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Exact Solution In a Scale Invariant Model With Vanishing Cosmological Constant

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We analyze a special scale invariant effective theory with vanishing cosmological constant. We show that, in this theory, the spatial-independence of the inflaton is implied by the equation of motion incorporated with the Robertson-Walker metric. Furthermore, we find that this theory can be solved exactly for $k = 0$ Robertson-Walker spaces. We also find *almost* exact solution in $k \neq 0$ spaces.

. Scale invariance[1] has been implemented in string theory to obtain the low energy effective action for massless string mode. It has also been proposed[2] to govern the cosmological evolution of our universe. Much progress[3] has been achieved in the past. It was recently shown[4] that certain asymptotic boundary condition (ABC) imposed on the scalar field ϕ, namely $\phi(r \to \infty) = v$, can be derived as a slow roller solution to the equation of motion. The prescribed spontaneous symmetry breaking effect can thus be derived as a background solution to the theory.

In the meantime, the scale invariant theory bears another completely different solution in the vanishing cosmological constant limit. This different spectrum can be an inflation solution only in an exceptional limit. We will present the details and discuss its implications. We will also present a sketch[5] of the proof that the associated scalar field ϕ must be a function oft only from the symmetry of the Robertson-Walker (RW) metric.[6]

The RW metric can be shown to describe all Riemannian four spaces with time-foilation of homogeneous and isotropic three spaces. Therefore, the observed isotropic and homogeneous universe in the cosmological scale must be a RW type space time. One notes that the Robertson-Walker metric can be read-off from the following definition of invariant scalar measure:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega\right) \tag{1}$$

Here $d\Omega = d\theta^2 + \sin^2\theta\, d\varphi^2$ denotes the solid angle. and $k = 0. \pm 1$ stands for a flat. closed or open universe respectively.

We will stud! the following scale invariant mode!:'

$$S = \int d^4x\sqrt{g}\,[\frac{-1}{2}\epsilon\phi^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{8}\phi^4]\ . \tag{2}$$

Here ϕ is a real scalar field, while λ and ϵ are dimensionless coupling constants. Indeed. this theory can be shown to be invariant under the scale transformation $g'_{\mu\nu} = s^2 g_{\mu\nu}$ and $\phi' = s^{-1}\phi$, with s denoting a constant scale parameter. Moreover. this theory is also known to provide a natural explanation for a universe with a dimensionful gravitational "constant" and cosmological "constant".

The solution to this model is very difficult to solve in fact. It can be solved only under a few seemly reasonable assumptions on the initial conditions on ϕ and $a(t)$. Recently it was shown[4] that the slow roller ϕ can be used to imply a prescribed spontaneous symmetry breaking mechanism. This dynamical symmetry breaking picture is equivalent to an imposed ABC' that has been introduced by hand. It depends. however. on the expanding rate of our universe to determine the symmetry breaking scale. This belongs to the class of solutions with $\lambda \neq 0$. The solution is. however. very different for the case with A = 0. In fact, we will be able to exactly solve the equations of motion in the case $\lambda = 0$ when $k = 0$, namely. on the flat three dimensional space-like slice. In the case $k \neq 0$, our solution is only an approximated solution that requires $\dot{a}^2 >> 1$. In fact. the solution we obtained satisfies the above assumption fairly well.

Moreover. the $\lambda = 0$ case prescribes a universe with vanishing cosmological constant. It is beyond the scope of this paper to address on the problem of cosmological constant. We will hereafter assume that the vanishing of the cosmological constant was settled, maybe due to the quantum cosmology approach, well before our model become active as an effective theory that is scale invariant manifestly. It is also beyond our purpose to discuss the physics beyond the Planck scale.

We can derive the classical equations of motion from the least action principle. The results are

$$D_\mu\partial^\mu\phi = \epsilon R\phi\ , \tag{3}$$

$$\epsilon\phi^2(\frac{1}{2}Rg^{\mu\nu} - R^{\mu\nu}) = \epsilon(D^\mu\partial^\nu - g^{\mu\nu}D_\alpha\partial^\alpha)\phi^2 + \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi \tag{4}$$

Our spatially isotropic and homogeneous universe implies that $\phi(x)$ must be a function oft and r only, nemely, $\phi(x) = \phi(t, r)$. Otherwise, its anisotropic contributions will affect the observational data. In fact, we are going to argue that the r dependence of ϕ can be shown to be absent. This can be proved by observing that the Euler-Lagrange equations from varying g_{0i} can be reduced to

$$\epsilon\partial_i(\partial_t\psi - \dot{\alpha}\psi) + \dot{\phi}\partial_i\phi = 0\ , \tag{5}$$

after inserting the RW metric. Moreover the $_{ii}$ equation of (4) can be regrouped as

$$(\epsilon D_i\partial_j\psi + \partial_i\phi\partial_j\phi)_3 = h_{ij}M(a, \phi, h_{ij}) \tag{6}$$

Here we have collected all h_{ij} proportional terms on the right hand side (RHS) and put the rests on its left hand side (LHS). Here we have also written $g_{ij}=a^2 h_{ij}$ and $\psi=\phi^2$. Furthermore, the subscript 3 means that all covariant derivatives on the LHS are to be evaluated on the three dimensional spatial Riemannian sub-manifold (M^3, h_{ij}).

