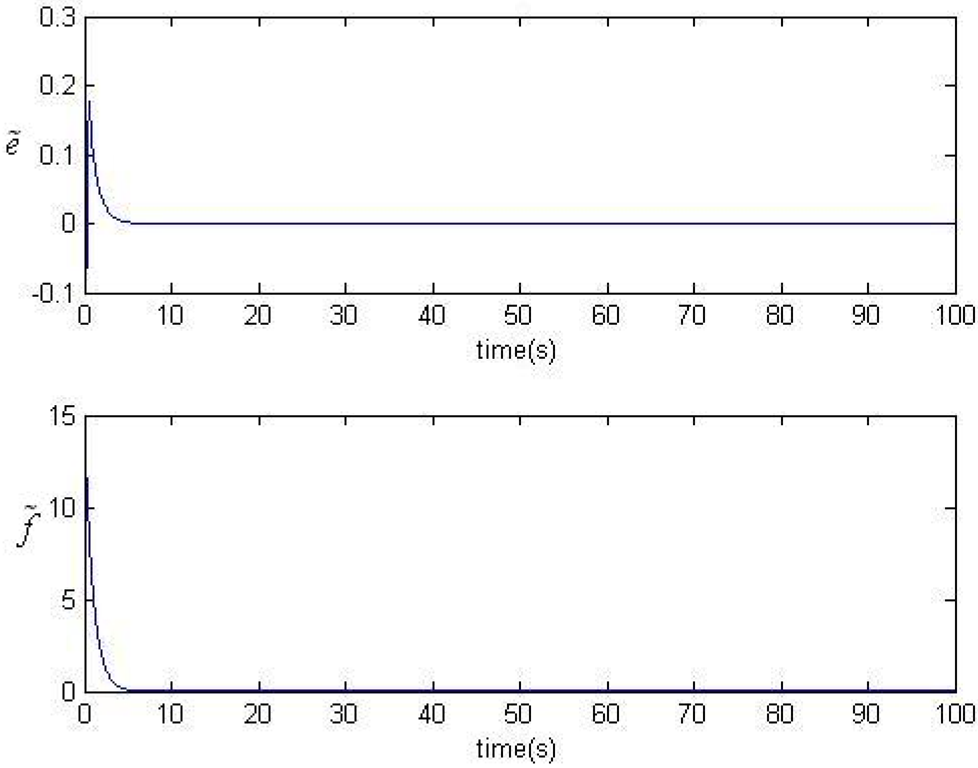

Fig. 4.6 The time histories of estimated parameter $\tilde{e}$ and $\tilde{f}$.

Chapter 5

Chaos Control of a New Mathieu- Duffing System by GYC Partial Region Stability Theory

5.1 Preliminaries

In this Chapter, a new strategy to achieve chaos control by GYC partial region stability [33, 34] is proposed. By using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of error states and the controllers are simpler than traditional controllers and so reduce the simulation error because they are in lower degree than that of traditional controllers.

This Chapter is organized as follows. In Section 2, chaos control scheme by GYC partial region stability theory is proposed. In Section 3, numerical Simulations of chaos control of new Mathieu – Duffing systems as simulated examples by GYC are achieved.

5.2 Chaos control scheme

Consider the following chaotic systems

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{5.1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$ is a the state vector, $\mathbf{f} : R_+ \times R^n \to R^n$ is a vector function.

The goal system which can be either chaotic or regular, is

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}) \tag{5.2}$$

where $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ is a state vector, $\mathbf{g}: R_+ \times R^n \to R^n$ is a vector function.

In order to make the chaos state $\mathbf{x}$ approaching the goal state $\mathbf{y}$, define $\mathbf{e} = \mathbf{x} - \mathbf{y}$ as the state error. The chaos control is accomplished in the sense that:

$$\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty} (\mathbf{x} - \mathbf{y}) = 0 \tag{5.3}$$

In this Chapter, we will use examples in which the e state is placed in the first quadrant of coordinate system and use on partial region stability theory. The Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler because they are in lower degree than that of traditional controllers and so reduce the simulation error because they are in lower degree than that of traditional controllers.

5.3 Numerical simulations of chaos control by GYC

The following chaotic system is a Mathieu – Duffing system of which the old origin is translated to $(x_1, x_2, x_3, x_4) = (50, 50, 50, 50)$:

$$\begin{cases} \frac{d}{dt} x_1 = x_2 - 50 \\ \frac{d}{dt} x_2 = -(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\ \qquad - c(x_2 - 50) + d(x_3 - 50) \\ \frac{d}{dt} x_3 = x_4 - 50 \\ \frac{d}{dt} x_4 = -(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) \end{cases} \tag{5.4}$$

and the chaotic motion always happens in the first quadrant of coordinate system (x_1, x_2, x_3, x_4). This translated Mathieu – Duffing system is presented as simulated examples where the initial conditions is $x_1(0)=49$, $x_2(0)=61$, $x_3(0)=49$, $x_4(0)=61$. The chaotic motion is shown in Fig. 5.1.

In order to lead (x_1, x_2, x_3, x_4) to the goal, we add control terms u_1, u_2 and u_3 to each equation of Eq. (5.4), respectively.

$$\begin{cases} \frac{d}{dt}x_1 = x_2 - 50 + u_1 \\ \frac{d}{dt}x_2 = -(a+b(x_3-50))(x_1-50) - (a+b(x_3-50))(x_1-50)^3 \\ \qquad -c(x_2-50) + d(x_3-50) + u_2 \\ \frac{d}{dt}x_3 = x_4 - 50 + u_3 \\ \frac{d}{dt}x_4 = -(x_3-50) - (x_3-50)^3 - e(x_4-50) + f(x_1-50) + u_4 \end{cases} \tag{5.5}$$

CASE I. Control the chaotic motion to zero.

In this case we will control the chaotic motion of the Mathieu – Duffing system (5.4) to zero. The goal is $y=0$. The state error is $e = x - y = x$ and error dynamics becomes

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 = x_2 - 50 + e_1^2 - e_1^2 + u_1 \\ \dot{e}_2 &= \dot{x}_2 = -(a+b(x_3-50))(x_1-50) - (a+b(x_3-50))(x_1-50)^3 \\ &\qquad -c(x_2-50) + d(x_3-50) + e_1^2 - e_1^2 + u_2 \\ \dot{e}_3 &= \dot{x}_3 = x_4 - 50 + e_3^2 - e_3^2 + u_3 \\ \dot{e}_4 &= \dot{x}_4 = -(x_3-50) - (x_3-50)^3 - e(x_4-50) + f(x_1-50) + e_3^2 - e_3^2 + u_4 \end{aligned} \tag{5.6}$$

In Fig. 5.1, we see that the error dynamics always exists in first quadrant.

By GYC partial region stability, one can easily choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 + e_4 \tag{5.7}$$

Its time derivative through error dynamics (5.6) is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 \\ &= (x_2 - 50 + e_1^{\,2} - e_1^{\,2} + u_1) + (-(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\ &\quad - c(x_2 - 50) + d(x_3 - 50) + e_1^{\,2} - e_1^{\,2} + u_2) + (x_4 - 50 + e_3^{\,2} - e_3^{\,2} + u_3) \\ &\quad + (-(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) + e_3^{\,2} - e_3^{\,2} + u_4)\end{aligned} \tag{5.8}$$

Choose

$$\begin{aligned} u_1 &= -(x_2 - 50) + e_1^{\,2} - e_1 \\ u_2 &= -(-(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\ &\quad - c(x_2 - 50) + d(x_3 - 50)) - e_1^{\,2} - e_2 \\ u_3 &= -(x_4 - 50) + e_3^{\,2} - e_3 \\ u_4 &= -(-(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50)) - e_3^{\,2} - e_4 \end{aligned} \tag{5.9}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{5.10}$$

which is negative definite function. The numerical results are shown in Fig.5.2. After 30 sec, the motion trajectories approach the origin.

CASE II. Control the chaotic motion to a sine function.

In this case we will control the chaotic motion of the Mathieu – Duffing system (5.5) to sine function of time. The goal is $y = m \sin wt$. The error equation

$$e = x - y = x - m \sin wt \tag{5.11}$$

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} (x_i - m_i \sin w_i t) = 0, \qquad i = 1,2,3,4 \tag{5.12}$$

and $\dot{e}_i = \dot{x}_i - w_i m_i \cos w_i t \quad (i = 1,2,3,4)$, and $m_1 = m_2 = m_3 = m_4$. The error dynamics is

$$
\begin{aligned}
e_1 &= \dot{x}_1 - w_1 m\cos w_1 t = x_2 - 50 - w_1 m\cos w_1 t + e_1^{\,2} - e_1^{\,2} + u_1 \\
\dot{e}_2 &= \dot{x}_2 - w_2 m\cos w_2 t = -(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\
&\qquad - c(x_2 - 50) + d(x_3 - 50) - w_2 m\cos w_2 t + e_1^{\,2} - e_1^{\,2} + u_2 \\
\dot{e}_3 &= \dot{x}_3 - w_3 m\cos w_3 t = x_4 - 50 - w_3 m\cos w_3 t + e_3^{\,2} - e_3^{\,2} + u_3 \\
\dot{e}_4 &= \dot{x}_4 - w_4 m\cos w_4 t = -(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) \\
&\qquad - w_4 m\cos w_4 t + e_3^{\,2} - e_3^{\,2} + u_4
\end{aligned}
\tag{5.13}
$$

In Fig. 5.3, the error dynamics always exists in first quadrant.

By GYC partial region stability, one can easily choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 + e_4 \tag{5.14}$$

Its time derivative is

$$
\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 \\
&= (x_2 - 50 - w_1 m\cos w_1 t + e_1^{\,2} - e_1^{\,2} + u_1) + (-(a + b(x_3 - 50))(x_1 - 50) \\
&\quad - (a + b(x_3 - 50))(x_1 - 50)^3 - c(x_2 - 50) + d(x_3 - 50) \\
&\quad - w_2 m\cos w_2 t + e_1^{\,2} - e_1^{\,2} + u_2) + (x_4 - 50 - w_3 m\cos w_3 t + e_3^{\,2} - e_3^{\,2} + u_3) \\
&\quad + (-(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) - w_4 m\cos w_4 t + e_3^{\,2} - e_3^{\,2} + u_4)
\end{aligned}
\tag{5.15}
$$

Choose

$$
\begin{aligned}
u_1 &= -(x_2 - 50 - w_1 m\cos w_1 t) + e_1^{\,2} - e_1 \\
u_2 &= -(-(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\
&\qquad - c(x_2 - 50) + d(x_3 - 50) - w_2 m\cos w_2 t) - e_1^{\,2} - e_2 \\
u_3 &= -(x_4 - 50 - w_3 m\cos w_3 t) + e_3^{\,2} - e_3 \\
u_4 &= -(-(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) - w_4 m\cos w_4 t) - e_3^{\,2} - e_4
\end{aligned}
\tag{5.16}
$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{5.17}$$

which is negative definite function. The numerical results are shown in Fig.5.4 and Fig. 5.5, where $m = 0.5$ and $w_1 = w_2 = w_3 = w_4 = 2$. After 30 sec., the errors approach zero and the motion trajectories approach to sine functions.

CASE III. Control the chaotic motion of a new Mathieu – Duffing system to chaotic motion of a generalized Lorenz system.

In this case we will control chaotic motion of a new Mathieu – Duffing system (5.5) to that of a generalized Lorenz system. The goal system is generalized Lorenz system [22]:

$$\begin{cases} \frac{d}{dt} z_1 = a_1(z_2 - z_1) + dz_4 \\ \frac{d}{dt} z_2 = b_1 z_1 - z_1 z_3 - z_2 \\ \frac{d}{dt} z_3 = z_1 z_2 - c_1 z_3 \\ \frac{d}{dt} z_4 = -z_1 - a_1 z_4 \end{cases} \tag{5.18}$$

The error equation is $e = x - z$, Our aim is $\lim_{t\to\infty} e = 0$. The error dynamics become

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \dot{z}_1 = x_2 - 50 - (a_1(z_2 - z_1) + dz_4) + e_1^2 - e_1^2 + u_1 \\ \dot{e}_2 &= \dot{x}_2 - \dot{z}_2 = -(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\ &\qquad - c(x_2 - 50) + d(x_3 - 50) - (b_1 z_1 - z_1 z_3 - z_2) + e_1^2 - e_1^2 + u_2 \\ \dot{e}_3 &= \dot{x}_3 - \dot{z}_3 = x_4 - 50 - (z_1 z_2 - c_1 z_3) + e_3^2 - e_3^2 + u_3 \\ \dot{e}_4 &= \dot{x}_4 - \dot{z}_4 = -(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) - (-z_1 - a_1 z_4) + e_3^2 - e_3^2 + u_4 \end{aligned} \tag{5.19}$$

By Fig. 5.6, we know the error dynamics always exists in first quadrant.

By GYC partial region stability, one can easily choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 + e_4 \tag{5.20}$$

Its time derivative is

$$
\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 \\
&= \left(x_2 - 50 - (a_1(z_2 - z_1) + dz_4) + e_1^{\;2} - e_1^{\;2} + u_1 \right) \\
&\quad + (-(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\
&\quad - c(x_2 - 50) + d(x_3 - 50) - (b_1 z_1 - z_1 z_3 - z_2) + e_1^{\;2} - e_1^{\;2} + u_2) \\
&\quad + \left(x_4 - 50 - (z_1 z_2 - c_1 z_3) + e_3^{\;2} - e_3^{\;2} + u_3 \right) \\
&\quad + (-(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) - (-z_1 - a_1 z_4) + e_3^{\;2} - e_3^{\;2} + u_4)
\end{aligned}
\tag{5.21}
$$

Choose

$$
\begin{aligned}
u_1 &= -(x_2 - 50 - (a_1(z_2 - z_1) + dz_4) + e_1^{\;2} - e_1 \\
u_2 &= -(-(a + b(x_3 - 50))(x_1 - 50) - (a + b(x_3 - 50))(x_1 - 50)^3 \\
&\qquad - c(x_2 - 50) + d(x_3 - 50) - (b_1 z_1 - z_1 z_3 - z_2)) - e_1^{\;2} - e_2 \\
u_3 &= -(x_4 - 50 - (z_1 z_2 - c_1 z_3)) + e_3^{\;2} - e_3 \\
u_4 &= -(-(x_3 - 50) - (x_3 - 50)^3 - e(x_4 - 50) + f(x_1 - 50) - (-z_1 - a_1 z_4)) - e_3^{\;2} - e_4
\end{aligned}
\tag{5.22}
$$

We obtain

$$
\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{5.23}
$$

which is negative definite function. The numerical results are shown in Fig.5.7 and Fig.5.8 where $a = 0.2$, $b = 0.2$, and $c = 5.7$. After 30 sec, the errors approach zero and the chaotic trajectories of the new Mathieu – Duffing system approach to that of the generalized Lorenz system.

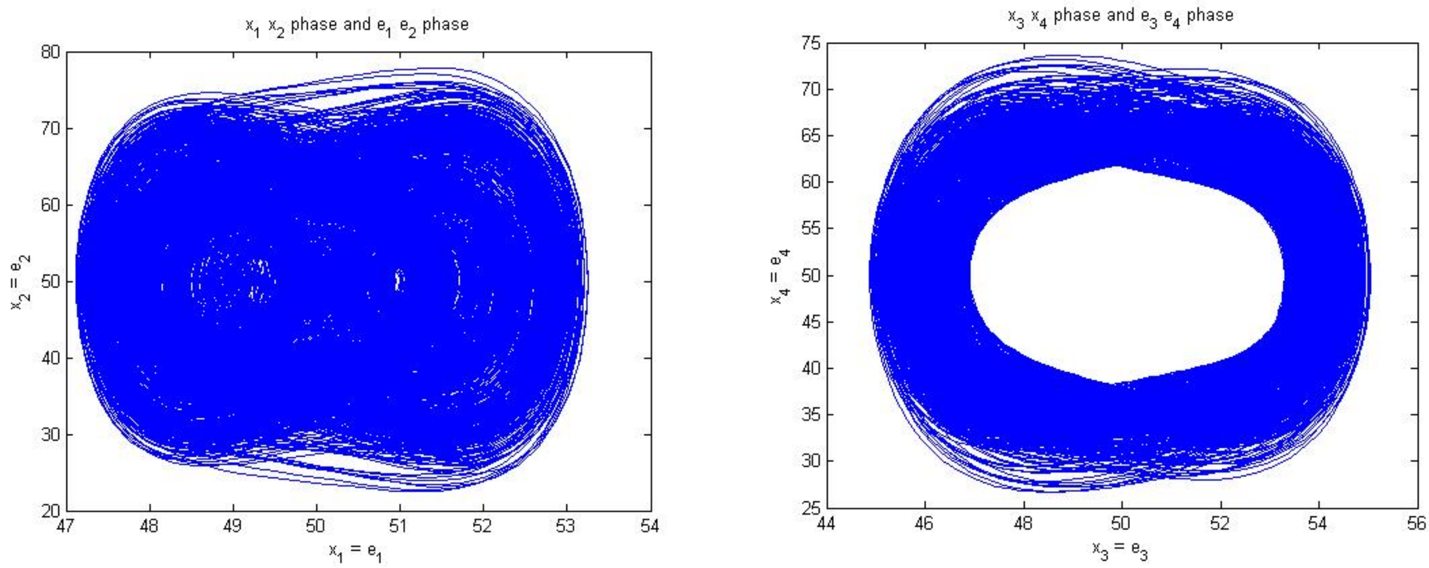

Fig. 5.1 Phase portrait of error dynamics for Case I.

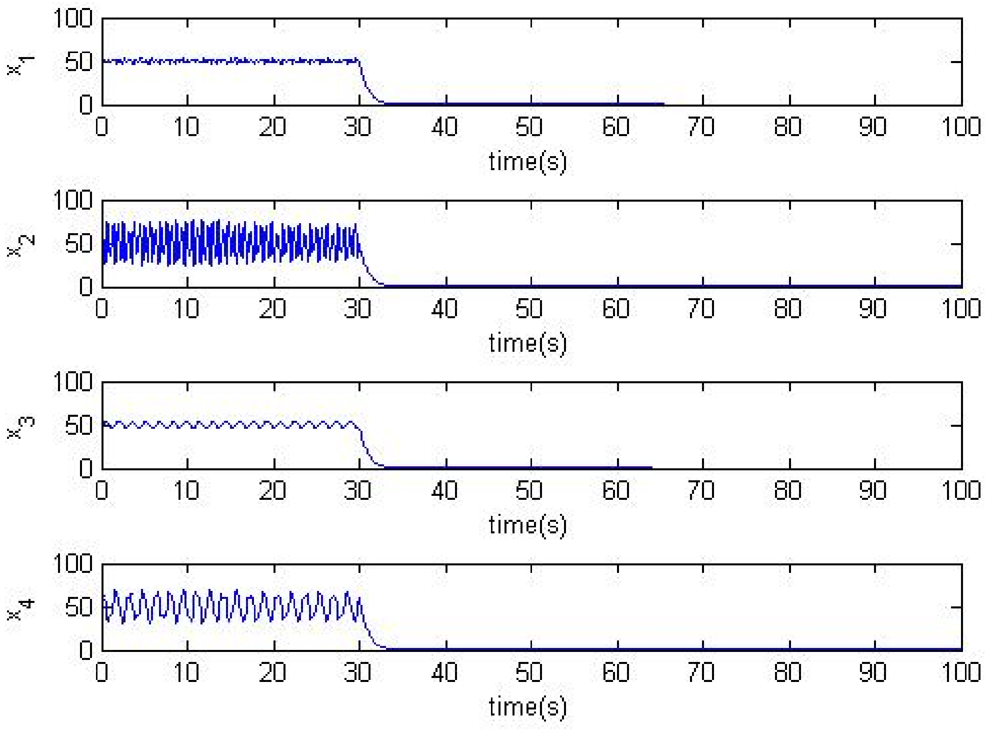

Fig.5.2 Time histories of x_1, x_2, x_3, x_4 for Case I.

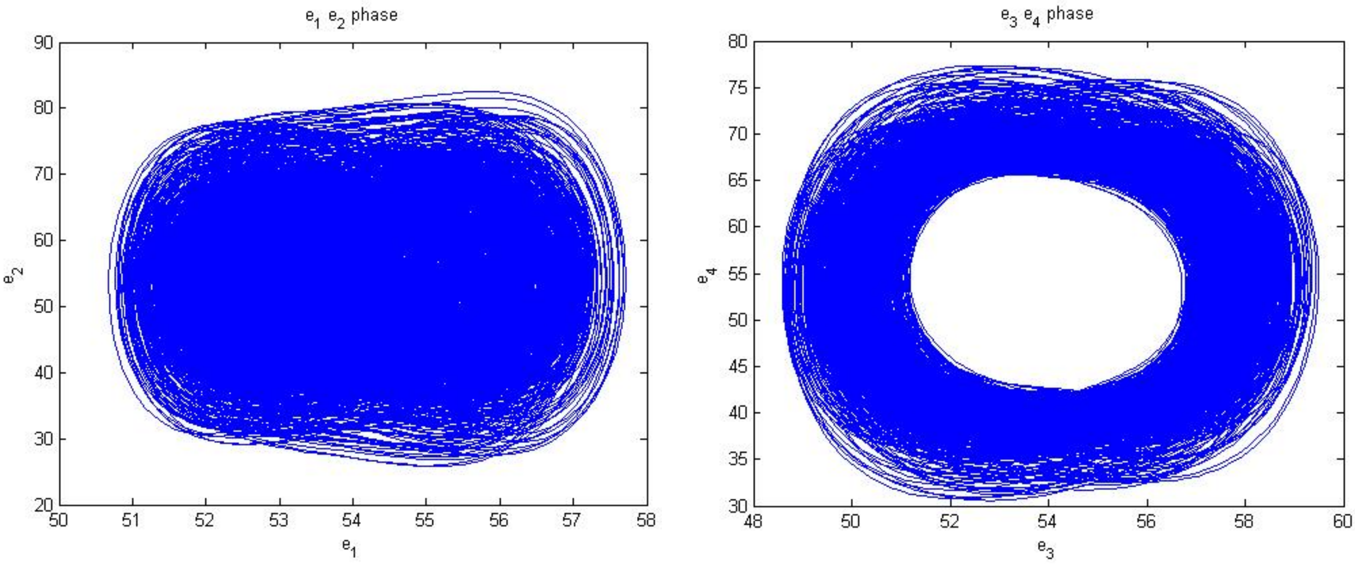

Fig. 5.3 Phase portrait of error dynamics for Case II.

