Nonlinear Analysis, Theory, Methods & Applications, Vol. 16, No. 3, pp. 283–297, 1991.
Printed in Great Britain.
0362–546X/91 \$3.00 + .00

ON THE NUMBER OF POSITIVE SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WHEN A PARAMETER IS LARGE

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(*Received* 21 *August* 1989; *received for publication* 30 *July* 1990)

Key words and phrases: Positive solution, semilinear eigenvalue problems, uniqueness theorem.

1. INTRODUCTION

THE MULTIPLICITY problems of positive solutions of a class of semilinear eigenvalue problems are studied

$$\begin{aligned} -\Delta u(x) &= \lambda f(u(x)) && \text{for } x \in \Omega \\ u(x) &= 0 && \text{for } x \in \partial\Omega \end{aligned} \tag{1.1}$$

where $f(u) > 0$ for $u > 0$ and $\lambda > 0$, and Ω is a bounded smooth domain in $\mathbb{R}^n$.

Equation (1.1) arises in nonlinear heat generation, in models of combustion, etc. We refer to the survey paper by Lions [12].

In [10], the author proved that the positive solution of (1.1) is unique for λ is large when f is bounded and satisfies a "concavity" condition. More precisely, (1.1) has an unique positive solution for large λ if f satisfies the following assumptions:

(i) $f \in C^1([0, \infty))$,
(ii) $f(u) \geq m > 0$ for each $u \geq 0$ and some $m > 0$,
(iii) $\lim_{u \to +\infty} f(u)/u = 0$,
(iv) $\lim_{u \to +\infty} \inf f(u) > \lim_{u \to +\infty} \sup f'(u)u$.

We proved that solutions u_λ of (1.1) satisfies $u_\lambda \geq \lambda C v_0$ if λ is large, where v_0 is the solution of

$$\begin{aligned} -\Delta v(x) &= 1 && \text{for } x \in \Omega \\ v(x) &= 0 && \text{for } x \in \partial\Omega \end{aligned} \tag{1.2}$$

and some constant $C > 0$. Then, condition (iv) implies that (1.1) has an unique positive solution when λ is large. Condition (iv) permits f has a logarithmic growth when u tends to infinite. In the case of an ODE, a similar result has been obtained by Shivaji [18].

In this paper, we shall study the multiplicity problem of (1.1) when f is sublinear. More precisely, $f(u) \sim u^\beta$, $0 < \beta < 1$, when u is large. By constructing some appropriate super- and subsolutions and applying Serrin's sweeping principle, we can prove that for "large solution" u_λ of (1.1), $u_\lambda \geq \lambda v_0$ for large λ. And then improve this estimate to $u_\lambda \sim \lambda^{1/(1-\beta)} v_\beta$ by a "singular perturbation method" used in [10], where v_β is the unique positive solution of

$$\begin{aligned} -\Delta u(x) &= u^\beta(x) && \text{for } x \in \Omega \\ u(x) &= 0 && \text{for } x \in \partial\Omega. \end{aligned} \tag{1.3}$$

The uniqueness of large solution can be obtained if f satisfies a concavity condition

$$\limsup_{u\to+\infty} \frac{f'(u)u}{f(u)} < 1.$$

If f has power growth at $u = 0$ and $+\infty$, then problem (1.1) can be classified into four types, (β, β), (α, β), (β, α) and (α, α), $0 < \beta < 1$ and $\alpha > 1$, where type (α, β) is

$$f(u) \sim u^{\alpha} \qquad \text{when } u \sim 0$$

and

$$f(u) \sim u^{\beta} \qquad \text{when } u \sim +\infty$$

etc. This classification makes it easier to identify large or small solution when it exists.

In this paper the existence and uniqueness problems for large solutions of types (β, β) and (α, β) are studied. Problems of type (β, α) and (α, α) are more difficult and complicated, depending on geometrical and/or topological properties of Ω. For reference, see e.g. Brezis and Nirenberg [5] and Lions [12].

In Section 2, we give a special case of Serrin's sweeping principle and recall some results in [10] which are useful in this paper.

In Section 3, we prove that $u_\lambda \sim \lambda^{1/(1-\beta)} v_\beta$ and an uniqueness theorem for large λ, in the case of type (β, β).

In Section 4, we prove that $u_\lambda \sim \lambda^{1/(1-\beta)} v_\beta$ for large solution u_λ and an uniqueness theorem for large solution in the case of type (α, β).

2. PRELIMINARIES

In this section we shall recall some results obtained in [10], which are useful in getting *a priori* estimates of u_λ when λ is large. We also give a theorem (with a proof) which is a special case of Serrin's sweeping principle. The principle can tell us where solutions do not exist, therefore gives us some *a priori* estimates of solutions. The principle is useful in various problems and is very useful in the case when f is sublinear.

In this paper it is always assumed that Ω is a bounded smooth domain in $\mathbb{R}^n$, for example, $\Omega \in C^3$. We denote by $t(x)$ the distance from $x \in \Omega$ to the boundary $\partial\Omega$, and by $s(x)$ the point of $\partial\Omega$ which is closest to x (which is uniquely defined if x is close enough to $\partial\Omega$). We choose sufficiently small $\delta_0 > 0$ such that the boundary strip Ω_{δ_0} of width δ_0, i.e.

$$\Omega_{\delta_0} = \{x \in \Omega \mid 0 < t(x) < \delta_0\}$$

is covered (and only covered) by the straight lines in the inner normal direction $-n_{s(x)}$ and emanating from $s(x)$.