If *we* assume that $\phi(x)=\phi(t,r)$, (5) and (6) can be reduced to

$$\epsilon(\partial_r\partial_t\psi - \dot{\alpha}\partial_r\psi) + \dot{\phi}\partial_r\phi = 0 \quad , \tag{7}$$

$$4\epsilon\psi(\partial_r^2\psi - \frac{1}{r(1-kr^2)}\partial_r\psi) + (\partial_r\psi)^2 = 0 \, , \tag{8}$$

after some algebra regrouping (6). Hereafter we will write $\psi\equiv e^{\theta(t,r)}$ and $a=e^{\alpha(t)}$ for convenience. Consequently, (7) and (8) can be written as a system of first order partial differential equations (PDEs) of $\partial_r\theta(t,r)$. These PDEs can thus be solved by observing that (7) and (8) are in fact two total partial derivatives, namely, $\partial_t(e^{-\alpha(t)+\mu\theta(t,r)}\partial_r\theta(t,r))=0$ and $\partial_r(e^{-g(r)+\mu\theta(t,r)}\partial_r\theta(t,r)) = 0$. Here $\mu = 1$ +&and $\partial_r g(r) = \frac{1}{r(1-kr^2)}$.—Indeed, these equations can be integrated accordingly. The result turns out to be very simple:

$$\partial_r\theta(t, r) = f_3 e^{\alpha(t)+g(r)-\mu\theta(t,r)} \quad . \tag{9}$$

Here f_3 is an integration constant. It can actually be absorbed by redefining $\alpha(t)$. Note that (9) can be integrated further. The final result reads:

$$\phi(t,r) = e^{\alpha(t)/2\mu}\left\{1+f_4[\frac{(1-|k|)}{2}r^2 + f^k|1-kr^2|^{1/2}]\right\}^{1/2\mu} \tag{10}$$

Here f_4 is the appropriate integration constant defined after $\alpha(t)$ is redefined properly. Also. f^k as a sign function defined as the sign of the coefficient of r^2 term of $|1-kr^2|^{1/2}$ after dropping the absolute sign. Note that ϕ is divergent as $r\to\infty$. Indeed. $\phi_{k=0}(r\to\infty) \propto r^{1/\mu}$ and $\phi_{k\neq 0}(r\to\infty)\propto r^{1/2\mu}$. Therefore. we prove that $\phi(x)=\phi(t)$.

One can therefore obtain the following equations. after inserting the Robertson-Walker metric into (3) and (4),

$$\frac{a''}{a} + \frac{(a')^2+k}{a^2} = \frac{1}{24\epsilon}[-(\frac{\psi'}{\psi})^2 + 6\frac{a'\psi'}{a\psi}] \quad . \tag{11}$$

$$\frac{(a')^2+k}{a^2} + \frac{a'\psi'}{a\psi} = \frac{1}{24\epsilon}[(\frac{\psi'}{\psi})^2] \quad . \tag{12}$$

$$2\frac{a''}{a} + \frac{(a')^2+k}{a^2} + 2\frac{a'\psi'}{a\psi} + \frac{\psi''}{\psi} = \frac{1}{24\epsilon}[-3(\frac{\psi'}{\psi})^2] \tag{13}$$

Here prime denotes the differentiation with respect to t. Furthermore. (12)+(13)-2x (11) gives a simplified equation

$$\psi'' + 3\frac{a}{a}\psi' = 0 \ . \tag{14}$$

This is in fact the conserved current equation. $D_\mu J^\mu = 0$. Here $J_\mu = \phi\partial_\mu\phi$ is the Noether current for the scale symmetry. By introducing the integration factor $e^{3\alpha}$, (14) can be written as $(\psi' e^{3\alpha})' = 0$. Hence Eq. (15) can be integrated to give

$$\psi' e^{3\alpha} = \text{constant} \tag{15}$$

In terms of $\psi \equiv e^{\theta(t)}$ and $a = e^{\alpha(t)}$, (12) and (13) become

$$\alpha'^2 + \alpha'\theta' = \frac{1}{24\epsilon}\theta'^2 \tag{16}$$

$$2\alpha'' + 3\alpha'^2 - \alpha'\theta' = \frac{1}{8\epsilon}\theta'^2 \tag{17}$$

Also (16) and (17) implies

$$a'' + 2\alpha'^2 = -\frac{1}{24\epsilon}\theta'^2 \ . \tag{18}$$

Note that we are considering a set of 3 differential equations. i.e. equations (15, 16. 17). for 2 unknown variables. Even it is in general unlikely to admit a consistent solution without a magic, we will show that (15) actually is derivable from (16) and (18). Hence it is adequate to analyze the system by solving (16) and (18). Note also that we have assumed that $k = 0$ (or $a'^2 >> 1$ for $k \neq 0$) in deriving (16) and (18).