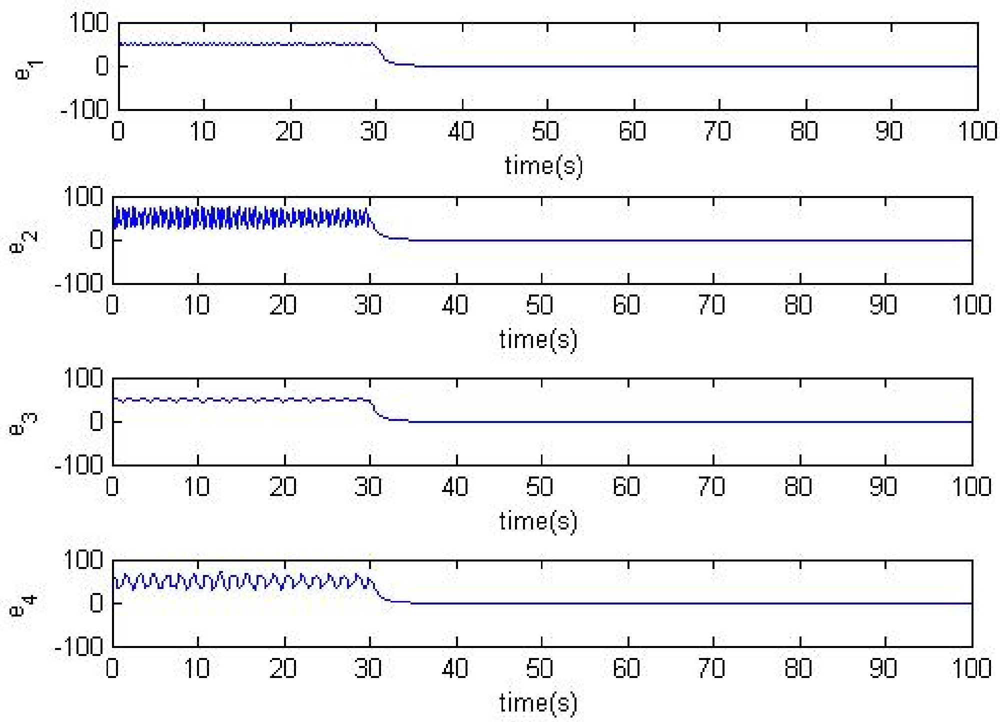

Fig. 5.4 Time histories of errors for Case II.

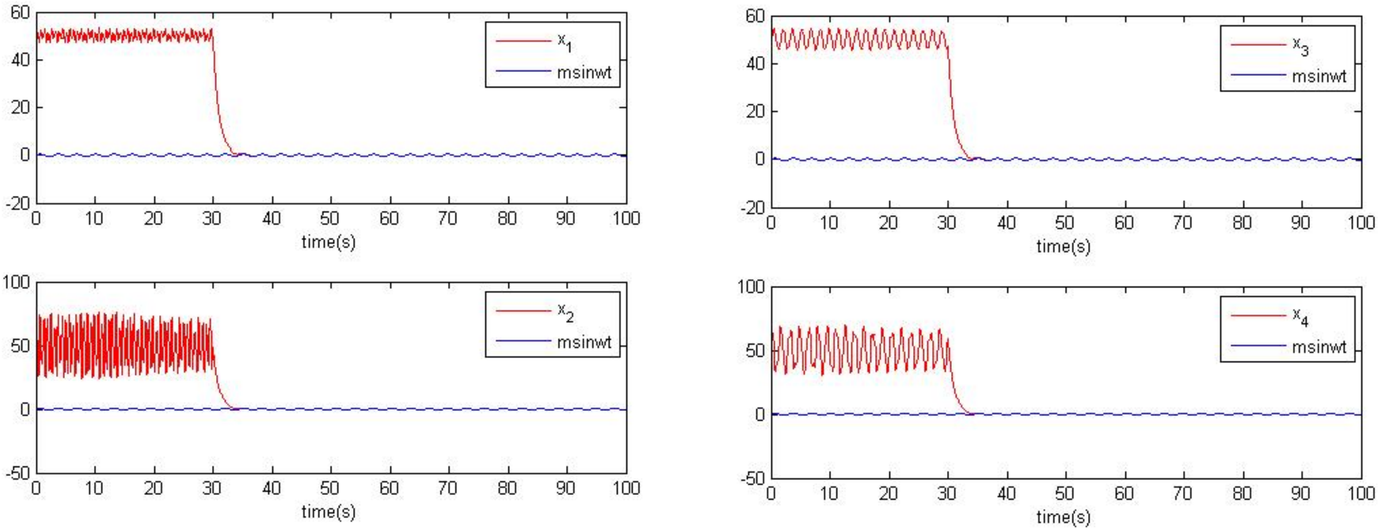

Fig. 5.5 Time histories of x_1, x_2, x_3, x_4 for Case II.

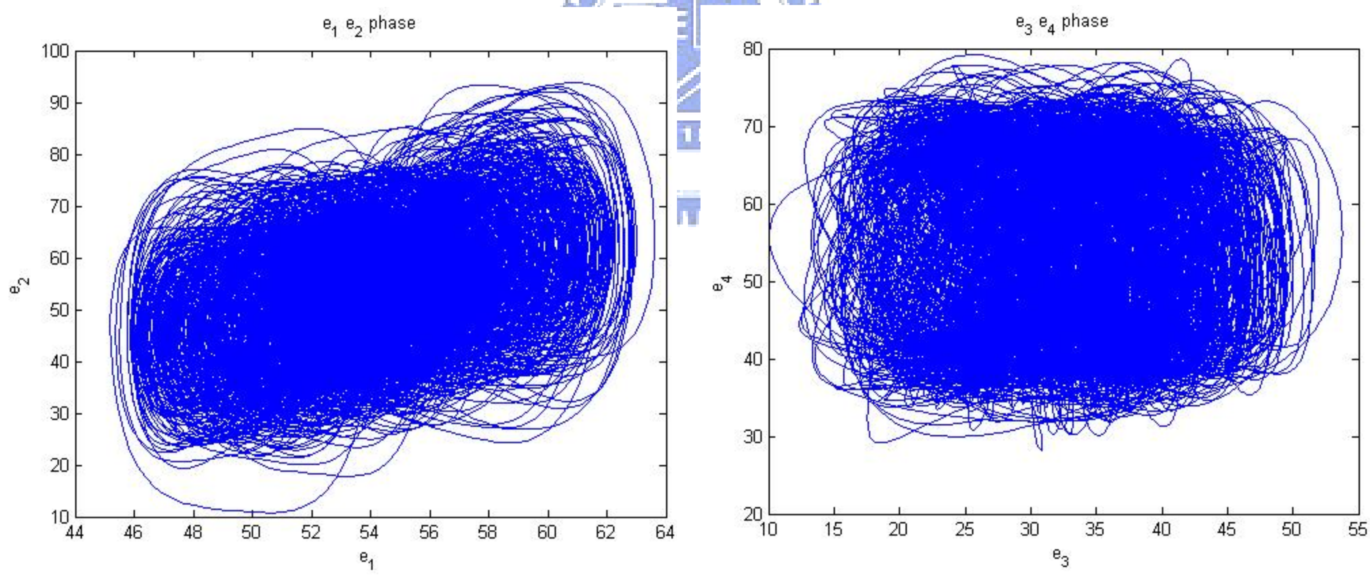

Fig. 5.6 Phase portrait of error dynamics for Case III.

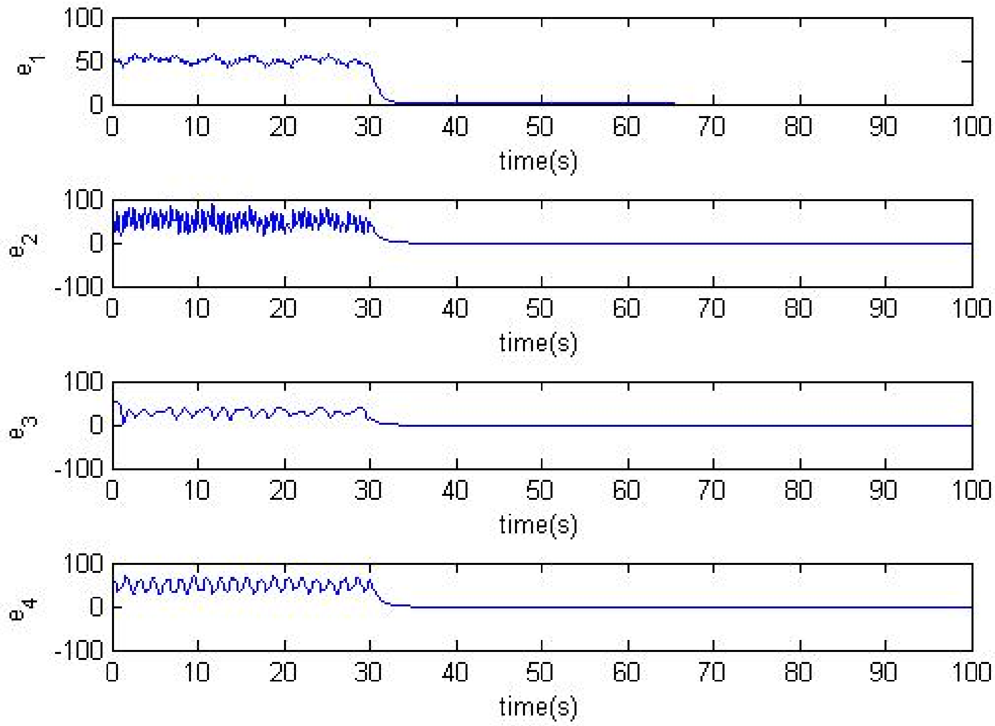

Fig. 5.7 Time histories of errors for Case III.

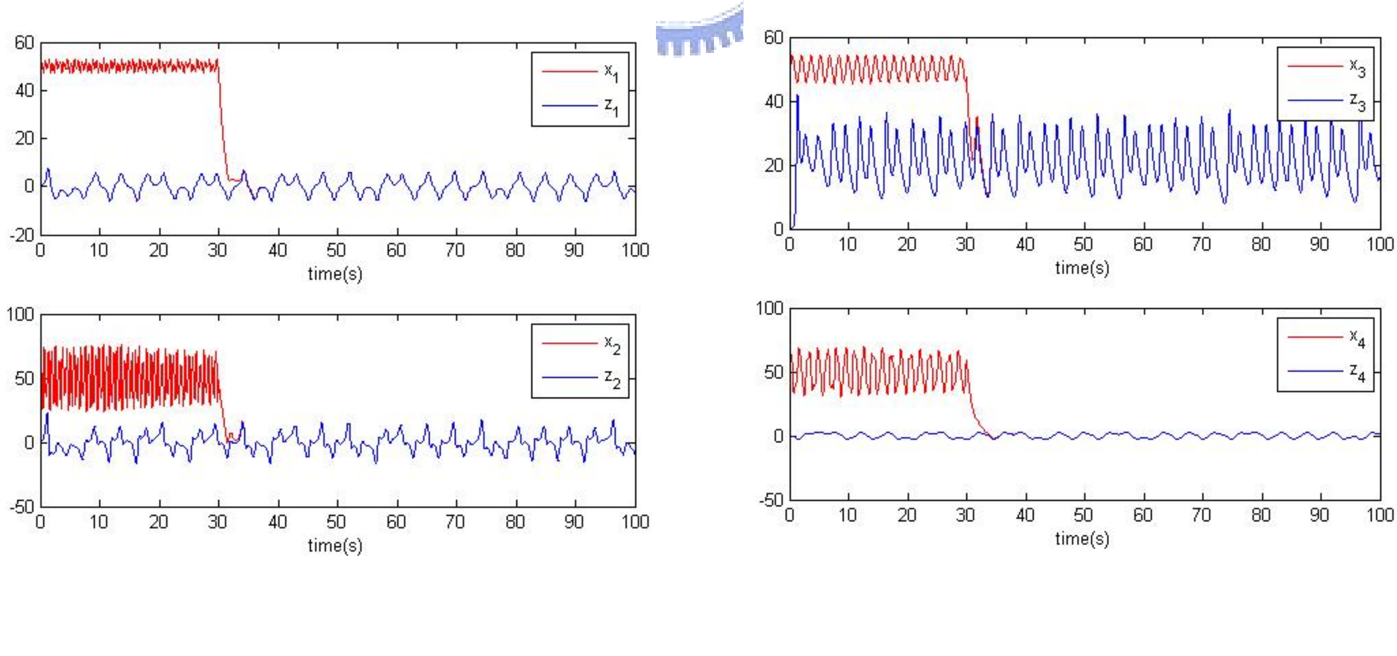

Fig. 5.8 Time histories of $x_{1,}\ x_{2,}\ x_{3,}\ x_{4,}\ z_{1,}\ z_{2,}\ z_{3,}\ z_{4}$ for Case III.

Chapter 6

Chaos Generalized Synchronization of New Mathieu- Duffing Systems and Chaotization by GYC Partial Region Stability Theory

6.1 Preliminaries

Among many kinds of synchronizations, the generalized synchronization is investigated. This means that we can give a function relationship between the state vector x of the master and the state vector y of slave: $y = G(x)$. In this chapter, a new chaos generalized synchronization strategy by GYC partial region stability theory is proposed.

This Chapter is organized as follow. In Section 2, chaos generalized synchronization strategy are proposed. In Section 3, numerical simulations of chaos generalized synchronization of Mathieu – Duffing systems as simulated examples by GYC are achieved. In Section 4, chaotization of a regular motion to the chaotic motion of a new Mathieu –Duffing systemis studied.

6.2 Chaos generalized synchronization strategy

Consider the following unidirectional coupled chaotic systems

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{h}(t, \mathbf{y}) + \mathbf{u} \end{aligned} \tag{6.1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$, $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ denote two state vectors, $\mathbf{f}$ and $\mathbf{h}$ are nonlinear vector functions, and $\mathbf{u} = [u_1, u_2, \cdots, u_n]^T \in R^n$ is a control input vector.

The generalized synchronization can be accomplished when $t \to \infty$, the limit of the error vector $\mathbf{e} = [e_1, e_2, \cdots, e_n]^T$ approaches zero:

$$\lim_{t\to\infty} \mathbf{e} = 0 \tag{6.2}$$

where

$$\mathbf{e} = \mathbf{G}(\mathbf{x}) - \mathbf{y} \tag{6.3}$$

$\mathbf{G}(\mathbf{x})$ is given rector function of $\mathbf{x}$.

By using the GYC partial region stability theory, the Lyapunov function is easier to find, a homogeneous function of first degree can be used to construct a positive definite Lyapunov function and the controllers can be designed in lower degree.

6.3 Numerical simulations of chaos generalized synchronization by GYC theory

Two new Mathieu – Duffing systems, with the unidirectional coupling appear as

$$\begin{cases} \frac{d}{dt}x_1 = x_2 \\ \frac{d}{dt}x_2 = -(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3 \\ \frac{d}{dt}x_3 = x_4 \\ \frac{d}{dt}x_4 = -x_3 - x_3^3 - ex_4 + fx_1 \end{cases} \tag{6.4}$$

$$\begin{cases} \dfrac{d}{dt} y_1 = y_2 + u_1 \\ \dfrac{d}{dt} y_2 = -(a + by_3) y_1 - (a + by_3) y_1^3 - cy_2 + dy_3 + u_2 \\ \dfrac{d}{dt} y_3 = y_4 + u_3 \\ \dfrac{d}{dt} y_4 = -y_3 - y_3^3 - ey_4 + fy_1 + u_4 \end{cases} \tag{6.5}$$

CASE I. The generalized synchronization error function is $e_i = x_i - y_i + k_i$,$(i = 1, 2, 3, 4)$.

$$\begin{cases} e_1 = x_1 - y_1 + k_1 \\ e_2 = x_2 - y_2 + k_2 \\ e_3 = x_3 - y_3 + k_3 \\ e_4 = x_4 - y_4 + k_4 \end{cases} \tag{6.6}$$

where k_i , $(i = 1, 2, 3, 4)$ is positive constants, we choose k_1=10 , k_2=75 , k_3=15 , k_4=60 in order that the error dynamics always happens in first quadrant.

Our goal is $\mathbf{y} = \mathbf{x} + k$, i.e. the controlling goal is that

$$\lim_{t \to \infty} e_i = \lim_{t \to \infty} (x_i - y_i + k_i) = 0 \text{ , } (i = 1, 2, 3, 4) \tag{6.7}$$

The error dynamics becomes

$$\begin{cases} \dot{e}_1 = x_2 - y_2 + e_1{}^2 - e_1{}^2 - u_2 \\ \dot{e}_2 = -(a + bx_3) x_1 - (a + bx_3) x_1^3 - cx_2 + dx_3 \\ \qquad -(-(a + by_3) y_1 - (a + by_3) y_1^3 - cy_2 + dy_3) + e_1{}^2 - e_1{}^2 - u_2 \\ \dot{e}_3 = x_4 - y_4 + e_3{}^2 - e_3{}^2 - u_3 \\ \dot{e}_4 = -x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) + e_3{}^2 - e_3{}^2 - u_4 \end{cases} \tag{6.8}$$

where

$$\dot{e}_i = \dot{x}_i - \dot{y}_i \text{ , } (i = 1, 2, 3, 4) \tag{6.9}$$

Let initial states be (x_1, x_2, x_3, x_4) = (-2, 10, -2, 10), (y_1, y_2, y_3, y_4) = (-1, 11, -1, 11), we find that the error dynamics always exists in first quadrant as shown in Fig. 6.1. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{6.10}$$

Its time derivative is

$$\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 \\
&= \left(x_2 - y_2 + e_1^{\ 2} - e_1^{\ 2} - u_1 \right) \\
&\quad + \left(-(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 \right. \\
&\quad \left. - (-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) + e_1^{\ 2} - e_1^{\ 2} - u_2 \right) \\
&\quad + \left(x_4 - y_4 + e_3^{\ 2} - e_3^{\ 2} - u_3 \right) \\
&\quad + (-x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) + e_3^{\ 2} - e_3^{\ 2} - u_4)
\end{aligned} \tag{6.11}$$

Choose

$$\begin{aligned}
u_1 &= x_2 - y_2 + e_1^{\ 2} + e_1 \\
u_2 &= -(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 \\
&\quad - (-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) - e_1^{\ 2} + e_2 \\
u_3 &= x_4 - y_4 + e_3^{\ 2} + e_3 \\
u_4 &= -x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) - e_3^{\ 2} + e_4
\end{aligned} \tag{6.12}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{6.13}$$

which is negative definite function in first quadrant. After 30 sec, four error states approach zero versus time as shown in Fig. 6.2. Time histories of states are shown in Fig. 6.3.

CASE II. The generalized synchronization error function is $e_i = x_i - y_i + m \sin wt + k_i$, , $(i = 1, 2, 3, 4)$.

Our goal is $y_i = x_i + m\sin wt + k_i$, i.e. $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + m\sin wt + k_i) = 0$, , $(i = 1, 2, 3, 4)$

The error dynamics become

$$\begin{cases} \dot{e}_1 = x_2 - y_2 + mw\cos wt + e_1^{\ 2} - e_1^{\ 2} - u_2 \\ \dot{e}_2 = -(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 \\ \quad -(-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) + mw\cos wt + e_1^{\ 2} - e_1^{\ 2} - u_2 \\ \dot{e}_3 = x_4 - y_4 + mw\cos wt + e_3^{\ 2} - e_3^{\ 2} - u_3 \\ \dot{e}_4 = -x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) + mw\cos wt + e_3^{\ 2} - e_3^{\ 2} - u_4 \end{cases} \tag{6.14}$$

where

$$\dot{e}_i = \dot{x}_i + mw\cos wt - \dot{y}_i , \quad (i = 1, 2, 3, 4) \tag{6.15}$$

Let initial states be (x_1, x_2, x_3, x_4) = (-2, 10, -2, 10), (y_1, y_2, y_3, y_4) = (-1, 11, -1, 11), and w=1, m=2, $k_1 = 100$, $k_2 = 100$, $k_3 = 100$, $k_4 = 100$, we find the error dynamic always exists in first quadrant as shown in Fig. 6.4. By GYC partial region asymptoical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{6.16}$$

Its time derivative is

$$\begin{aligned} \dot{V} &= \left(x_2 - y_2 - u_1 + mw\cos wt + e_1^{\ 2} - e_1^{\ 2}\right) \\ &+ \left(-(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 + e_1^{\ 2} - e_1^{\ 2}\right. \\ &\left. -(-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) - u_2 + mw\cos wt\right) \\ &+ \left(x_4 - y_4 - u_3 + mw\cos wt + e_3^{\ 2} - e_3^{\ 2}\right) \\ &+ (-x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) - u_4 + mw\cos wt + e_3^{\ 2} - e_3^{\ 2}) \end{aligned} \tag{6.17}$$

Choose

$$
\begin{aligned}
u_1 &= x_2 - y_2 + mw\cos wt + e_1^{\ 2} + e_1 \\
u_2 &= -(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3 - e_1^{\ 2} \\
&\quad -(-(a+by_3)y_1 - (a+by_3)y_1^3 - cy_2 + dy_3) + mw\cos wt + e_2 \\
u_3 &= x_4 - y_4 + mw\cos wt + e_3^{\ 2} + e_3 \\
u_4 &= -x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) + mw\cos wt - e_3^{\ 2} + e_4
\end{aligned}
\tag{6.18}
$$

We obtain

$$
\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{6.19}
$$

which is negative definite function in first quadrant. After 30 sec, four error states approach zero versus time as shown in Fig. 6.5. Time histories of $x_i - y_i + k_i$ are shown in Fig. 6.6.

CASE III. The generalized synchronization error function is $e_i = \frac{1}{2}x_i^2 - y_i + k_i$, , $(i = 1,2,3,4)$ where $k_1 = 10, k_2 = 700, k_3 = 20, k_4 = 350$.

Our goal is $y = \frac{1}{2}x^2 + k$, i.e. $\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty}(\frac{1}{2}x_i^2 - y_i + k_i) = 0$, $(i = 1,2,3,4)$

The error dynamics become

$$
\begin{cases}
\dot{e}_1 = x_1 x_2 - y_2 + e_1^{\ 2} - e_1^{\ 2} - u_2 \\
\dot{e}_2 = x_2(-(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3) \\
\qquad -(-(a+by_3)y_1 - (a+by_3)y_1^3 - cy_2 + dy_3) + e_1^{\ 2} - e_1^{\ 2} - u_2 \\
\dot{e}_3 = x_3 x_4 - y_4 + e_3^{\ 2} - e_3^{\ 2} - u_3 \\
\dot{e}_4 = x_4(-x_3 - x_3^3 - ex_4 + fx_1) - (-y_3 - y_3^3 - ey_4 + fy_1) + e_3^{\ 2} - e_3^{\ 2} - u_4
\end{cases}
\tag{6.20}
$$

where

$$
\dot{e}_i = x_i \dot{x}_i - \dot{y}_i, \quad (i = 1,2,3,4) \tag{6.21}
$$

Let initial states be (x_1, x_2, x_3, x_4) = (-2, 10, -2, 10), (y_1, y_2, y_3, y_4) = (-1, 11, -1, 11), we find that the error dynamics always exists in first quadrant as shown in Fig. 6.7. By GYC partial region asymptotial stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{6.22}$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 \\ &= x_1 x_2 - y_2 + e_1{}^2 - e_1{}^2 - u_1 \\ &\quad + \big(x_2(-(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3) \\ &\quad - (-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) + e_1{}^2 - e_1{}^2 - u_2 \big) \\ &\quad + x_3 x_4 - y_4 + e_3{}^2 - e_3{}^2 - u_3 \\ &\quad + (x_4(-x_3 - x_3^3 - ex_4 + fx_1) - (-y_3 - y_3^3 - ey_4 + fy_1) + e_3{}^2 - e_3{}^2 - u_4)\end{aligned} \tag{6.23}$$

Choose

$$\begin{aligned} u_1 &= x_1 x_2 - y_2 - e_1{}^2 + e_1 \\ u_2 &= x_2(-(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3) \\ &\quad - (-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) + e_1{}^2 + e_2 \\ u_3 &= x_3 x_4 - y_4 + e_3{}^2 + e_3 \\ u_4 &= x_4(-x_3 - x_3^3 - ex_4 + fx_1) - (-y_3 - y_3^3 - ey_4 + fy_1) - e_3{}^2 + e_4 \end{aligned} \tag{6.24}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{6.25}$$

which is negative definite function in first quadrant. After 30 sec, four error states approach zero versus time as shown in Fig. 6.8. Time histories of $\frac{1}{2} x_i{}^2 - y_i + k$ are shown in Fig. 6.9.

CASE IV. The generalized synchronization error function is $\mathbf{e} = \mathbf{x} - \mathbf{y} + \mathbf{z} + \mathbf{k}$, $\mathbf{z} = [z_1, z_2, z_3, z_4]$ is the state vector of generalized Lorenz system.

The goal system for synchronization is generalized chaotic Lorenz system [12]:

$$
\begin{cases}
\dfrac{d}{dt}z_1 = a_1(z_2 - z_1) + dz_4 \\
\dfrac{d}{dt}z_2 = b_1 z_1 - z_1 z_3 - z_2 \\
\dfrac{d}{dt}z_3 = z_1 z_2 - c_1 z_3 \\
\dfrac{d}{dt}z_4 = -z_1 - a_1 z_4
\end{cases}
\tag{6.26}
$$

Here initial states are (0.1, 0.1, 0.1, 0.1), system parameters $a_1 = 1$, $b_1 = 26$, $c_1 = 0.7$, $d_1 = 1.5$. We have $\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty}(\mathbf{x} - \mathbf{y} + \mathbf{z} + \mathbf{k}) = 0$, where $\mathbf{k} = [70\ 70\ 70\ 70]^{\mathrm{T}}$.

The error dynamics becomes

$$
\begin{cases}
\dot{e}_1 = x_2 - y_2 + a_1(z_2 - z_1) + dz_4 + e_1^{\ 2} - e_1^{\ 2} - u_2 \\
\dot{e}_2 = -(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 + b_1 z_1 - z_1 z_3 - z_2 \\
\qquad -(-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) + e_1^{\ 2} - e_1^{\ 2} - u_2 \\
\dot{e}_3 = x_4 - y_4 + z_1 z_2 - c_1 z_3 + e_3^{\ 2} - e_3^{\ 2} - u_3 \\
\dot{e}_4 = -x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) - z_1 - a_1 z_4 + e_3^{\ 2} - e_3^{\ 2} - u_4
\end{cases}
\tag{6.27}
$$

Let initial states be $(x_1, x_2, x_3, x_4) =$ (-2, 10, -2, 10), $(y_1, y_2, y_3, y_4) =$ (-1, 11, -1, 11), we find the error dynamics always exists in first quadrant as shown in Fig. 6.10. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$
V = e_1 + e_2 + e_3 + e_4 \tag{6.28}
$$

Its time derivative is

$$
\begin{aligned}
\dot{V} = &\left(x_2 - y_2 + a_1(z_2 - z_1) + dz_4 + e_1^{\ 2} - e_1^{\ 2} - u_2\right) \\
&+ (-(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 + b_1 z_1 - z_1 z_3 - z_2 \\
&- (-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) + e_1^{\ 2} - e_1^{\ 2} - u_2) \\
&+ x_4 - y_4 + z_1 z_2 - c_1 z_3 + e_3^{\ 2} - e_3^{\ 2} - u_3 \\
&+ \left(-x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) - z_1 - a_1 z_4 + e_3^{\ 2} - e_3^{\ 2} - u_4\right)
\end{aligned}
\tag{6.29}
$$

Choose

$$
\begin{aligned}
u_1 &= x_2 - y_2 + a_1(z_2 - z_1) + dz_4 + e_1^{\ 2} + e_1 \\
u_2 &= -(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 + b_1 z_1 - z_1 z_3 - z_2 \\
&\quad - (-(a + by_3)y_1 - (a + by_3)y_1^3 - cy_2 + dy_3) - e_1^{\ 2} + e_2 \\
u_3 &= x_4 - y_4 + z_1 z_2 - c_1 z_3 + e_3^{\ 2} + e_3 \\
u_4 &= -x_3 - x_3^3 - ex_4 + fx_1 - (-y_3 - y_3^3 - ey_4 + fy_1) - z_1 - a_1 z_4 - e_3^{\ 2} + e_4
\end{aligned}
\tag{6.30}
$$

We obtain

$$
\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{6.31}
$$

which is negative definite function in first quadrant. After 30 sec, four error states approach zero versus time as shown in Fig. 6.11. Time histories of $x_i - y_i + k_i$ are shown in Fig. 6.12.

6.4 Chaotization to a new Mathieu –Duffing system

In this section we will control a periodic motion of a periodic system to the chaotic motion of the new Mathieu – Duffing system. The periodic system is

$$
\begin{cases}
\dfrac{d}{dt} y_1 = y_2 + y_3 \\
\dfrac{d}{dt} y_2 = m_2 \omega_2 \cos \omega_2 t - y_4 \\
\dfrac{d}{dt} y_3 = y_4 + y_2 \\
\dfrac{d}{dt} y_4 = m_4 \omega_4 \cos \omega_4 t + y_2
\end{cases}
\tag{6.32}
$$

where m_2, m_4, ω_2, ω_4 are parameters. This system exhibits periodic motion when the parameters of system are $m_2 = 15$, $m_4 = 20$, $\omega_2 = 2$, $\omega_4 = 1.5$ and the initial states of system are $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 0$, $y_4(0) = 0$. Periodic motion is shown in Fig. 6.13.

In order to lead (y_1, y_2, y_3, y_4) to the goal, we add control terms u_1, u_2 and u_3 to each equation of Eq. (6.32), respectively.

$$\begin{cases} \frac{d}{dt}y_1 = y_2 + y_3 + u_1 \\ \frac{d}{dt}y_2 = m_2\omega_2 \cos\omega_2 t - y_4 + u_2 \\ \frac{d}{dt}y_3 = y_4 + y_2 + u_3 \\ \frac{d}{dt}y_4 = m_4\omega_4 \cos\omega_4 t + y_2 + u_4 \end{cases} \tag{6.33}$$

The goal system is new Mathieu – Duffing system:

$$\begin{cases} \frac{d}{dt}x_1 = x_2 \\ \frac{d}{dt}x_2 = -(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3 \\ \frac{d}{dt}x_3 = x_4 \\ \frac{d}{dt}x_4 = -x_3 - x_3^3 - ex_4 + fx_1 \end{cases} \tag{6.34}$$

The generalized synchronization error function is $e_i = y_i - x_i + k_i \quad ,(i = 1,2,3,4)$.

$$\begin{cases} e_1 = y_1 - x_1 + k_1 \\ e_2 = y_2 - x_2 + k_2 \\ e_3 = y_3 - x_3 + k_3 \\ e_4 = y_4 - x_4 + k_4 \end{cases} \tag{6.35}$$

where k_i, $(i = 1,2,3,4)$ is positive constants, we choose $\mathrm{k_1{=}k_2{=}k_3{=}k_4{=}100}$ in order that the error dynamics always happens in first quadrant.

Our goal is $\mathbf{x} = \mathbf{y} + k$, i.e. the controlling goal is that

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - x_i + k_i) = 0 \ , (i = 1,2,3,4) \tag{6.36}$$

The error dynamics becomes

$$\begin{cases} \dot{e}_1 = y_2 + y_3 - x_2 + e_1^{\;2} - e_1^{\;2} + u_1 \\ \dot{e}_2 = m_2\omega_2 \cos\omega_2 t - y_4 \\ \qquad -(-(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3) + e_1^{\;2} - e_1^{\;2} + u_2 \\ \dot{e}_3 = y_4 + y_2 - x_4 + e_3^{\;2} - e_3^{\;2} + u_3 \\ \dot{e}_4 = m_4\omega_4 \cos\omega_4 t + y_2 - (-x_3 - x_3^3 - ex_4 + fx_1) + e_3^{\;2} - e_3^{\;2} + u_4 \end{cases} \tag{6.37}$$

where

$$\dot{e}_i = \dot{y}_i - \dot{x}_i\,, \quad (i = 1,2,3,4) \tag{6.38}$$

Let initial states be (x_1, x_2, x_3, x_4) = (-2, 10, -2, 10), (y_1, y_2, y_3, y_4) = (0, 0, 0, 0), we find the error dynamics always exists in first quadrant as shown in Fig. 6.14. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{6.39}$$

Its time derivative is

$$\begin{aligned} \dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 \\ &= \left(y_2 + y_3 - x_2 + e_1^{\;2} - e_1^{\;2} + u_1\right) \\ &\quad + \left(m_2\omega_2 \cos\omega_2 t - y_4\right. \\ &\quad \left. -(-(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3) + e_1^{\;2} - e_1^{\;2} + u_2\right) \\ &\quad + \left(y_4 + y_2 - x_4 + e_3^{\;2} - e_3^{\;2} + u_3\right) \\ &\quad + (m_4\omega_4 \cos\omega_4 t + y_2 - (-x_3 - x_3^3 - ex_4 + fx_1) + e_3^{\;2} - e_3^{\;2} + u_4) \end{aligned} \tag{6.40}$$

Choose

$$\begin{aligned} u_1 &= -y_2 - y_3 + x_2 + e_1^{\;2} - e_1 \\ u_2 &= -m_2\omega_2 \cos\omega_2 t + y \\ &\qquad + (-(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3) - e_1^{\;2} - e_2 \\ u_3 &= -y_4 - y_2 + x_4 + e_3^{\;2} - e_3 \\ u_4 &= -m_4\omega_4 \cos\omega_4 t - y_2 + (-x_3 - x_3^3 - ex_4 + fx_1) - e_3^{\;2} - e_4 \end{aligned} \tag{6.41}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{6.42}$$

which is negative definite function. After 30 sec, four error states approach zero versus time as shown in Fig. 6.15. Time histories of states are shown in Fig. 6.16.

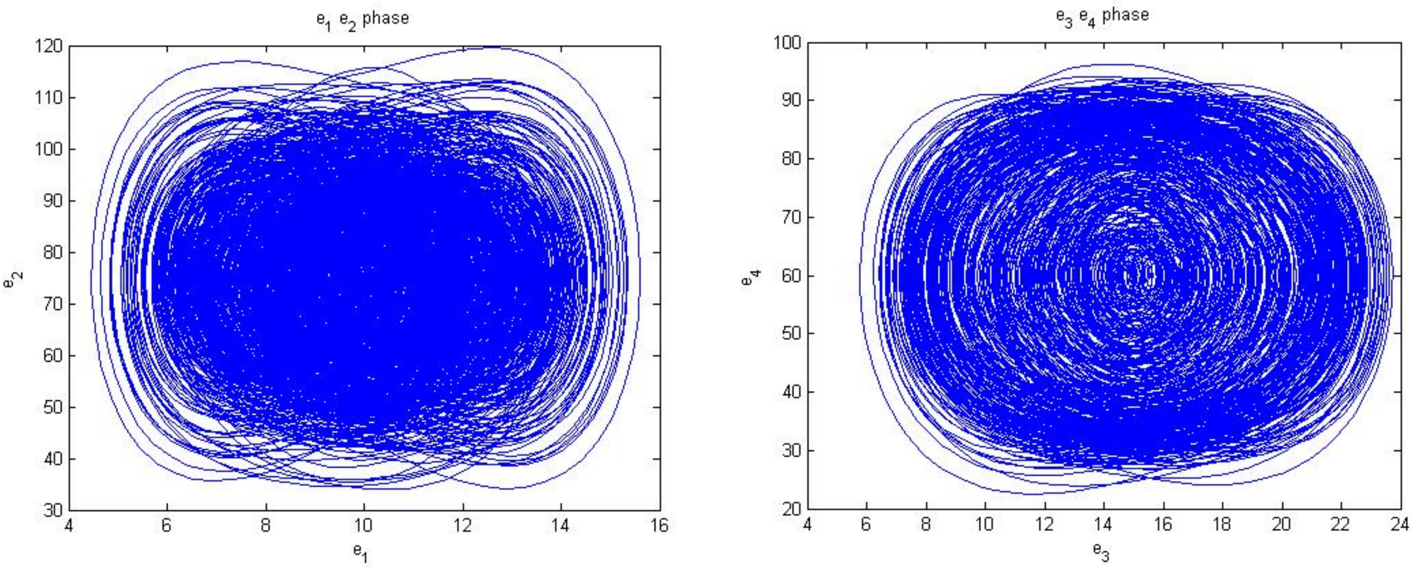

Fig. 6.1 Phase portrait of four errors dynamics for Case I.

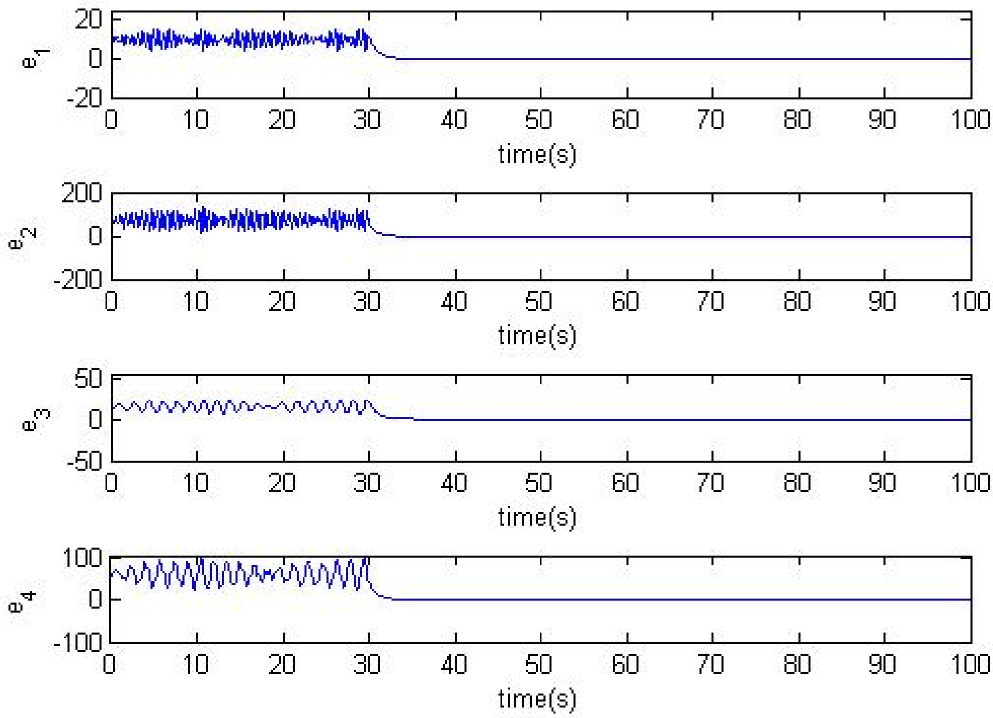

Fig.6.2 Time histories of errors for Case I.

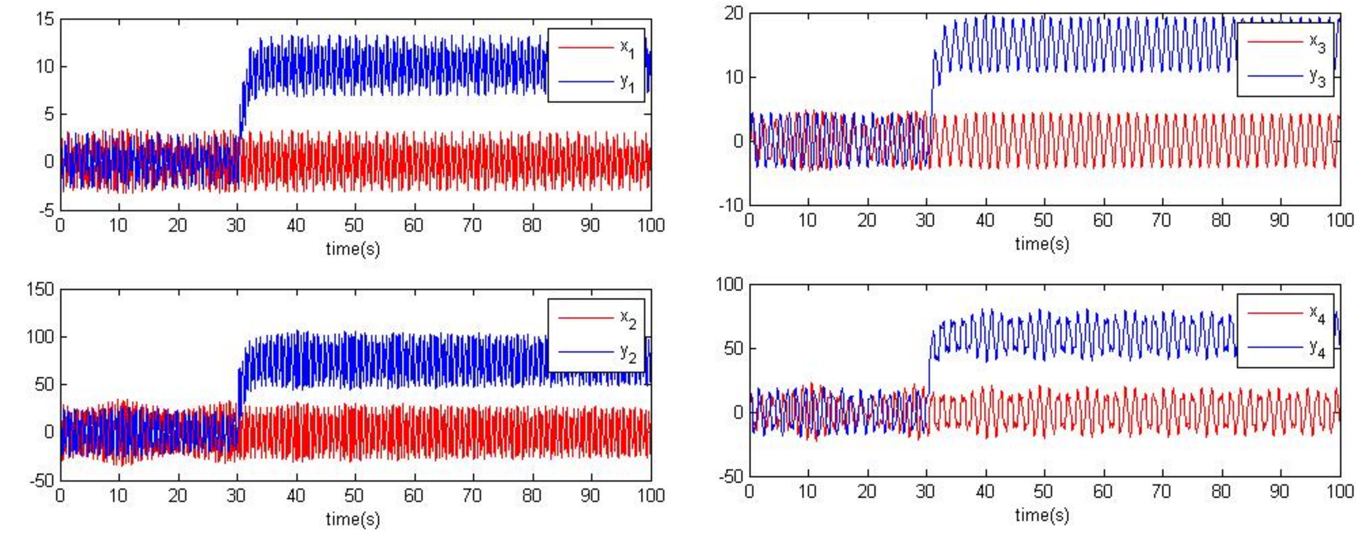

Fig. 6.3 Time histories of $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ for Case I.

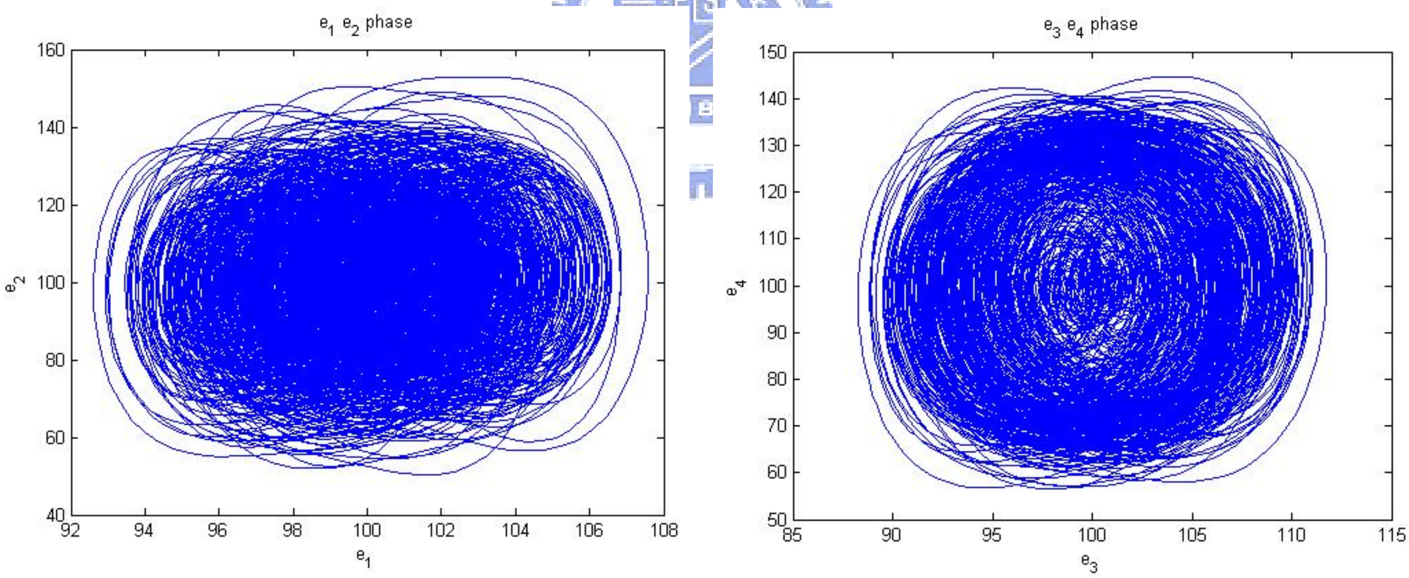

Fig. 6.4 Phase portrait of error dynamics for Case II.