In fact, (16) is a simple algebraic equation of θ' and α'. It can be solved to give the following simple equation

$$\alpha' = -l\theta' \ . \tag{19}$$

or equivalently $\alpha + l\theta =$ constant. Here $l = l_\pm = \frac{\sqrt{6\epsilon} \pm \sqrt{1+6\epsilon}}{2\sqrt{6\epsilon}}$. Note also that $l^2 = l + \frac{1}{24\epsilon}$. We will also write $m = 2 + \frac{1}{24\epsilon l^2}$ such that $m = 3 - \frac{1}{l}$. To be more specifically, $m_\pm = 3 - \frac{1}{l_\pm}$. In fact, (18) can be written as

$$(\alpha' e^{m\alpha})' = \frac{1}{m}(e^{m\alpha})'' = 0 \tag{20}$$

This implies $e^{m\alpha} = \alpha_0 + \alpha_1 t$. Therefore.

$$\alpha = \ln(\alpha_0 + \alpha_1 t)^{1/m} \ , \tag{21}$$

$$e = \ln(\alpha_0 + \alpha_1 t)^{-1/m} + \text{constant} \tag{22}$$

Hence

$$\psi = \psi_0 (1 + \alpha_1 \, t)^{-1/lm} \quad , \tag{23}$$

$$a = a_0 (1 + \alpha_1 t)^{1/m} \tag{24}$$

Here we have used the parameters $\psi_0 \equiv \psi(t = 0)$, a, $\equiv a(t = 0)$ in (23) and (24). Moreover Eq. (15) can be shown to be redundant since

$$\psi' e^{3\alpha} = \psi'_0 a_0^3 = -\frac{\alpha_1}{lm} \psi_0 a_0^3 \quad , \tag{25}$$

with ψ' and a given by (23) and (24) respectively. Here $\psi'_0 \equiv \psi'(t = 0) = -\frac{\alpha_1}{lm} \psi_0$. Consequently, $\alpha_1 = -lm\frac{\psi'_0}{\psi_0}$. Therefore (23) and (24) can be written as

$$\psi = \psi_0 (1 - ml \frac{\psi'_{00}}{\psi_0} \, t)^{-1/lm} \quad , \tag{26}$$

$$a = a_0 (1 - ml \frac{\psi'_0}{\psi_0} \, t)^{1/m} \tag{27}$$

Note that $m \cong 3$ and 1, $\cong \pm\sqrt{\frac{1}{24\epsilon}}$ for $\epsilon << 1$. Therefore, the inflationary growth of a(t) will require (i) $l\psi'_0 < 0$ (ii) $m\frac{|l\psi'_0|}{\psi_0} >> 1$. There are two possibilities: (i) $\psi' < 0$ if $\psi'_0 < 0$, hence ϕ will be decreasing monotonically. (ii) $\psi' > 0$ if $\psi'_0 > 0$, hence ϕ will be increasing monotonically. Enough inflation can thus be achieved by tuning ϵ, ψ_0 and ψ'_0 properly. The condition $m\frac{|l\psi'_0|}{\psi_0} << 1$ is, however, quite unusual even ψ is in fact almost stationary due to the $\frac{-1}{lm}$ power. In fact, ψ is almost static shortly after the inflation turns on. Since (26) and (27) are exact solutions, they will govern the whole domain wherever the action (2) remains effective. It is only when $l\psi'_0 > 0$ (such that $\alpha_1 < 0$), the solution can no longer be valid when $t > \frac{1}{|\alpha_1|}$.

In summary, we analyze a special scale invariant effective theory with vanishing cosmological constant. We show that, in this theory, the spatial-independence of the inflaton is implied by the equation of motion incorporated with the Robertson-Walker metric. Furthermore, we find that this theory can be solved exactly for $k = 0$ Robertson-Walker spaces. We also find *almost* exact solution in $k \neq 0$ spaces. One remarks here that we consider the action (2) as an effective theory that remains effective only during some era (or energy scale) in the evolution of our universe. In fact, we haven't found, and we don't expect there can exist, an all mighty theory that can describe all physics in one.

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