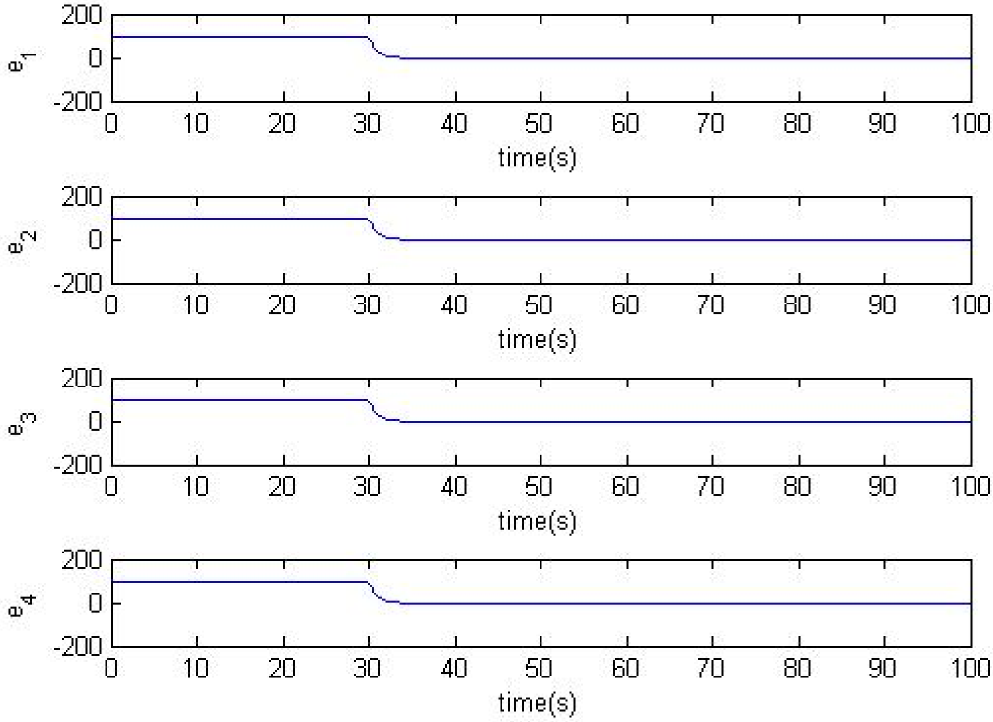

Fig. 6.5 Time histories of errors for Case II.

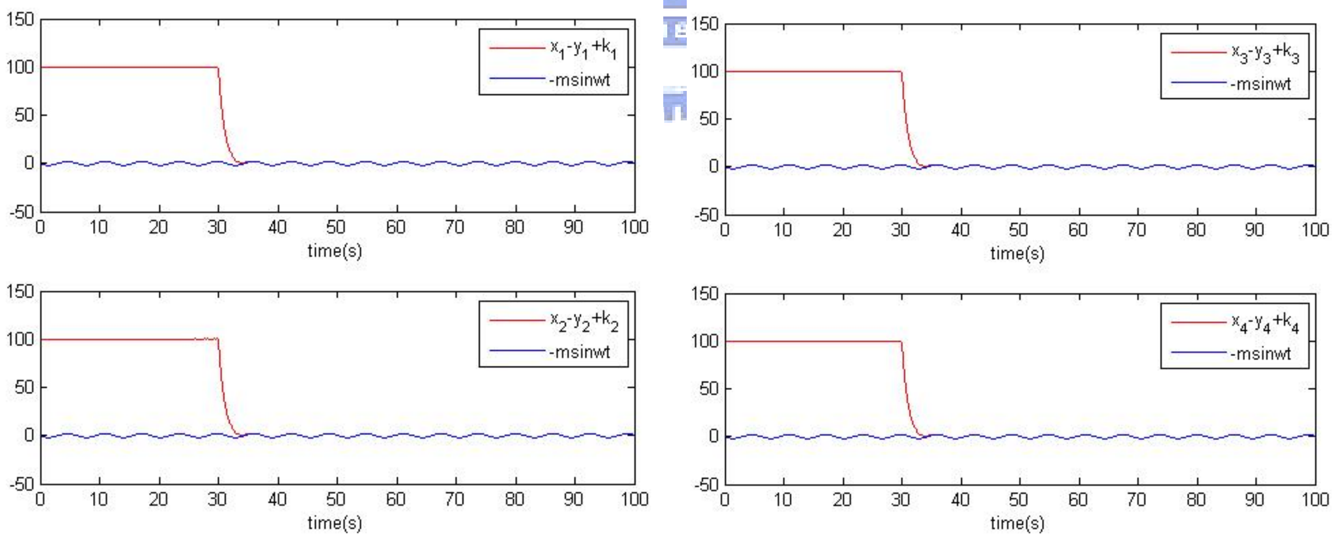

Fig. 6.6 Time histories of $x_i - y_i + k_i$ and $-m\sin\omega t$ for Case II.

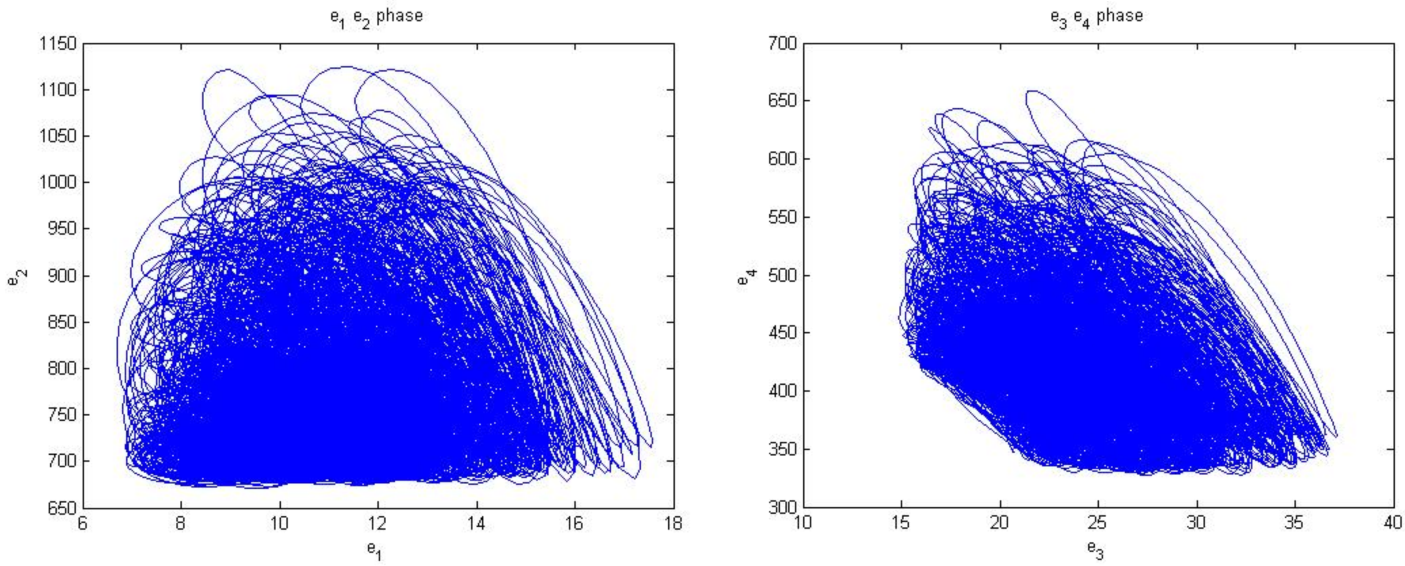

Fig. 6.7 Phase portrait of error dynamics for Case III.

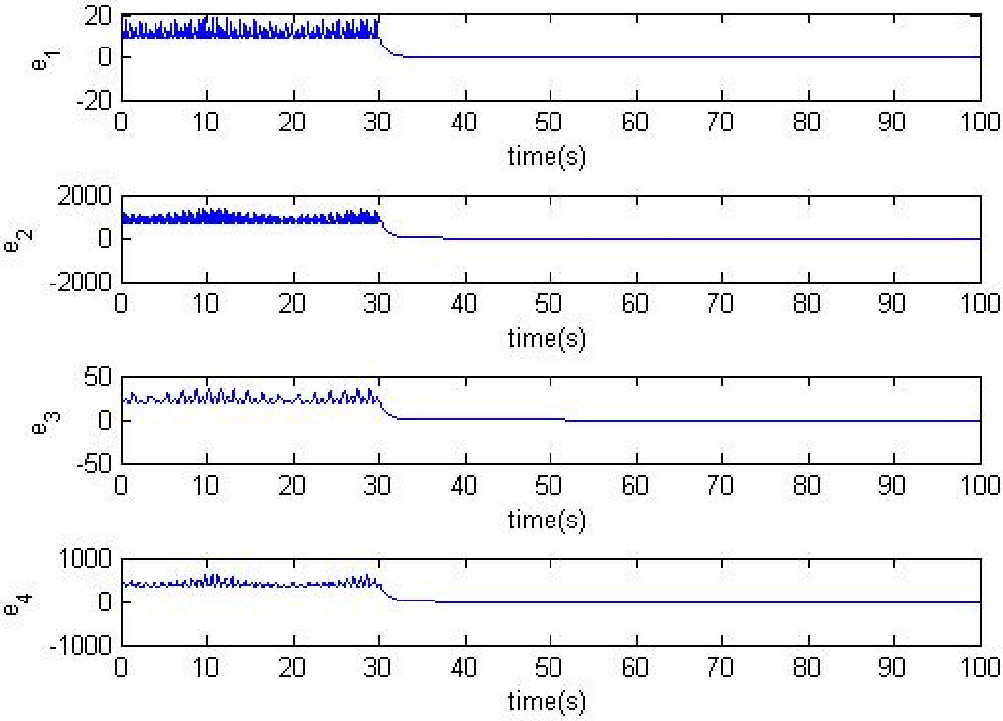

Fig. 6.8 Time histories of errors for Case III.

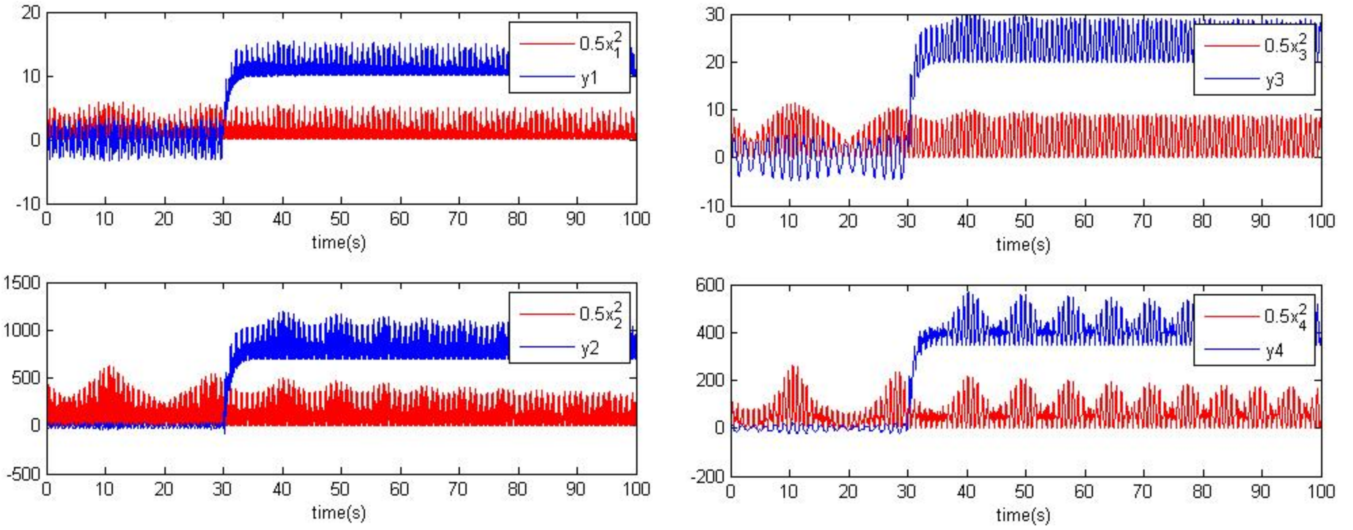

Fig. 6.9 Time histories of $\frac{1}{2}x_i^2$ and y_i for Case III.

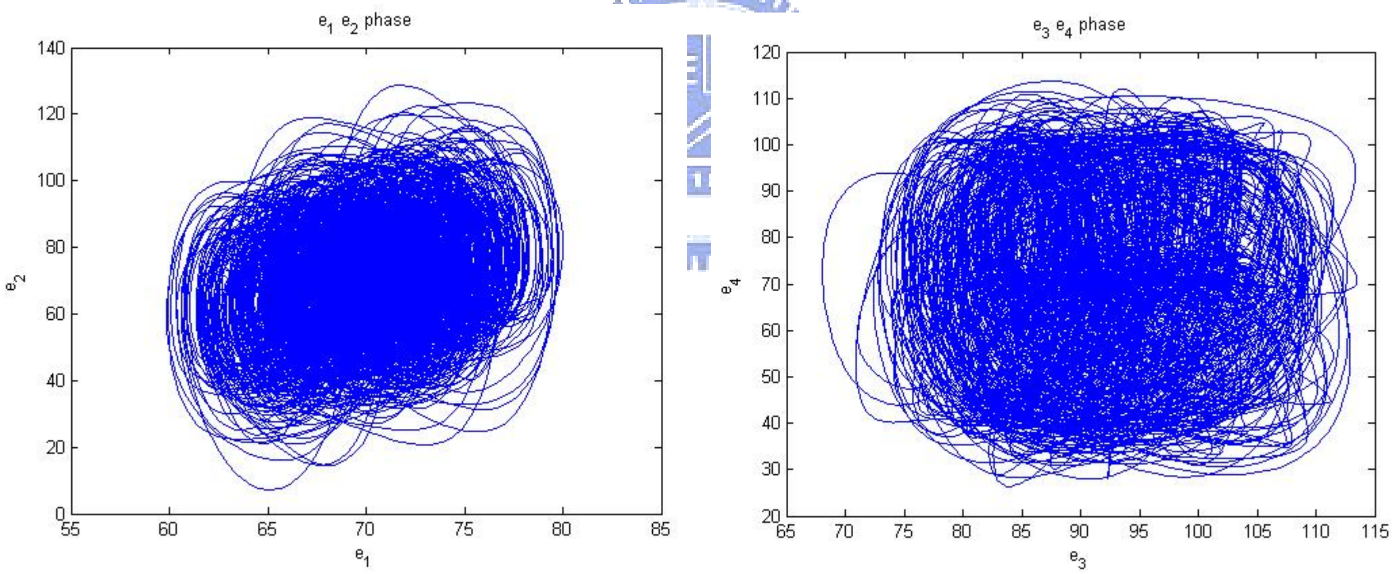

Fig. 6.10 Phase portrait of error dymanics for Case IV.

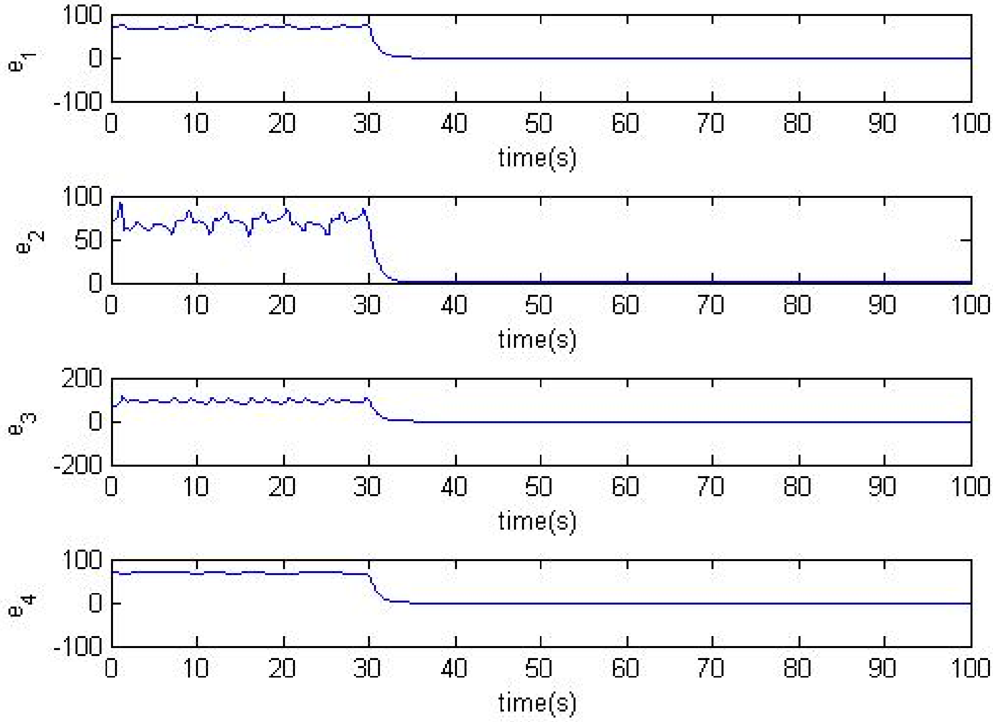

Fig. 6.11 Time histories of errors for Case IV.

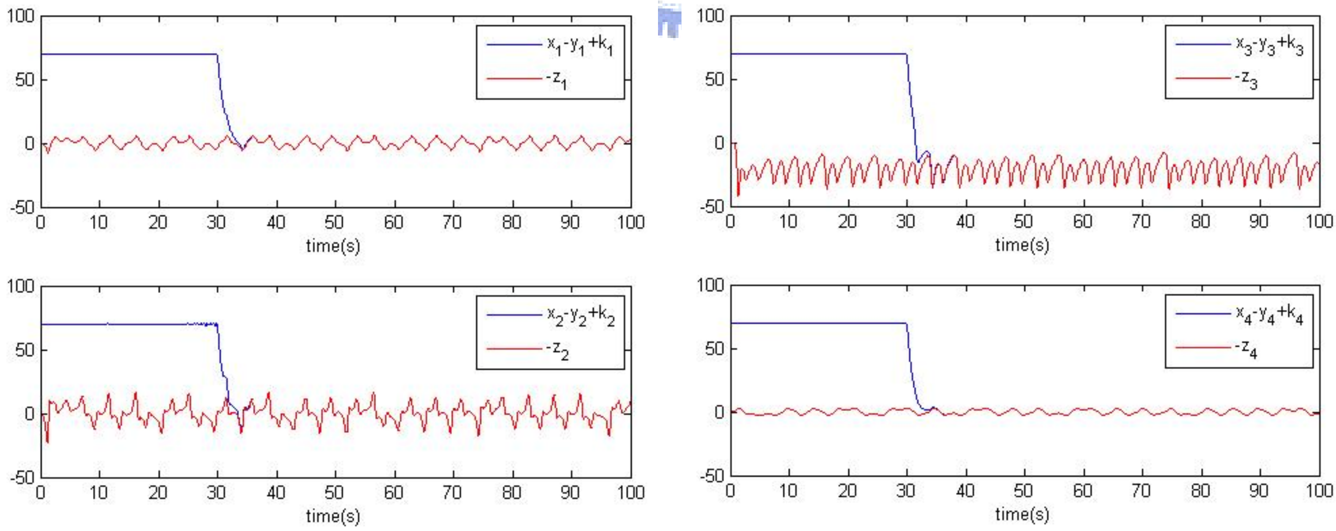

Fig. 6.12 Time histories of $x_i - y_i + k_i$ and $-z_i$ for Case IV.

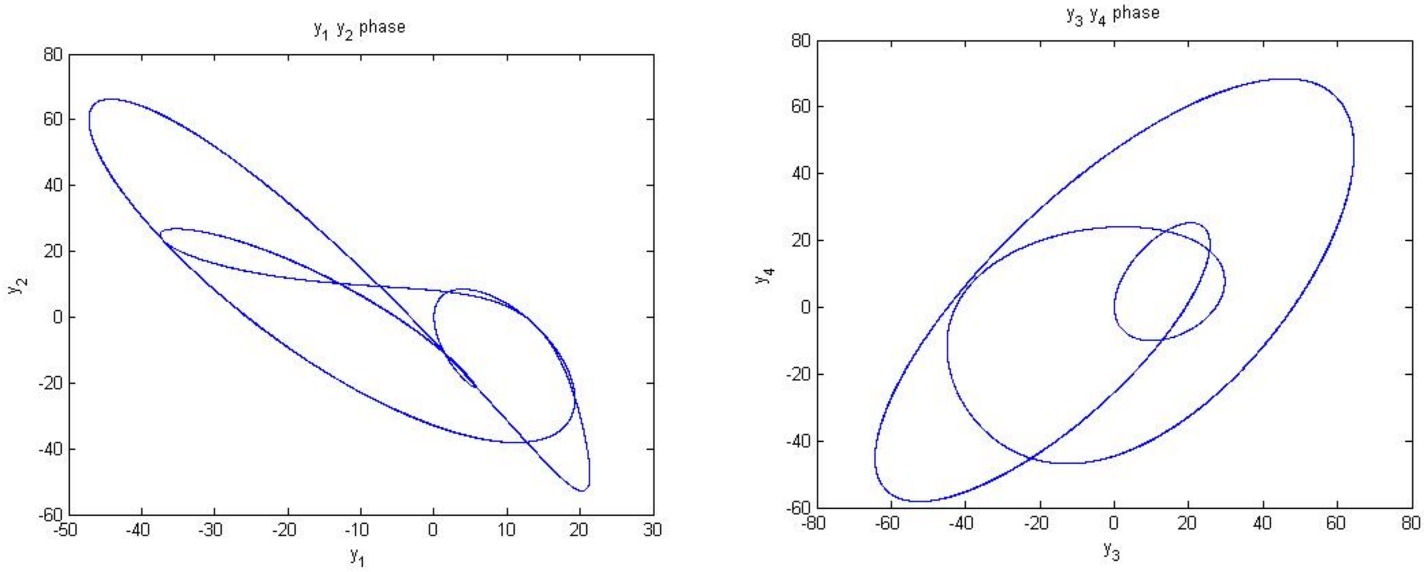

Fig. 6.13 Phase portraits of a new periodic system.

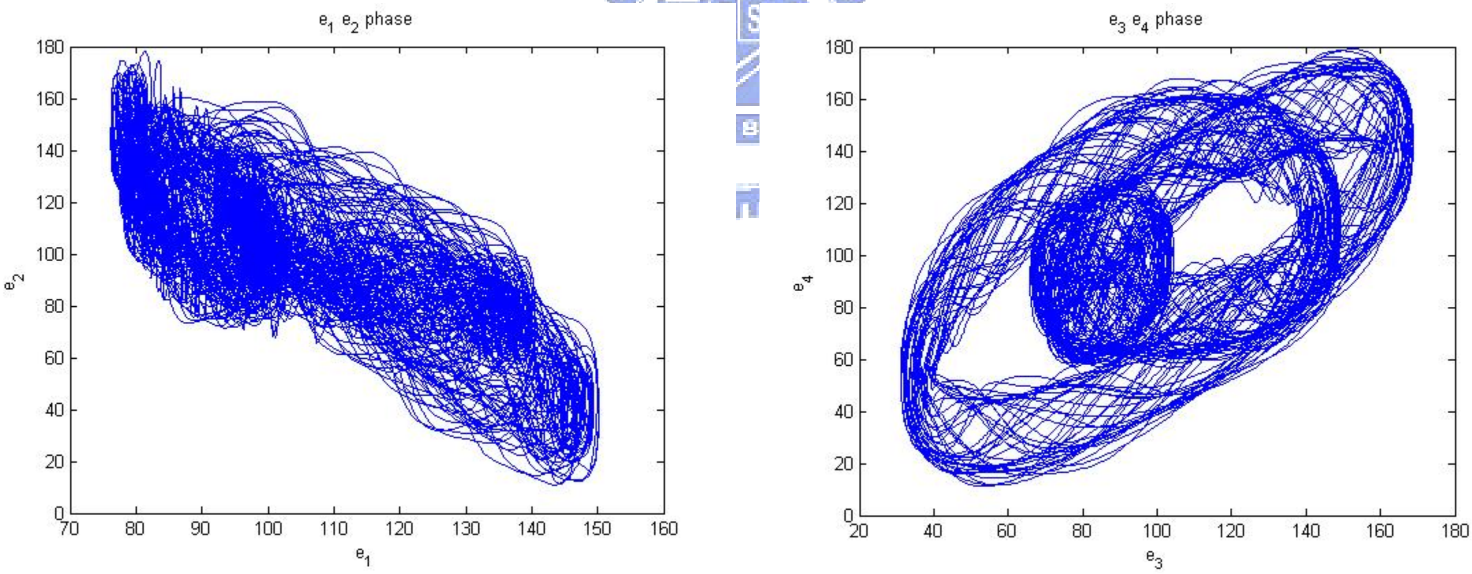

Fig. 6.14 Phase portraits of error dynamics for Section 6.4.

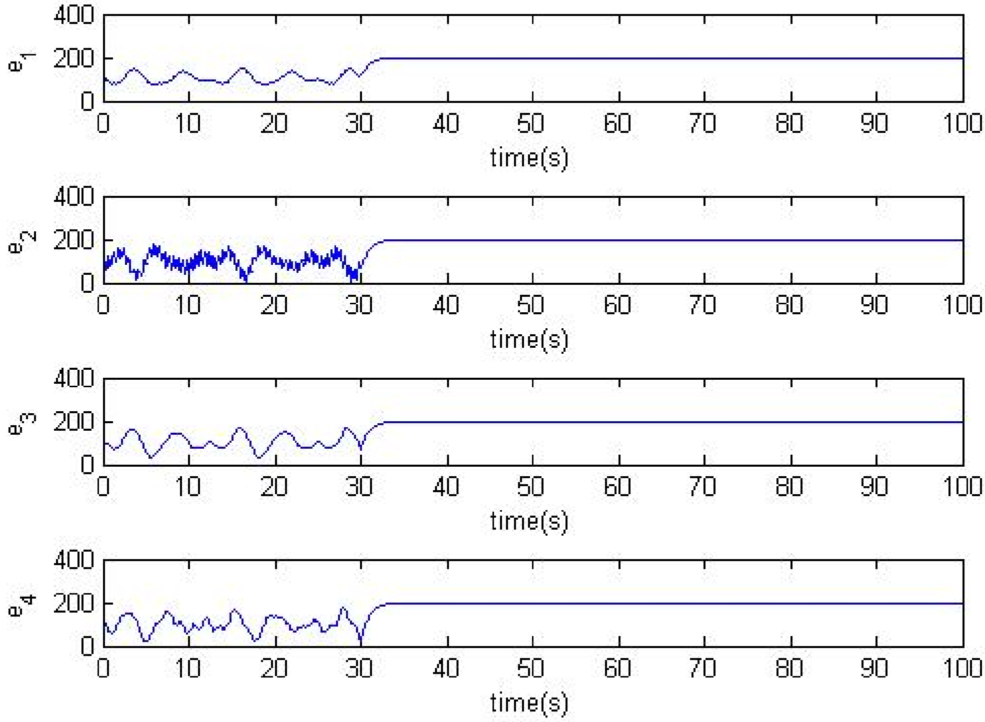

Fig. 6.15 Time histories of errors.

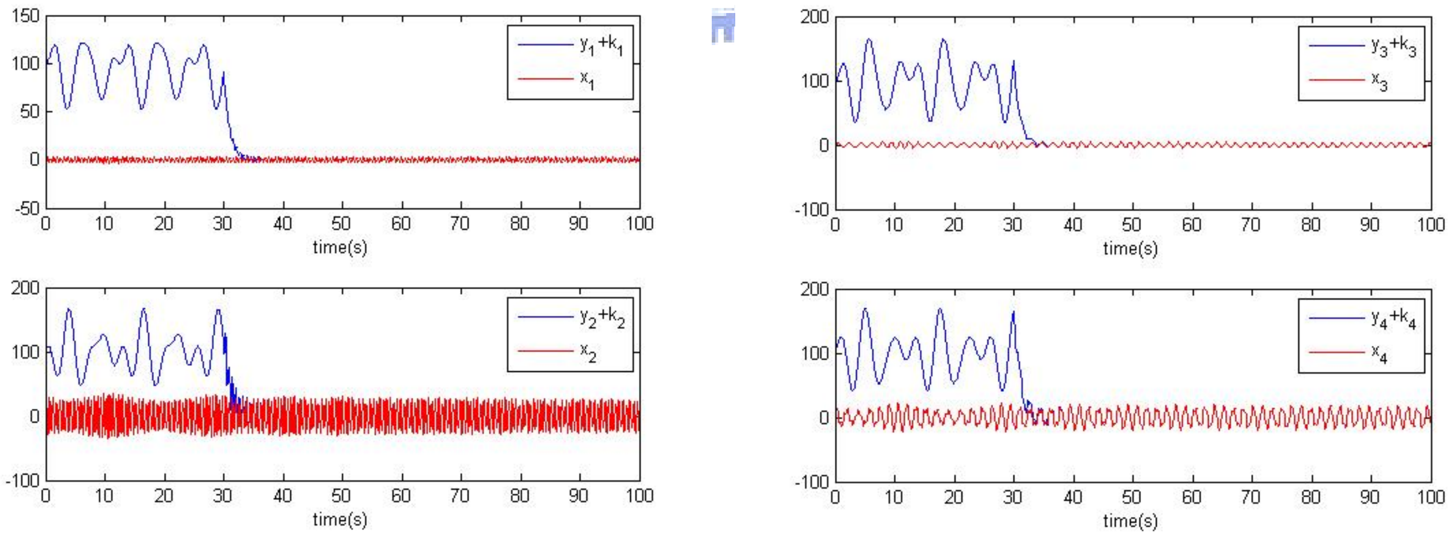

Fig. 6.16 Time histories of $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$.

Chapter 7

Chaos of a New Mathieu – Duffing System with Bessel Function Parameters

7.1 Preliminaries

The chaotic behaviors in a new Mathieu - Duffing system with Bessel function parameters is studied numerically by phase portraits, Poincaré maps and Lyapunov exponent diagram. It is found that chaos exists.

7.2 A new Mathieu – Duffing system with Bessel function parameters

Mathieu system and Duffing system are two typical nonlinear nonautonomous systems:

$$\begin{cases} \frac{d}{dt}x_1 = x_2 \\ \frac{d}{dt}x_2 = -(a + b\sin\omega t)x_1 - (a + b\sin\omega t)x_1^3 - cx_2 + d\sin\omega t \end{cases} \tag{7.1}$$

$$\begin{cases} \frac{d}{dt}x_3 = x_4 \\ \frac{d}{dt}x_4 = -x_3 - x_3^3 - ex_4 + f\sin\omega t \end{cases} \tag{7.2}$$

Exchanging $\sin\omega t$ term in Eq.(7.1) with x_3 and $\sin\omega t$ in Eq.(7.2) with x_1, we obtain a new autonomous Mathieu – Duffing system:

$$\begin{cases} \frac{d}{dt}x_1 = x_2 \\ \frac{d}{dt}x_2 = -(a+bx_3)x_1 - (a+bx_3)x_1^3 - cx_2 + dx_3 \\ \frac{d}{dt}x_3 = x_4 \\ \frac{d}{dt}x_4 = -x_3 - x_3^3 - ex_4 + fx_1 \end{cases} \tag{7.3}$$

where x_1, x_2, x_3, x_4 are state variables, and a, b, c, d, e, f are constant parameters. When a, b, c, d, e, f are given as:

$$\begin{cases} a(t) = J_0(t) + 15 \\ b(t) = Y_0(t+0.01) + 1 \\ c(t) = K_1(t+0.01) + 0.005 \\ d(t) = H_1^1(t+0.01) - 23 \\ e(t) = K_0(t+0.01) + 0.002 \\ f(t) = J_1(t) + 14 \end{cases} \tag{7.4}$$

A new Mathieu – Duffing system with Bessel function [75] parameters is obtained, where

$$\begin{cases} J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1)} (\frac{t}{2})^{2n} \\ Y_0(t+0.01) = \lim_{\mu \to 0} \frac{\cos \mu\pi J_\mu(t+0.01) - J_{-\mu}(t+0.01)}{\sin \mu\pi} \\ K_1(t+0.01) = \frac{(-1)^1}{2} \left[\frac{\partial I_{-\mu}(t+0.01)}{\partial \mu} - \frac{\partial I_\mu(t+0.01)}{\partial \mu} \right]_{\mu=1} \\ H_1^1(t+0.01) = J_1(t+0.01) + iY_1(t+0.01) \\ K_0(t+0.01) = \frac{(-1)^o}{2} \left[\frac{\partial I_{-\mu}(t+0.01)}{\partial \mu} - \frac{\partial I_\mu(t+0.01)}{\partial \mu} \right]_{\mu=0} \\ J_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+2)} (\frac{t}{2})^{2n+1} \end{cases} \tag{7.5}$$

where Γ is Gamma function, $J_{\pm\mu}$ is Bessel function of the first kind, Y_0 is Bessel function of the second kind, $I_{\pm\mu}$ is modified Bessel function of the first kind, K_μ is modified Bessel function of the second kind, H_1^1 is Bessel function of the third kind. The time histories of $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, $f(t)$ are shown in Figs 7.1-7.6. The numerical simulations are carried out by MATLAB with using the fractional operator in the Simulink environment.

7.3 Simulation results

The time history of four states, phase portraits, Poincaré maps, power spectrum, bifurcation and Lyapunov exponents of the new Mathieu – Duffing system Bessel function parameters are showed in Fig. 7.7~Fig. 7.12. Chaos exists for all cases.

For Figs. 7.7-7.12, we vary the system parameter *d*, other system parameters are fixed, which are given $a(t) = J_0(t) + 15$, $b(t) = Y_0(t + 0.01) + 1$, $c(t) = K_1(t + 0.01) + 0.005$, $d(t) = H_1^1(t + 0.01) - 23$, $e(t) = K_0(t + 0.01) + 0.002$, $f(t) = J_1(t) + 14$. The numerical simulations are carried out by FORTRAN with using the Runge-Kutta method for four dimensions in Fig.7.12. .

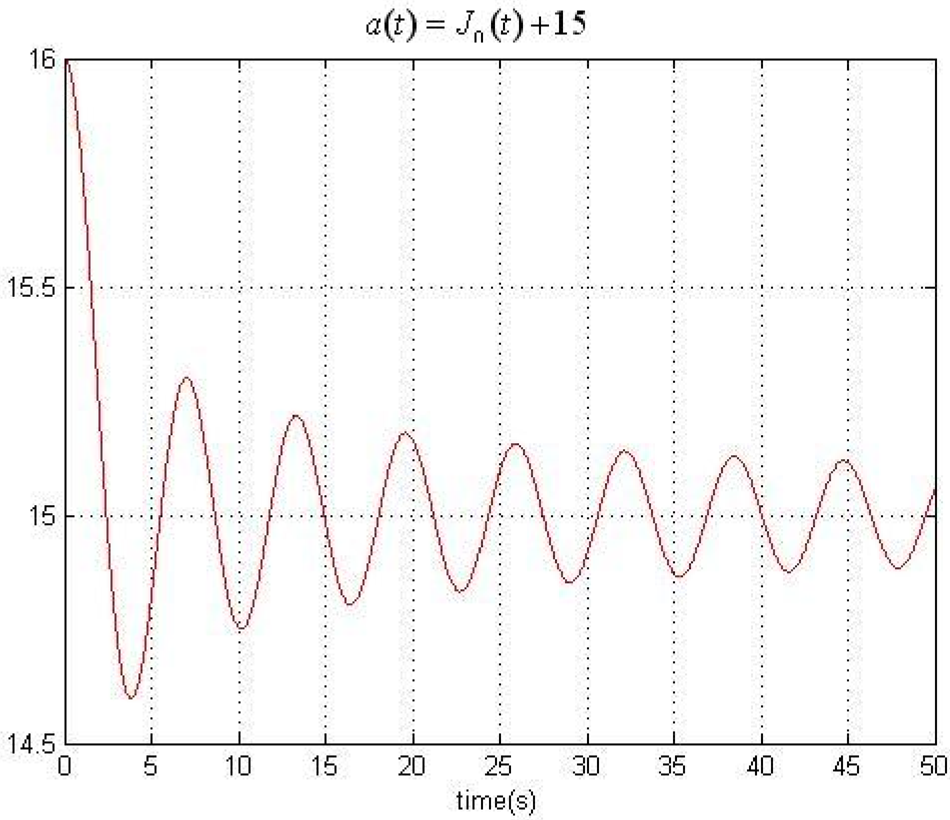

Fig. 7.1 The time history of *a(t)* with 50 sec.

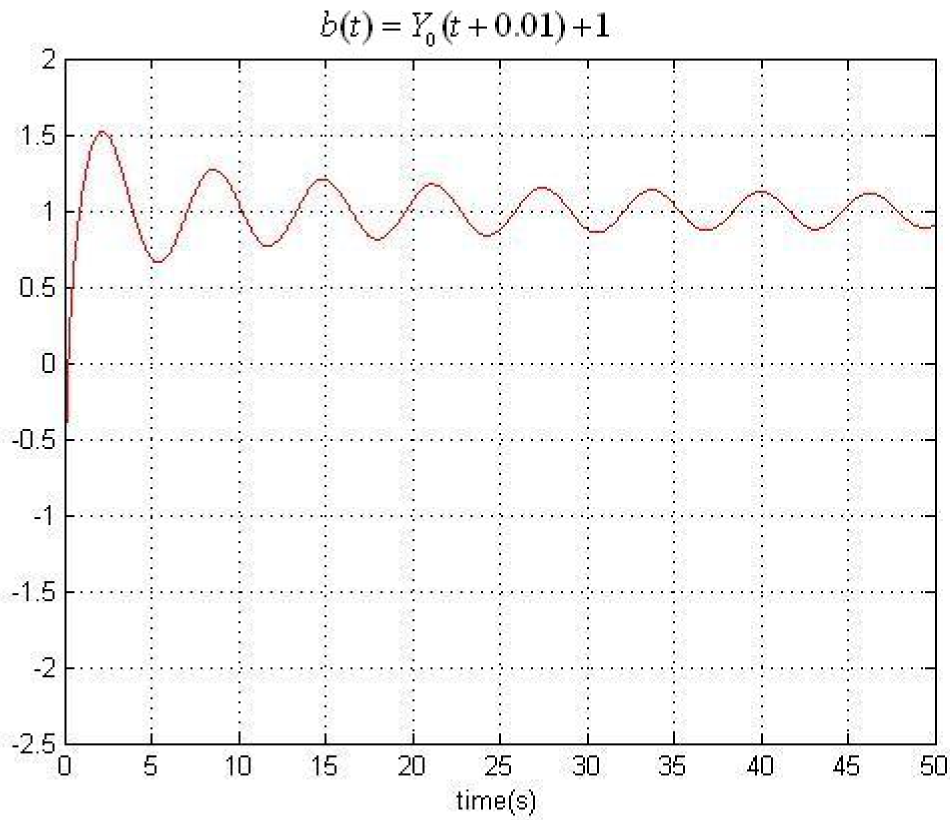

Fig. 7.2 The time history of *b(t)* with 50 sec.

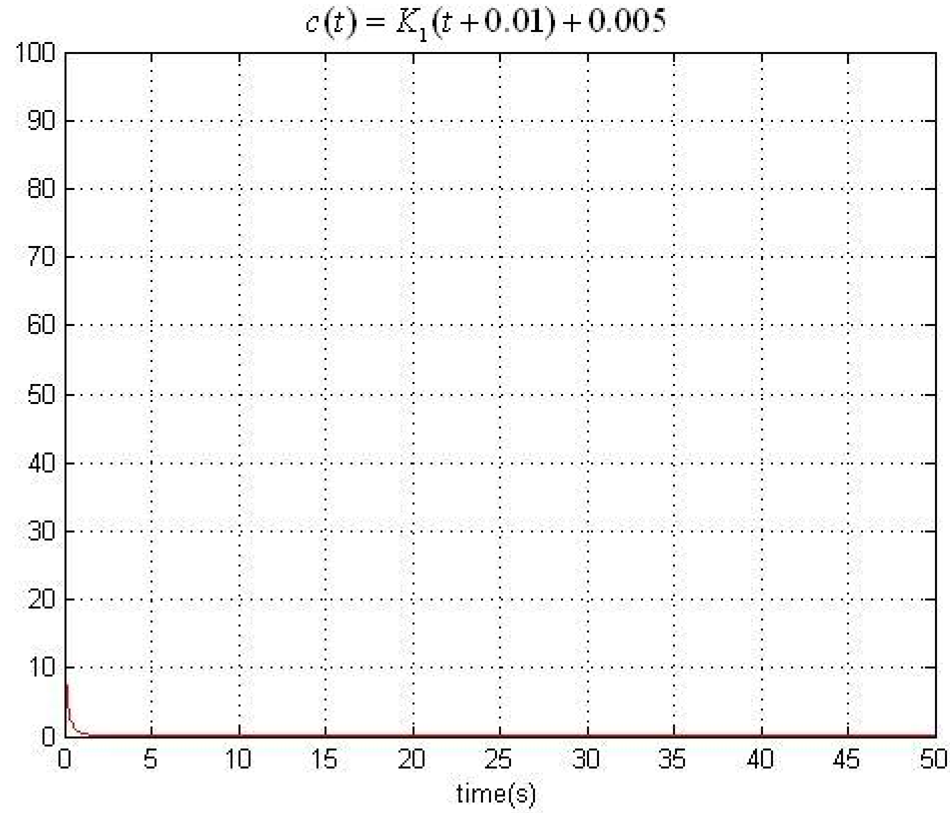

Fig. 7.3 The time history of *c(t)* with 50 sec.

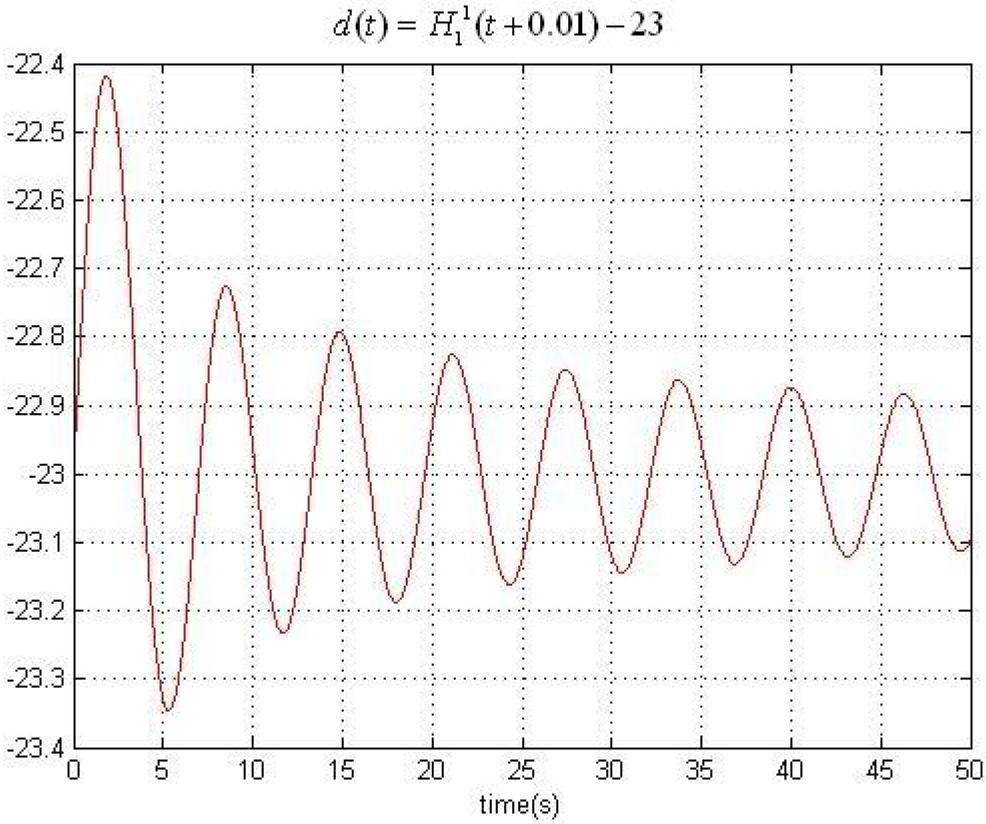

Fig. 7.4 The time history of *d(t)* with 50 sec.

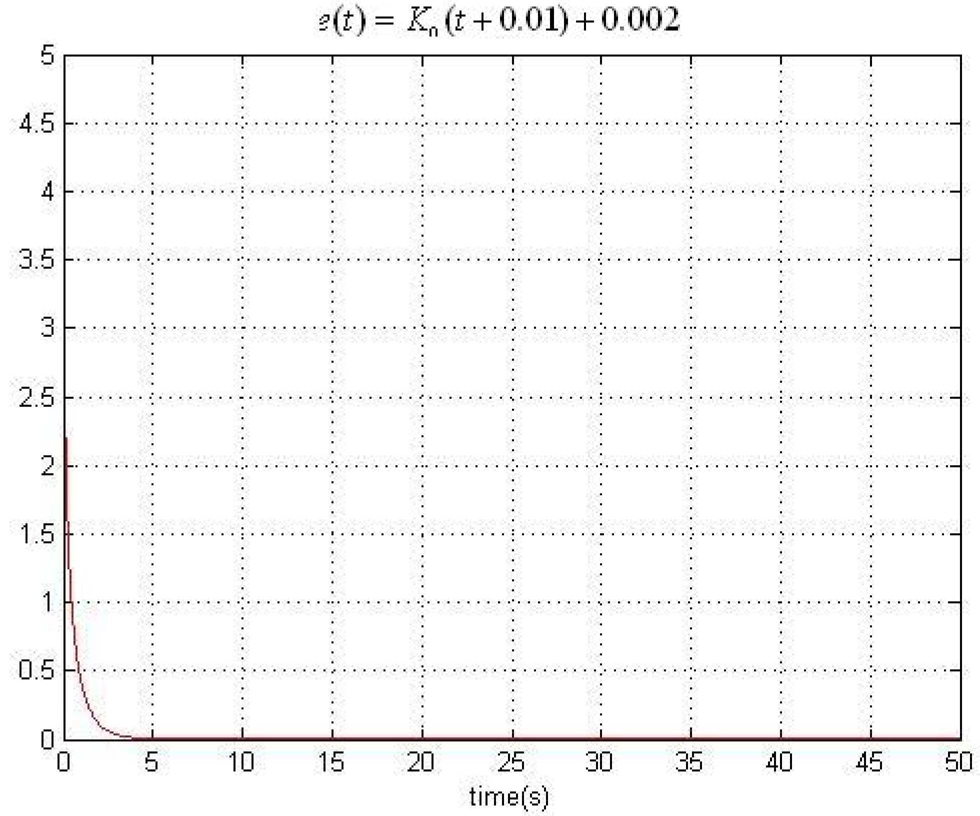

Fig. 7.5 The time history of *e(t)* with 50 sec.

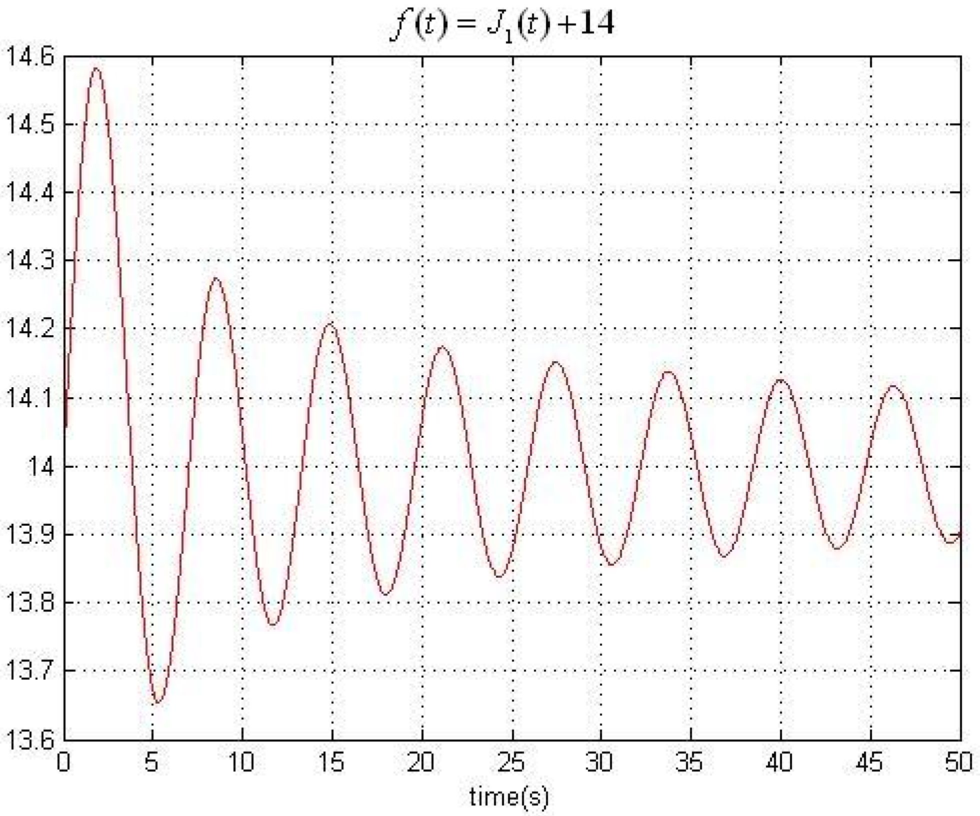

Fig. 7.6 The time history of *f(t)* with 50 sec.

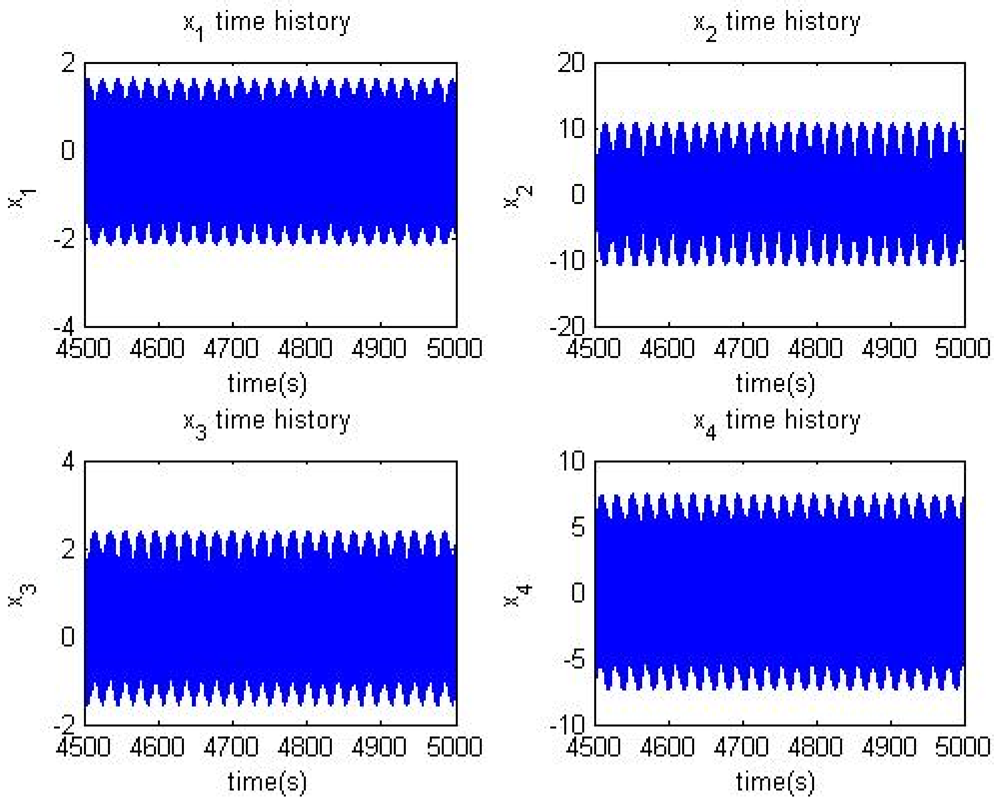

Fig. 7.7 The time history of the four states with 5000 sec.

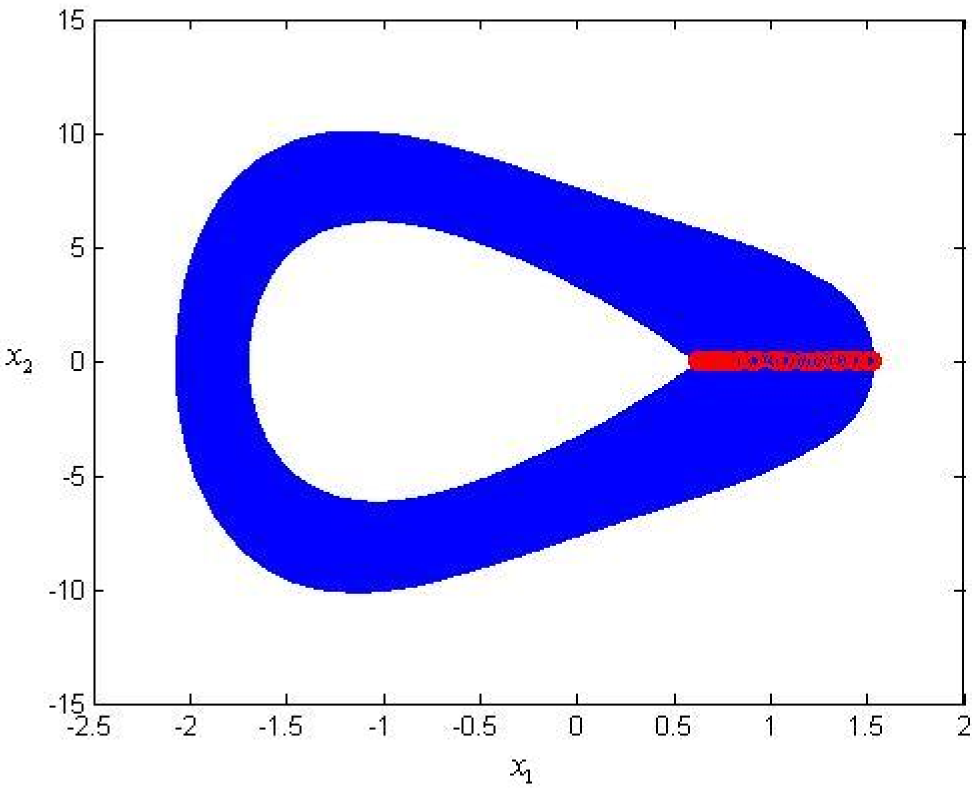

Fig. 7.8 The phase portrait and Poincaré map of x_1, x_2 dimensions.

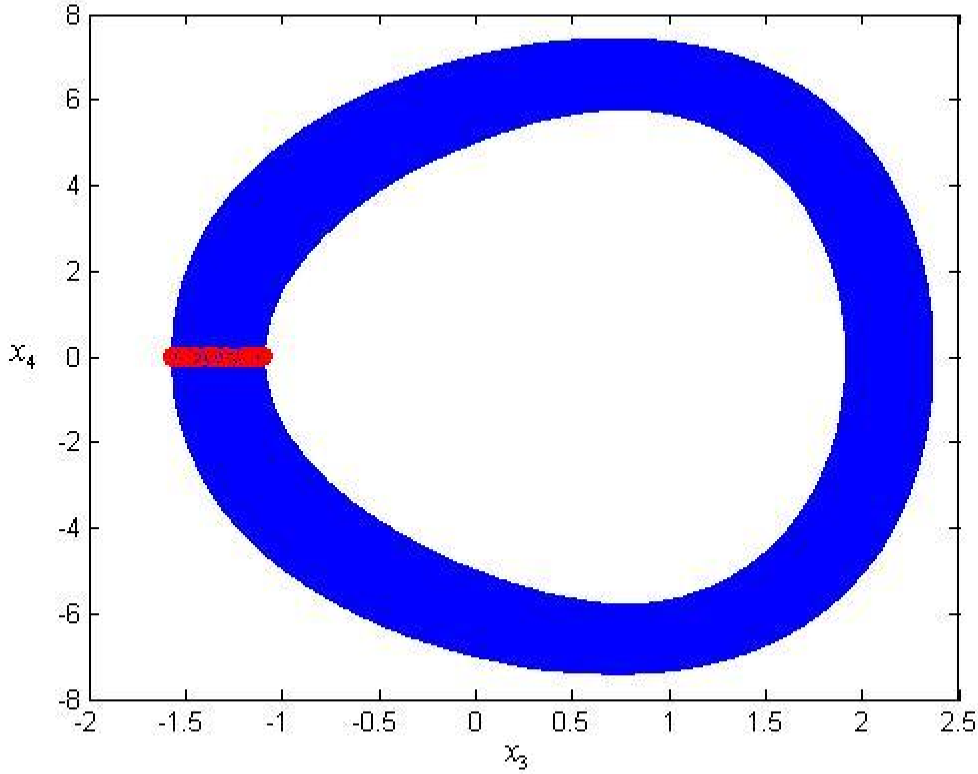

Fig. 7.9 The phase portrait and Poincaré map of x_3, x_4 dimensions.

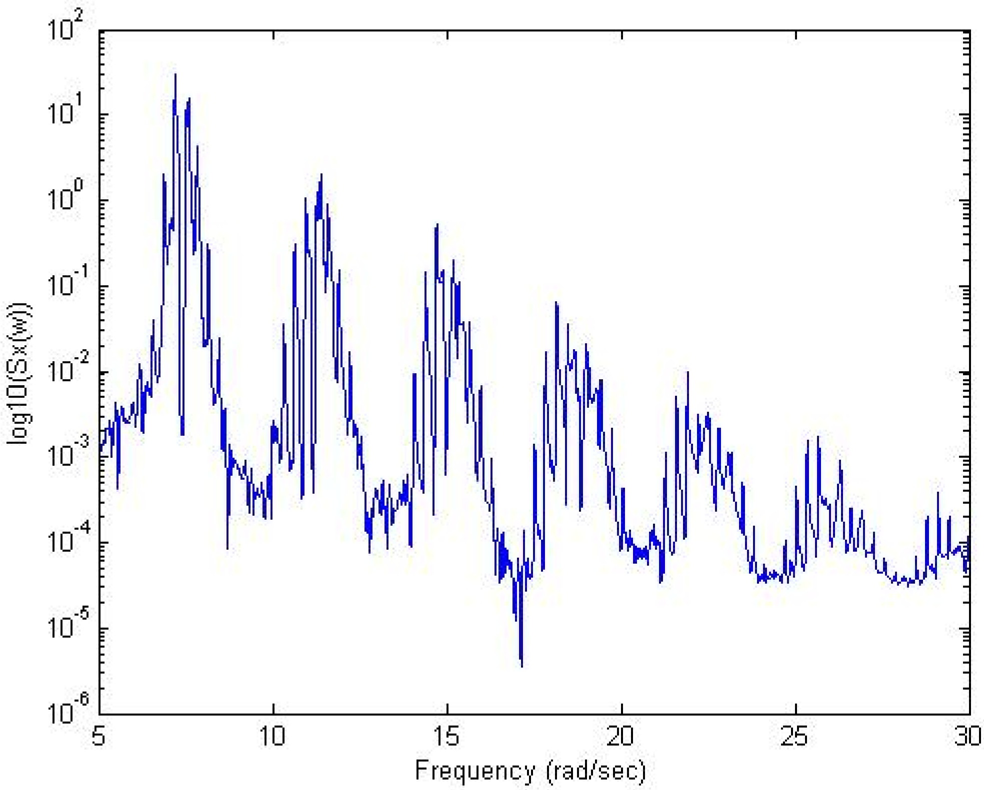

Fig. 7.10 The chaotic power spectrum of x_1 for Mathieu-Duffing system.

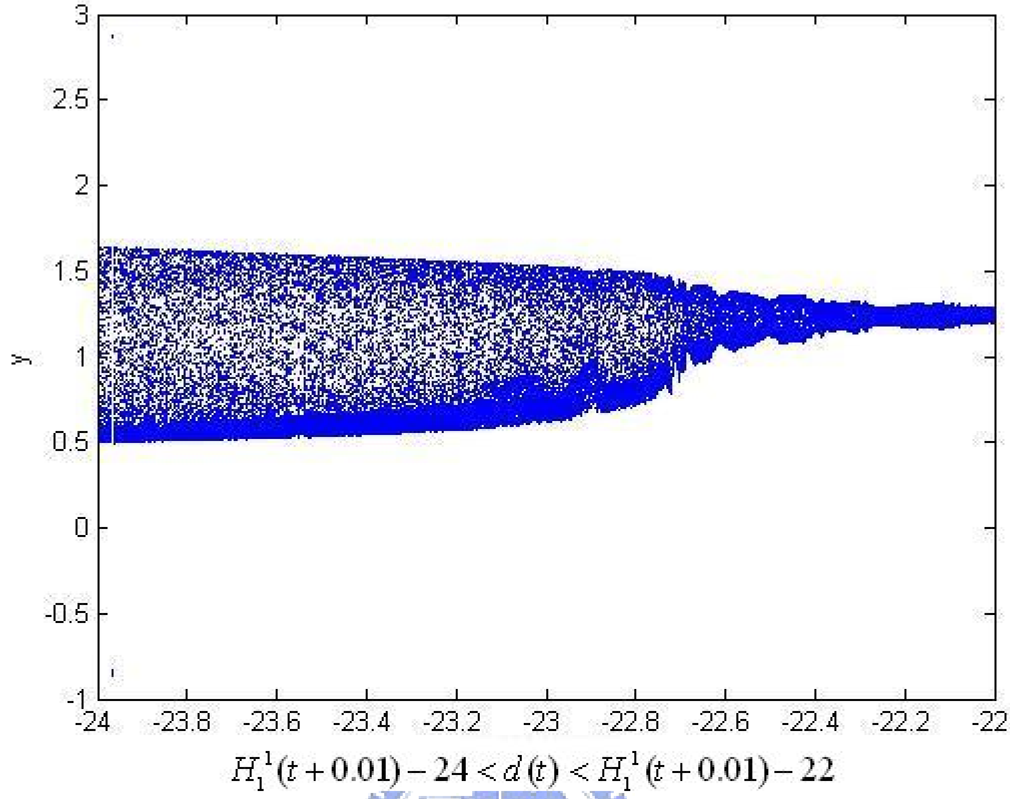

Fig. 7.11 Bifurcation of chaotic Mathieu-Duffing system.

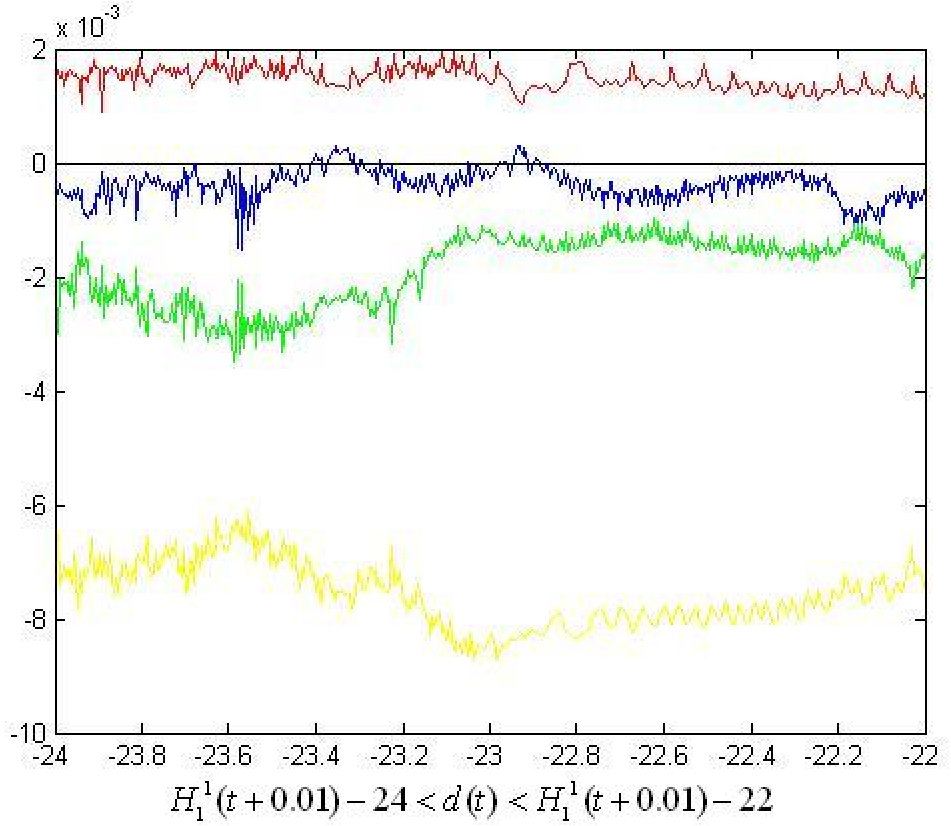

Fig. 7.12 Lyapunov exponents of chaotic Mathieu-Duffing system (+,0,-,-).

Chapter 8

Pragmatical Hybrid Projective Generalized Synchronization of New Mathieu- Duffing Systems with Bessel Function Parameters by Adaptive Control and GYC Partial Region Stability Theory

8.1 Preliminaries

In this Chapter, our study is devoted to a pragmatical hybrid projective generalized synchronization of two chaotic systems of four-dimension, two new Mathieu-Duffing systems with Bessel function parameters. The parameters of master system of the new Mathieu-Duffing system are fully unknown or uncertain. Based on GYC pragmatical asymptotical stability theorem, GYC partial region stability theory, and adaptive control the master-slave systems are in pragmatical hybrid projective generalized synchronization (PHPGS). Both projective synchronization and projective anti-synchronization are obtained. Numerical simulations prove the effectiveness of the scheme.

8.2 Synchronization scheme

Among many kinds of synchronizations [18-24], the generalized synchronization is investigated [25-30]. This means that we can give a function relationship between the states of the master and slave: $\mathbf{y} = \mathbf{G}(\mathbf{x})$. In this Chapter, a special case of hybrid projective generalized synchronizations

$$\mathbf{y} = \mathbf{G(x)} = g\,\mathbf{x(t)} \tag{8.1}$$

is studied, where $\mathbf{x}$ and $\mathbf{y}$ are state variable vectors of the master and slave, respectively, and $\mathbf{G}$ is a given vector function. Since g is a constant rector with both positive and negative entries, hybrid projective synchronization is named. GYC pragmatical asymptotical stability theorem is used, pragmatical synchronization is named. As a whole, pragmatical hybrid projective generalized synchronization (PHPGS) is named.

The master system is

$$\dot{\mathbf{x}} = \mathbf{f(x)} \tag{8.2}$$

where $\mathbf{x} = [x_1, x_2, ..., x_n]^T \in \Re^n$ is a state vector and all parameters of Eq.(8.2) are uncertain. The slave system is

$$\dot{\mathbf{y}} = \mathbf{f(y)} + \mathbf{u} \tag{8.3}$$

where $\mathbf{y} = [y_1, y_2, ..., y_n]^T \in \Re^n$ is also a state vector, u is a controlled vector.

The control task is to force the slave state vector to track an n-dimensional desired vector

$$\mathbf{h}(t) = [h_1(t), h_2(t), ..., h_n(t)] = [g_1 x_1(t), g_2 x_2(t), ..., g_n x_n(t)] \tag{8.4}$$

where $\mathbf{g} = [g_1, g_2, ..., g_n]$ are constant vector with positive and negative entries.

Define the error as

$$\mathbf{e(t)} = \mathbf{y(t)} - \mathbf{h(t)} = \mathbf{y} - \mathbf{gx} \tag{8.5}$$

where $\mathbf{e}(t) = [e_1, e_2, ..., e_n]^T \in \Re^n$ denotes an error vector. The controlling goal is that

$$\lim_{t\to\infty} \mathbf{e} = 0 \tag{8.6}$$

can be accomplished on the base of GYC pragmatical asymptotical stability theorem and GYC partial region stability theory. By using the GYC partial region stability theory, the

Lyapunov function is easier to find as a homogeneous function of first degree of error states, which is a positive definite Lyapunov function of error states in first quadrant. The controllers can be designed in lower degree.

8.3. Numerical results of PHPGS by GYC partial region stability theory

Take new Mathieu-Duffing system with Bessel function parameters (7.3) as master system, where a, b, c, d, e, f are uncertain parameters and following new Mathieu-Duffing system as slave system:

$$\begin{cases} \frac{d}{dt}y_1 = y_2 \\ \frac{d}{dt}y_2 = -(\hat{a}+\hat{b}y_3)y_1 - (\hat{a}+\hat{b}y_3)y_1^3 - \hat{c}y_2 + \hat{d}y_3 \\ \frac{d}{dt}y_3 = y_4 \\ \frac{d}{dt}y_4 = -y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 \end{cases} \tag{8.7}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}$ *and* $\hat{f}$ are estimated parameters.

Define error states $e_i (i = 1,2,3,4)$

$$\begin{cases} e_1 = y_1 - g_1x_1 + k_1 \\ e_2 = y_2 - g_2x_2 + k_2 \\ e_3 = y_3 - g_3x_3 + k_3 \\ e_4 = y_4 - g_4x_4 + k_4 \end{cases} \tag{8.8}$$

where $g_1, g_2, g_3, and\ g_4$ are partly positive and partly negative constants, to form hybrid projective synchronization, and k_i, $(i = 1,2,3,4)$ is positive constants, we choose $k_1=k_2=k_3=k_4=100$, in order that the error dynamics always happens in first quadrant.

In order to lead (y_1, y_2, y_3, y_4) to $(g_1x_1 + k_1, g_2x_2 + k_2, g_3x_3 + k_3, g_4x_4 + k_4)$, u_1, u_2, u_3 and u_4 are added to each equation of Eq.(8.7), respectively:

$$\begin{cases} \frac{d}{dt}y_1 = y_2 + u_1 \\ \frac{d}{dt}y_2 = -(\hat{a} + \hat{b}y_3)y_1 - (\hat{a} + \hat{b}y_3)y_1^3 - \hat{c}y_2 + \hat{d}y_3 + u_2 \\ \frac{d}{dt}y_3 = y_4 + u_3 \\ \frac{d}{dt}y_4 = -y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 + u_4 \end{cases} \tag{8.9}$$

Our goal is $\mathbf{y} = g\mathbf{x} + k$, i.e. the controlling goal is that

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(g_i x_i - y_i + k_i) = 0 \text{ , } (i = 1,2,3,4) \tag{8.10}$$

The error dynamics becomes

$$\begin{cases} \dot{e}_1 = g_1x_2 - y_2 + e_1^{\,2} - e_1^{\,2} - u_1 \\ \dot{e}_2 = g_2(-(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3) \\ \qquad -(-(\hat{a} + \hat{b}y_3)y_1 - (\hat{a} + \hat{b}y_3)y_1^3 - \hat{c}y_2 + \hat{d}y_3 + e_1^{\,2} - e_1^{\,2} + u_2) \\ \dot{e}_3 = g_3x_4 - y_4 + e_3^{\,2} - e_3^{\,2} - u_3 \\ \dot{e}_4 = g_4(-x_3 - x_3^3 - ex_4 + fx_1) \\ \qquad -(-y_3 - y_3^3 - \hat{e}y_4 + \hat{f}y_1 + e_3^{\,2} - e_3^{\,2} + u_4) \end{cases} \tag{8.11}$$

where

$$\dot{e}_i = g_i\dot{x}_i - \dot{y}_i \text{ , } (i = 1,2,3,4) \tag{8.12}$$

Let initial states be (x_1, x_2, x_3, x_4) = (-2, 10, -2, 10), (y_1, y_2, y_3, y_4) = (-2.000001, 10.000001, -2.000001, 10.00001) and $a(t) = J_0(t) + 15$, $b(t) = Y_0(t + 0.01) + 1$, $c(t) = K_1(t + 0.01) + 0.005$, $d(t) = H_1^{\,1}(t + 0.01) - 23$, $e(t) = K_0(t + 0.01) + 0.002$, $f(t) = J_1(t) + 14$, we find the error dynamics always exists in first quadrant as shown in Fig. 8.1. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function of e_1, e_2, e_3, e_4, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 + \tilde{a} + \tilde{b} + \tilde{c} + \tilde{d} + \tilde{e} + \tilde{f} \tag{8.13}$$

where $\tilde{a} = a - \hat{a}$, $\tilde{b} = b - \hat{b}$, $\tilde{c} = c - \hat{c}$, $\tilde{d} = d - \hat{d}$, $\tilde{e} = e - \hat{e}$, $\tilde{f} = f - \hat{f}$ and $\hat{a}$,$\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{e}$, $\hat{f}$ are estimated values of the unknown parameters a, b, c, d, e, f respectively. We have

$$\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 + \dot{\tilde{a}} + \dot{\tilde{b}} + \dot{\tilde{c}} + \dot{\tilde{d}} + \dot{\tilde{e}} + \dot{\tilde{f}} \\
&= \left(g_1 x_2 - y_2 + e_1^{\,2} - e_1^{\,2} - u_1\right) \\
&\quad + \left(g_2(-(a + b x_3) x_1 - (a + b x_3) x_1^3 - c x_2 + d x_3)\right. \\
&\quad \left. - (-(\hat{a} + \hat{b} y_3) y_1 - (\hat{a} + \hat{b} y_3) y_1^3 - \hat{c} y_2 + \hat{d} y_3) + e_1^{\,2} - e_1^{\,2} - u_2\right) \\
&\quad + \left(g_3 x_4 - y_4 + e_3^{\,2} - e_3^{\,2} - u_3\right) \\
&\quad + (g_4(-x_3 - x_3^3 - e x_4 + f x_1) - (-y_3 - y_3^3 - \hat{e} y_4 + \hat{f} y_1) + e_3^{\,2} - e_3^{\,2} - u_4) \\
&\quad + \dot{\tilde{a}} + \dot{\tilde{b}} + \dot{\tilde{c}} + \dot{\tilde{d}} + \dot{\tilde{e}} + \dot{\tilde{f}}
\end{aligned} \tag{8.14}$$

Choose

$$\begin{aligned}
\dot{\tilde{a}} &= \dot{a} - \dot{\hat{a}} = g_2 (a - \hat{a})(x_1 + x_1^3) - \tilde{a} \\
\dot{\tilde{b}} &= \dot{b} - \dot{\hat{b}} = g_2 (b - \hat{b})(x_3 x_1 + x_3 x_1^3) - \tilde{b} \\
\dot{\tilde{c}} &= \dot{c} - \dot{\hat{c}} = g_2 (c - \hat{c})(x_2) - \tilde{c} \\
\dot{\tilde{d}} &= \dot{b} - \dot{\hat{d}} = g_2 (d - \hat{d})(-x_3) - \tilde{d} \\
\dot{\tilde{e}} &= \dot{e} - \dot{\hat{e}} = g_4 (e - \hat{e})(x_4) - \tilde{e} \\
\dot{\tilde{f}} &= \dot{f} - \dot{\hat{f}} = g_4 (f - \hat{f})(-x_1) - \tilde{f} \\
u_1 &= g_1 x_2 - y_2 + e_1^{\,2} - e_1 \\
u_2 &= g_2(-(\hat{a} + \hat{b} x_3) x_1 - (\hat{a} + \hat{b} x_3) x_1^3 - \hat{c} x_2 + \hat{d} x_3) \\
&\quad - (-(\hat{a} + \hat{b} y_3) y_1 - (\hat{a} + \hat{b} y_3) y_1^3 - \hat{c} y_2 + \hat{d} y_3) - \tilde{a} - \tilde{b} - \tilde{c} - \tilde{d} - e_1^{\,2} - e_2 \\
u_3 &= g_3 x_4 - y_4 - e_3^{\,2} - e_3 \\
u_4 &= g_4(-x_3 - x_3^3 - \hat{e} x_4 + \hat{f} x_1) - (-y_3 - y_3^3 - \hat{e} y_4 + \hat{f} y_1) + e_3^{\,2} - \tilde{e} - \tilde{f} - e_4
\end{aligned} \tag{8.15}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{8.16}$$

which is negative definite function of e_1, e_2, e_3, e_4, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ in first quadrant.

The Lyapunov asymptotical stability theorem can not be satisfied in this case. The common origin of error dynamics and parameter update dynamics cannot be determined to be asymptotically stable. By GYC pragmatical asymptotical stability theorem (see Appendix) D is a 10 -manifold, n = 10 and the number of error state variables p = 4. When $e_1 = e_2 = e_3 = e_4 = 0$ and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ take arbitrary values , $\dot{V} = 0$, so X is of 6 dimensions, m = n – p = 10 – 4 = 6. m + 1 ≤ n are satisfied. By the GYC pragmatical asymptotical stability theorem, not only error vector e tends to zero but also all estimated parameters approach their uncertain parameters. PHPGS of chaotic systems by GYC partial region stability theory is accomplished. The equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{a} = \tilde{b} = \tilde{c} = \tilde{d} = \tilde{e} = \tilde{f} = 0$ is asymptotically stable. The numerical results are shown in Figs 8.3-8.5. The generalized synchronization is accomplished with $g_1 = -1.5$, $g_2 = 0.5$, $g_3 = -1.5$, and $g_4 = 2$ while $e_1(0) = 105.000001$, $e_2(0) = 94.999999$, $e_3(0) = 105.000001$ and $e_4(0) = 119.99999$. Four error states versus time are shown in Fig. 8.2. The estimated parameters approach the uncertain parameters of the chaotic system as shown in Figs. 8.3-5. The initial values of estimated parameters are $\hat{a}(0) = 15$, $\hat{b}(0) = 1$, $\hat{c}(0) = 0.005$, $\hat{d}(0) = -23$, $\hat{e}(0) = 0.002$ and $\hat{f}(0) = 14$.

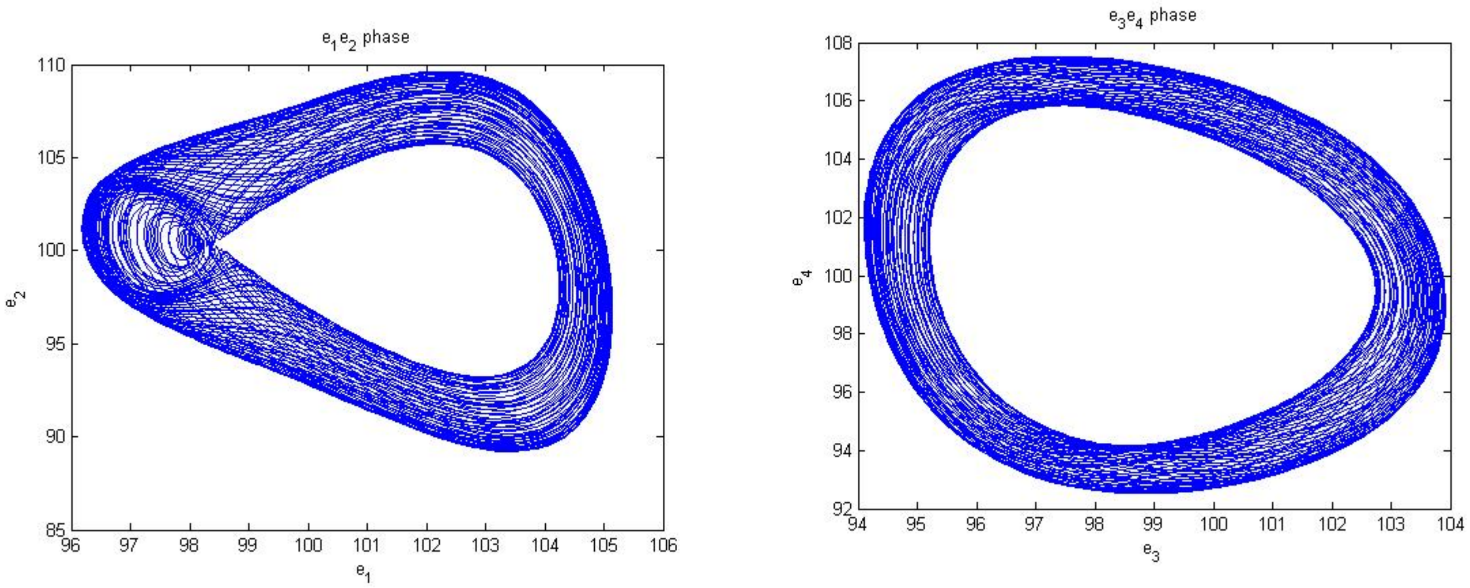

Fig. 8.1 Phase portraits of error dynamics.

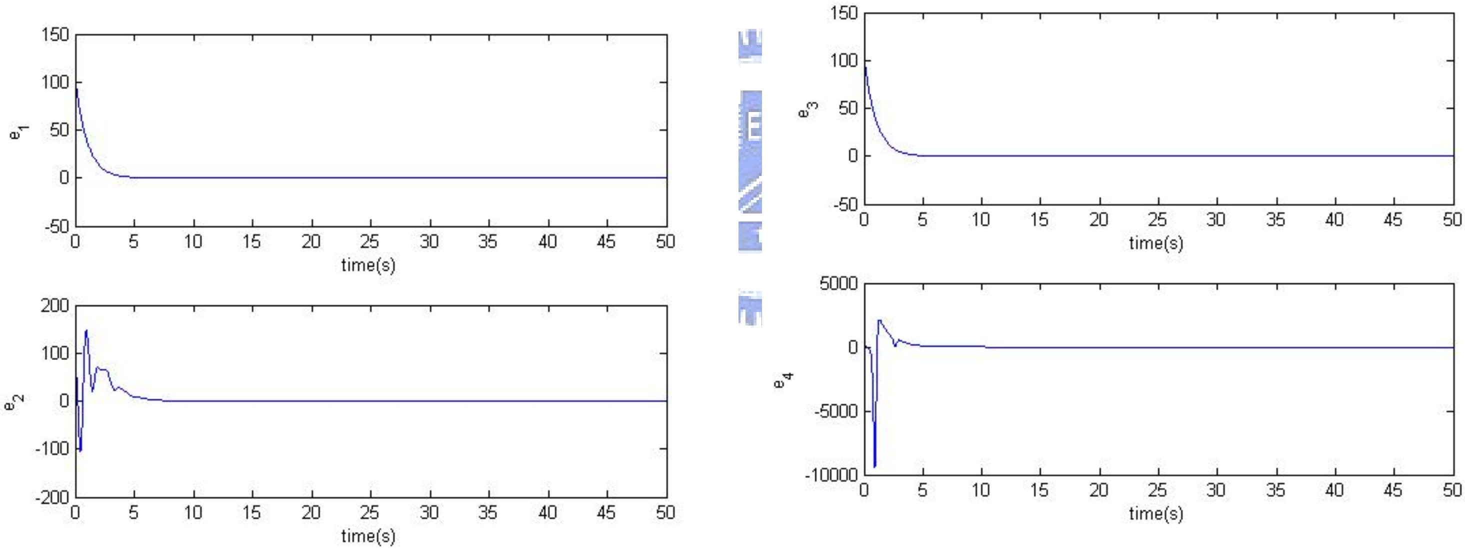

Fig. 8.2 The time histories of errors (e_1, e_2, e_3, e_4).

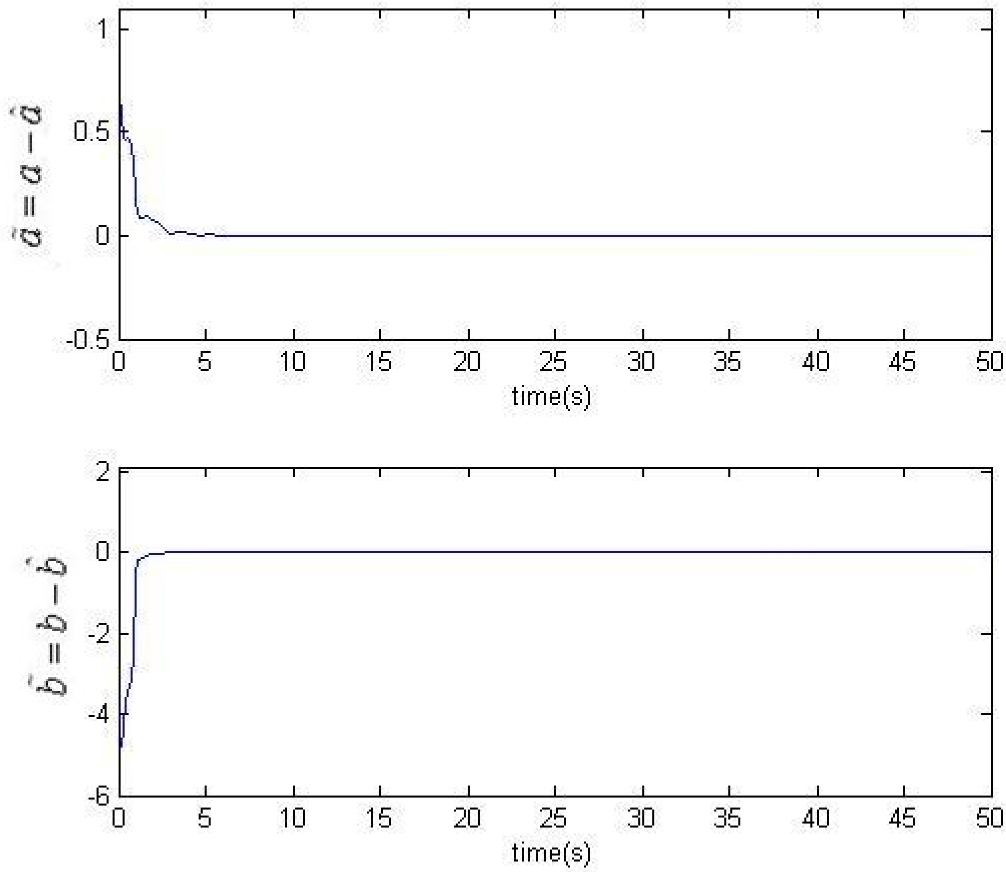

Fig. 8.3 The time histories of $\tilde{a}$ and $\tilde{b}$.

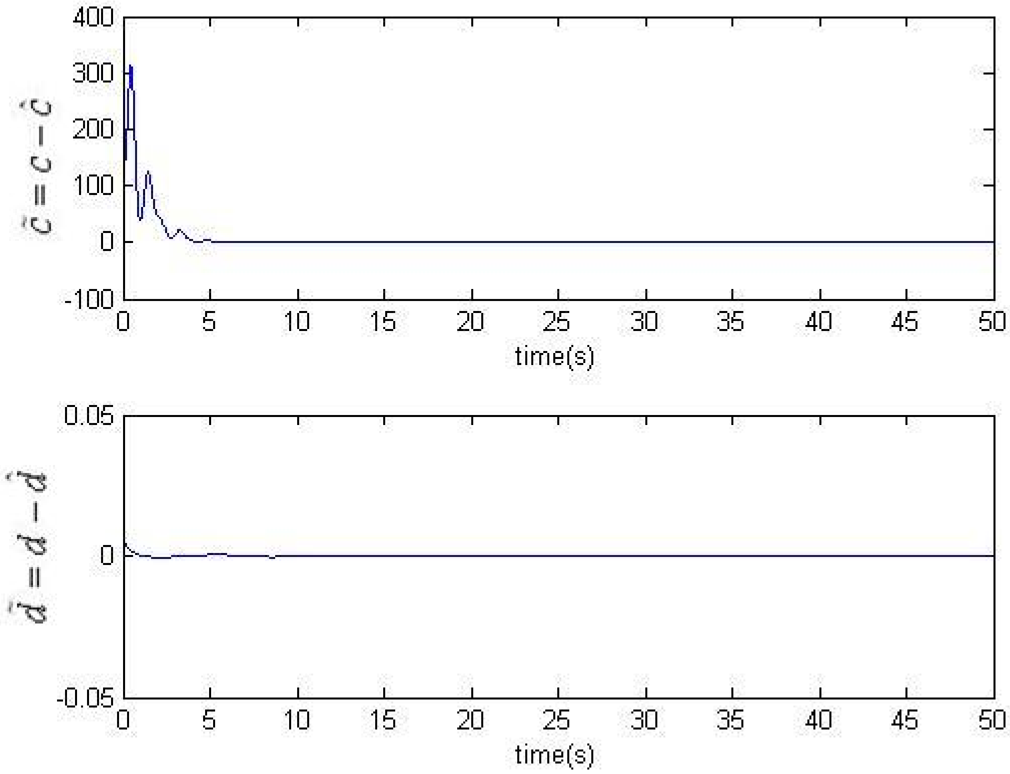

Fig. 8.4 The time histories of $\tilde{c}$ and $\tilde{d}$.

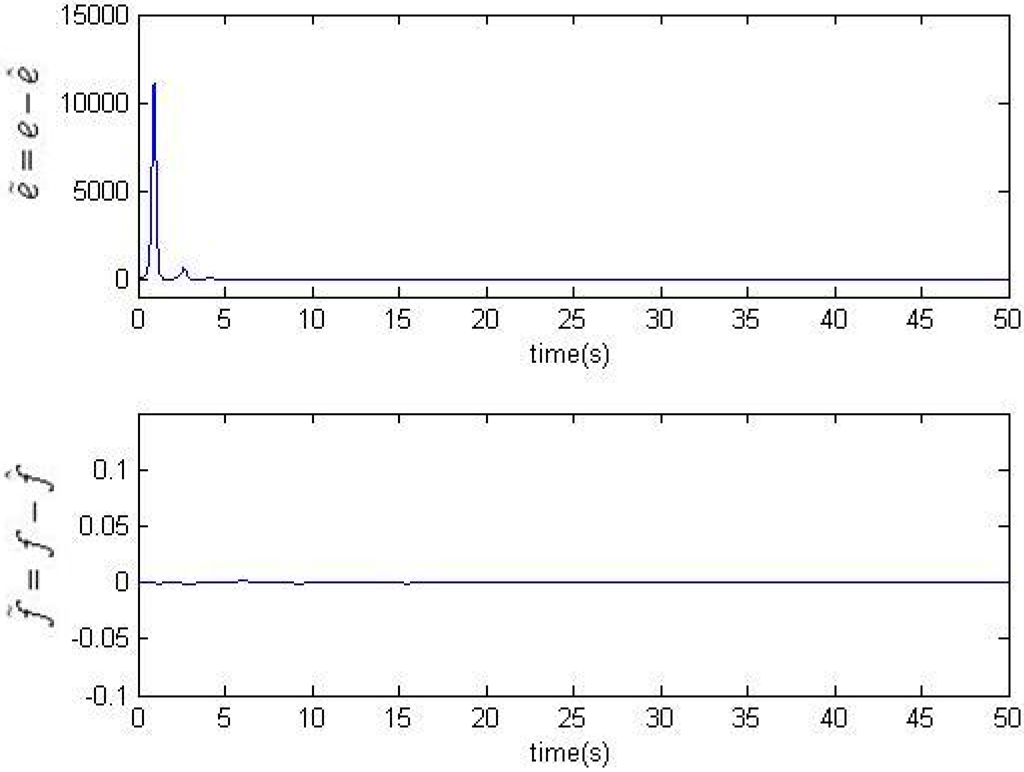

Fig. 8.5 The time histories of $\tilde{e}$ and $\tilde{f}$.

Chapter 9

Conclusions

Chaotic system features that it has complex dynamical behaviors and sensitive behavior dependence initial conditions. Because of this property, chaotic systems are thought difficult to be synchronized or controlled. In practice, some or all of the system parameters are uncertain. Additionally, these parameters change at every time. A lot of researchers have studied to solve this problem by different control theories. There are many control techniques which are presented to synchronize and control chaotic systems, such as backstepping design method [2], impulsive control method [3], invariant manifold method [4], adaptive control method [5], linear and nonlinear feedback control method [6], and active control approach [7], PC method [8], etc. In this thesis, we have studied the chaos of a new Mathieu – Duffing system by phase portraits, Poincaré maps, power spectrum and Lyapunov exponent diagram in Chapter 2.

In Chapter 3, by the pragmatical asymptotical stability theorem, the estimated parameters approach uncertain parameters can be answered strictly. In the current scheme of adaptive synchronization [12-15], the traditional Lyapunov stability theorem and Babalat lemma are used to prove that the error vector approaches zero, as time approaches infinity. But the question of that why the estimated parameters also approach uncertain parameters remains unanswered. By the pragmatical asymptotical stability theorem, the question can be answered strictly. A new Mathieu – Duffing system and a new Duffing – van der Pol system are used as simulated example. Pragmatical hybrid projective hyper chaotic generalized synchronization of chaotic systems by adaptive backstepping control is accomplished.

In Chapter 4, a new kind of symplectic synchronization and a new control Lyapunov function are proposed. A new kind of symplectic synchronization plays a “interwined” role, so we call the “master” system partner A, the “slave” system partner B. Using the new control Lyapunov function

$$V(e) = \exp(ke^T e) - 1 \tag{9.1}$$

, the error tolerance introduced by using this new control Lyapunov function can be reduced marvelously to 10^{-17} of that using traditional control Lyapnov function

$$V(e) = e^T e \tag{9.2}$$

In Chapter 5, by the GYC partial region stability theory, chaos control is achieved. In GYC partial region stability theory, Lyapunov function is simpler a traditional Lyapunov function of error states, which is a linear homogenous function of error states. The simulation error can be reduced by using the GYC partial region stability and simple controllers. A new Mathieu – Duffing system in the first quadrant is used as simulated examples which effectively confirm the scheme.

In Chapter 6, by using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of error states and the controllers are simpler than traditional controllers and so reduce the simulation error. Two new Mathieu – Duffing systems are in chaos generalized synchronization successfully.

In Chapter 7, the chaotic behaviors of a new Mathieu – Duffing systems Bessel function parameters are first proposed. The chaotic behaviors of a new Mathieu – Duffing systems with Bessel function parameters is studied numerically by phase portraits, Poincaré maps, bifurcation diagram and Lyapunov exponent diagram.

In Chapter 8, by the GYC pragmatical asymptotical stability theorem and GYC partial region stability theory, the error vector tends to zero and the estimated parameters approach uncertain values is guaranteed and the controllers of are simpler than traditional controllers and so reduce the simulation error. Pragmatical hybrid projective generalized synchronization of new Mathieu-Duffing systems with Bessel function parameters by adaptive control is achieved. The GYC pragmatical asymptotical stability theorem and GYC partial region stability theory are powerful to synchronize and control chaotic systems. The security of communication is greatly increased.

Appendix A

GYC Pragmatical asymptotical theorem

The stability for many problems in real dynamical systems is actual asymptotical stability, although it may not be mathematical asymptotical stability. The mathematical asymptotical stability demands that trajectories from all initial states in the neighborhood of zero solution must approach the origin as $t \to \infty$. If there are only a small part or even one of the initial states from which the trajectories or trajectory do not approach the origin as $t \to \infty$, the zero solution is not mathematically asymptotically stable. However, when the probability of occurrence of an event is zero, it means the event does not occur actually. If the probability of occurrence of the event that the trajectries from the initial states are that they do not approach zero when $t \to \infty$, is zero, the stability of zero solution is actual asymptotical stability though it is not mathematical asymptotical stability. In order to analyze the asymptotical stability of the equilibrium point of such systems, the pragmatical asymptotical stability theorem is used.

Let X and Y be two manifolds of dimensions m and n $(m < n)$, respectively, and φ be a differentiable map from X to Y; then $\varphi(X)$ is a subset of Lebesque measure 0 of Y [74] . For an autonomous system

$$\frac{dx}{dt} = f(x_1, x_2, \cdots x_n) \tag{A.1}$$

where $x = [x_1, x_2, \cdots x_n]^T$, the function $f = [f_1, f_2, \cdots f_n]^T$ is defined on $D \subset R^n$. Let $x = 0$ be an equilibrium point for the system (A.1), then

$$f(0) = 0 \tag{A.2}$$

For nonautonomous system,

$$\frac{dx}{dt} = f(x_1, x_2, \cdots x_{n+1}) \tag{A.3}$$

Where $t = x_{n+1} \subset R_+$. The equilibrium point is

$$f(0, x_{n+1}) = 0 \tag{A.4}$$

Definition : The equilibrium point for the dynamic system is pragmatically asymptotically stable provided that with initial points on *C* which is a subset of Lebesque measure 0 of *D*, the behaviors of the corresponding trajectories cannot be determined, while with initial points on $D - C$, the corresponding trajectories behave as that agree with traditional asymptotical stability [17, 18].

Theorem: Let $V = [x_1, x_2, \cdots x_n]^T : D \to R_+$ positive definite, analytic on *D*, where $x_1, x_2, \cdots x_n$ are all space coordinates such that the derivative of *V* through differential equation, $\dot{V}$, is negative semi-definite.

Let *X* be the m-manifold consisting of point set for which $\forall x \neq 0$, $\dot{V}(x) = 0$ and *D* is an n-manifold. If $m + 1 < n$, then the equilibrium point of the system is pragmatically asymptotically stable.

Proof : Since every point of *X* can be passed by a trajectory of Eq.(A.1), which is one dimensional, the collection of these trajectories, *C*, is a $(m+1)$-manifold [17, 18]. If $(m+1) < n$, then the collection *C* is a subset of Lebesque measure 0 of *D*. By the above definition, the equilibrium point of the system is pragmatically asymptotically stable. □

If an initial point is ergodicly chosen in *D*, *the probability of that the initial point falls on the collection C is zero*. *Here, equal probability is assumed for every point chosen as an initial point in the neighborhood of the equilibrium point.* Hence, the event that the initial point is chosen from collection *C does not occur actually*. Therefore, under the equal probability assumption, pragmatical asymptotical stability becomes actual asymptotical stability. When the initial point falls on $D - C$, $\dot{V}(x) < 0$, the corresponding trajectories behave as if they agree with traditional asymptotical stability because by the existence and uniqueness of the solution of initial-value problem, these trajectories never meet *C*.

For Eq.(3.39), Eq.(4.51) and Eq.(8.13), the Lyapunov function is a positive definite function of *n* variables, i.e. *p* error state variables and $n - p = m$ differences between unknown and estimated parameters, while $\dot{V} = e^T Ce$ is a negative semi-definite function of *n* variables. Since the number of error state variables is always more than one, $p > 1$, $(m+1) < n$ is always satisfied; by pragmatical asymptotical stability theorem we have

$$\lim_{t \to \infty} e = 0 \qquad \text{(A.5)}$$

and the estimated parameters approach the uncertain parameters. The pargmatical generalized synchronizations is obtained. Therefore, *the equilibrium point of the system is pragmatically asymptotically stable. Under the equal probability assumption, it is actually asymptotically stable for both error state variables and parameter variables.*

Appendix B

GYC Partial Region Stability Theory

Consider the differential equations of disturbed motion of a nonautonomous system in the normal form

$$\frac{dx_s}{dt} = X_s(t, x_1, \cdots, x_n), \qquad (s = 1, \cdots, n) \tag{B.1}$$

where the function X_s is defined on the intersection of the partial region Ω (shown in Fig. 1) and

$$\sum_s x_s^2 \le H \tag{B.2}$$

and $t > t_0$, where t_0 and H are certain positive constants. X_s which vanishes when the variables x_s are all zero, is a real valued function of t, $x_1, \cdots, x_n$. It is assumed that X_s is smooth enough to ensure the existence, uniqueness of the solution of the initial value problem. When X_s does not contain t explicitly, the system is autonomous.

Obviously, $x_s = 0 \quad (s = 1, \cdots n)$ is a solution of Eq.(B.1). We are interested to the asymptotical stability of this zero solution on partial region Ω (including the boundary) of the neighborhood of the origin which in general may consist of several subregions (Fig. B.1).

Definition 1:

For any given number $\varepsilon > 0$, if there exists a $\delta > 0$, such that on the closed given partial region Ω when

$$\sum_s x_{s0}^2 \le \delta, \quad (s = 1, \cdots, n) \tag{B.3}$$

for all $t \ge t_0$, the inequality

$$\sum_s x_s^2 < \varepsilon, \quad (s = 1, \cdots, n) \tag{B.4}$$

is satisfied for the solutions of Eq.(B.27) on Ω, then the disturbed motion $x_s = 0 \quad (s = 1, \cdots n)$ is stable on the partial region Ω.

Definition 2:

If the undisturbed motion is stable on the partial region Ω, and there exists a $\delta' > 0$, so that on the given partial region Ω when

$$\sum_s x_{s0}^2 \le \delta', \quad (s = 1, \cdots, n) \tag{B.5}$$

The equality

$$\lim_{t \to \infty} \left(\sum_s x_s^2 \right) = 0 \tag{B.6}$$

is satisfied for the solutions of Eq.(B.1) on Ω, then the undisturbed motion $x_s = 0 \quad (s = 1, \cdots n)$ is asymptotically stable on the partial region Ω.

The intersection of Ω and region defined by Eq.(B.2) is called the region of attraction.

Definition of Functions $V(t, x_1, \cdots, x_n)$:

Let us consider the functions $V(t, x_1, \cdots, x_n)$ given on the intersection Ω_1 of the partial region Ω and the region

$$\sum_s x_s^2 \le h, \quad (s = 1, \cdots, n) \tag{B.7}$$

for $t \ge t_0 > 0$, where t_0 and h are positive constants. We suppose that the functions are single-valued and have continuous partial derivatives and become zero when $x_1 = \cdots = x_n = 0$.

Definition 3:

If there exists $t_0 > 0$ and a sufficiently small $h > 0$, so that on partial region Ω_1 and $t \ge t_0$, $V \ge 0$ (or ≤ 0), then V is a positive (or negative) semidefinite, in general semidefinite, function on the Ω_1 and $t \geq t_0$.

Definition 4:

If there exists a positive (negative) definitive function $W(x_1 \ldots x_n)$ on Ω_1, so that on the partial region Ω_1 and $t \geq t_0$

$$V - W \geq 0 \; (or \; -V - W \geq 0), \tag{B.8}$$

then $V(t, x_1, \ldots, x_n)$ is a positive definite function on the partial region Ω_1 and $t \geq t_0$.

Definition 5:

If $V(t, x_1, \ldots, x_n)$ is neither definite nor semidefinite on Ω_1 and $t \geq t_0$, then $V(t, x_1, \ldots, x_n)$ is an indefinite function on partial region Ω_1 and $t \geq t_0$. That is, for any small $h > 0$ and any large $t_0 > 0$, $V(t, x_1, \ldots, x_n)$ can take either positive or negative value on the partial region Ω_1 and $t \geq t_0$.

Definition 6: Bounded function V

If there exist $t_0 > 0$, $h > 0$, so that on the partial region Ω_1, we have

$$|V(t, x_1, \ldots, x_n)| < L \tag{B.9}$$

where L is a positive constant, then V is said to be bounded on Ω_1.

Definition 7: Function with infinitesimal upper bound

If V is bounded, and for any $\lambda > 0$, there exists $\mu > 0$, so that on Ω_1 when $\sum_s x_s^2 \leq \mu$, and $t \geq t_0$, we have

$$|V(t, x_1, \ldots, x_n)| \leq \lambda \tag{B.10}$$

then V admits an infinitesimal upper bound on Ω_1.

Theorem 1 [33, 34]

If there can be found for the differential equations of the disturbed motion (Eq.(6.1)) a definite function $V(t, x_1, \ldots, x_n)$ on the partial region, and for which the derivative with respect to time based on these equations as given by the following :

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s \tag{B.11}$$

is a semidefinite function on the paritial region whose sense is opposite to that of V, or if it becomes zero identically, then the undisturbed motion is stable on the partial region.

Proof:

Let us assume for the sake of definiteness that *V* is a positive definite function. Consequently, there exists a sufficiently large number t_0 and a sufficiently small number *h* < *H*, such that on the intersection Ω_1 of partial region Ω and

$$\sum_s x_s^2 \le h, \quad (s = 1, \ldots, n) \tag{B.12}$$

and $t \ge t_0$, the following inequality is satisfied

$$V(t, x_1, \ldots, x_n) \ge W(x_1, \ldots, x_n) \tag{B.13}$$

where *W* is a certain positive definite function which does not depend on *t*. Besides that, Eq. (B.7) may assume only negative or zero value in this region.

Let ε be an arbitrarily small positive number. We shall suppose that in any case $\varepsilon < h$. Let us consider the aggregation of all possible values of the quantities $x_1, \ldots, x_n$, which are on the intersection ω_2 of Ω_1 and

$$\sum_s x_s^2 = \varepsilon, \tag{B.14}$$

and let us designate by $l > 0$ the precise lower limit of the function W under this condition. by virtue of Eq. (B.5), we shall have

$$V(t, x_1, \ldots, x_n) \ge l \quad \text{for } (x_1, \ldots, x_n) \text{ on } \omega_2. \tag{B.15}$$

We shall now consider the quantities x_s as functions of time which satisfy the differential equations of disturbed motion. We shall assume that the initial values x_{s0} of these functions for $t = t_0$ lie on the intersection Ω_2 of Ω_1 and the region

$$\sum_s x_s^2 \le \delta, \tag{B.16}$$

where δ is so small that

$$V(t_0, x_{10}, \ldots, x_{n0}) < l \tag{B.17}$$

By virtue of the fact that $V(t_0, 0, \ldots, 0) = 0$, such a selection of the number δ is obviously possible. We shall suppose that in any case the number δ is smaller than ε.Then the inequality

$$\sum_s x_s^2 < \varepsilon, \tag{B.18}$$

being satisfied at the initial instant will be satisfied, in the very least, for a sufficiently small $t - t_0$, since the functions $x_s(t)$ very continuously with time. We shall show that these inequalities will be satisfied for all values $t > t_0$. Indeed, if these inequalities were not satisfied at some time, there would have to exist such an instant *t*=*T* for which this inequality would become an equality. In other words, we would have

$$\sum_s x_s^2(T) = \varepsilon, \tag{B.19}$$

and consequently, on the basis of Eq. (B.9)

$$V(T, x_1(T), \ldots, x_n(T)) \ge l \tag{B.20}$$

On the other hand, since $\varepsilon < h$, the inequality (Eq.(B.4)) is satisfied in the entire interval of time [t_0, T], and consequently, in this entire time interval $\frac{dV}{dt} \le 0$. This yields

$$V(T, x_1(T), \ldots, x_n(T)) \le V(t_0, x_{10}, \ldots, x_{n0}), \tag{B.21}$$

which contradicts Eq. (B.12) on the basis of Eq. (B.11). Thus, the inequality (Eq.(B.1)) must be satisfied for all values of $t > t_0$, hence follows that the motion is stable.

Finally, we must point out that from the view-point of mathenatics, the stability on partial region in general does not be related logically to the stability on whole region. If an undisturbed solution is stable on a partial region, it may be either stable or unstable on the whole region and vice versa. From the viewpoint of dynamics, we wre not interesting to the solution starting from Ω_2 and going out of Ω.

Theorem 2 [33, 34]

If in satisfying the conditions of theorem 1, the derivative $\frac{dV}{dt}$ is a definite function on the partial region with opposite sign to that of V and the function V itself permits an infinitesimal upper limit, then the undisturbed motion is asymptotically stable on the partial region.

Proof:

Let us suppose that V is a positive definite function on the partial region and that consequently, $\frac{dV}{dt}$ is negative definite. Thus on the intersection Ω_1 of Ω and the region defined by Eq. (B.4) and $t \geq t_0$ there will be satisfied not only the inequality (Eq.(B.5)), but the following inequality as will:

$$\frac{dV}{dt} \leq -W_1(x_1,\dots x_n), \tag{B.22}$$

where W_1 is a positive definite function on the partial region independent of t.

Let us consider the quantities x_s as functions of time which satisfy the differential equations of disturbed motion assuming that the initial values $x_{s0} = x_s(t_0)$ of these quantities satisfy the inequalities (Eq. (B.10)). Since the undisturbed motion is stable in any case, the magnitude δ may be selected so small that for all values of $t \geq t_0$ the quantities x_s remain within Ω_1. Then, on the basis of Eq. (B.13) the derivative of function $V(t, x_1(t),\dots, x_n(t))$ will be negative at all times and, consequently, this function will approach a certain limit, as *t* increases without limit, remaining larger than this limit at all times. We shall show that this limit is equal to some positive quantity different from zero. Then for all values of $t \geq t_0$ the following inequality will be satisfied:

$$V(t, x_1(t), \ldots, x_n(t)) > \alpha \tag{B.23}$$

where $\alpha > 0$.

Since V permits an infinitesimal upper limit, it follows from this inequality that

$$\sum_s x_s^2(t) \geq \lambda, \quad (s = 1, \ldots, n), \tag{B.24}$$

where λ is a certain sufficiently small positive number. Indeed, if such a number λ did not exist, that is , if the quantity $\sum_s x_s(t)$ were smaller than any preassigned number no matter how small, then the magnitude $V(t, x_1(t), \ldots, x_n(t))$, as follows from the definition of an infinitesimal upper limit, would also be arbitrarily small, which contradicts (B.14).

If for all values of $t \geq t_0$ the inequality (Eq. (B.15)) is satisfied, then Eq. (B.13) shows that the following inequality will be satisfied at all times:

$$\frac{dV}{dt} \leq -l_1, \tag{B.25}$$

where l_1 is positive number different from zero which constitutes the precise lower limit of the function $W_1(t, x_1(t), \ldots, x_n(t))$ under condition (Eq. (B.15)). Consequently, for all values of $t \geq t_0$ we shall have:

$$V(t, x_1(t), \ldots, x_n(t)) = V(t_0, x_{10}, \ldots, x_{n0}) + \int_{t_0}^{t} \frac{dV}{dt} dt \leq V(t_0, x_{10}, \ldots, x_{n0}) - l_1(t - t_0),$$

which is, obviously, in contradiction with Eq.(B.14). The contradiction thus obtained shows that the function $V(t, x_1(t), \ldots, x_n(t))$ approached zero as t increase without limit. Consequently, the same will be true for the function $W(x_1(t), \ldots, x_n(t))$ as well, from which it follows directly that

$$\lim_{t\to\infty} x_s(t) = 0, \quad (s = 1,\ldots,n), \tag{B.26}$$

which proves the theorem.

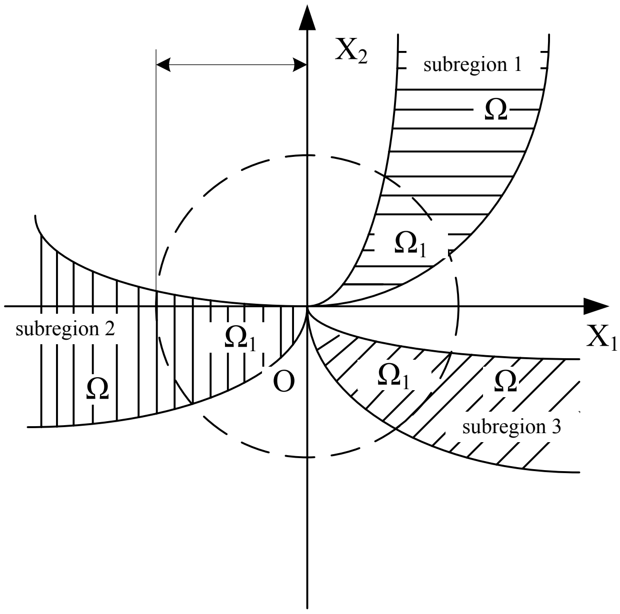

Fig. B.1 Partial regions Ω and Ω_1.

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