Given $\rho \in L^p(\Omega)$, $p > 1$, let $u \in W^{2,p}(\Omega)$ be the solution of Poisson equation with Dirichlet boundary condition, i.e.

$$\begin{aligned} -\Delta u(x) &= \rho(x) && \text{for } x \in \Omega \\ u(x) &= 0 && \text{for } x \in \partial\Omega. \end{aligned} \tag{2.1}$$

We prove the following proposition by using L^p-estimates and applying maximum and strong maximum principle.

PROPOSITION 2.1. Assume that $\rho \geq 0$ and $\rho \not\equiv 0$ on Ω.

(i) Suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is the solution of (2.1). Then there exist constants $l \geq k > 0$, such that

$$kt(x) \leq u(x) \leq lt(x) \qquad \text{on } \bar{\Omega}. \tag{2.2}$$

(ii) Suppose that $\rho \in L^\infty(\Omega)$ and $\operatorname{supp}(\rho) \subseteq \Omega_\delta$, $0 < \delta < \delta_0$, then for $p > n$, solution u of (2.1) satisfies

$$u(x) \leq C_p \delta^{1/p} \|\rho\|_\infty t(x) \qquad \text{on } \bar{\Omega} \tag{2.3}$$

where constants C_p depends only on p, n, Ω, but not ρ.

(iii) For $0 < \delta < \delta_0$, define the function ρ_δ by

$$\rho_\delta(x) = \begin{cases} 1 & \text{for } x \in \Omega_\delta \\ 0 & \text{for } x \in \Omega - \Omega_\delta. \end{cases} \tag{2.4}$$

Then the solution u_δ of (2.1) with $\rho(x) = \rho_\delta(x)$ satisfies

$$u_\delta(x) \leq C_p \delta^{1/p} v_0(x) \qquad \text{on } \bar{\Omega}. \tag{2.5}$$

Remark. u_δ is given by

$$u_\delta(x) = \int_{\Omega_\delta} G(x, y)\, dy \qquad \text{on } \bar{\Omega} \tag{2.6}$$

where $G(x, y)$ is the Green's function of $-\Delta$ with zero Dirichlet boundary condition.

Proof. (i) Since $\rho \geq 0$ and $\rho \not\equiv 0$. By maximum principle, $u(x) > 0$ in Ω, and by strong maximum principle

$$\frac{\partial u}{\partial n_s}(s) < 0 \qquad \text{on } \partial\Omega.$$

Since $\partial\Omega$ is compact, there exists δ', $0 < \delta' < \delta_0$, and $l' \geq k' > 0$ such that

$$-l' \leq \frac{\partial u}{\partial n_{s(x)}}(x) \leq -k' < 0 \tag{2.7}$$

for any $x \in \Omega_{\delta'}$. (2.2) follows by

$$u(x) = -t(x) \int_0^1 \frac{\partial u}{\partial n_{s(x)}}(x + \tau t(x) n_{s(x)})\, d\tau \tag{2.8}$$

for any $x \in \Omega_{\delta'}$, and by choosing $l \geq k > 0$ appropriately for $x \in \Omega - \Omega_{\delta'}$. It is clear that k and l depend on u and then on ρ.

(ii) In the following, constants C_i, $i = 1, 2, 3, \ldots$, may vary but depend only on p, n and Ω.

By L^p-estimate, there exists a constant C_1 such that

$$\begin{aligned} \|u\|_{2,p} &\leq C_1 \|\rho\|_p \\ &\leq C_2 \|\rho\|_\infty \operatorname{vol}(\Omega_\delta)^{1/p} \\ &\leq C_3 \|\rho\|_\infty \operatorname{area}(\partial\Omega)^{1/p}\, \delta^{1/p} \\ &= C_4 \|\rho\|_\infty \delta^{1/p}. \end{aligned}$$

For $p > n$, by Sobolev imbedding theorem, $u \in C^{1+\alpha}(\bar{\Omega})$, $\alpha = 1 - n/p$. Therefore, for any $x \in \bar{\Omega}$

$$|u(x)| + \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i}(x) \right| \le C_5 \|u\|_{2,p} \le C_6 \|\rho\|_\infty \delta^{1/p}.$$

Since (2.8) is also valid for u, $u(x) \le C_p \delta^{1/p} \|\rho\|_\infty t(x)$ on Ω, with C_p depends on p, n and Ω only.

(iii) This follows by (i) and (ii). The proof is complete.

We need the following comparison lemma in order to prove Serrin's sweeping principle.

LEMMA 2.2. Suppose that $f \in C^1$ and u and $v \in C^2(\Omega)$ satisfying

$$\Delta u + f(u) \ge \Delta v + f(v) \qquad \text{in } \Omega$$

and

$$v \ge u \qquad \text{in } \Omega.$$

Then, either $u \equiv v$ or $v > u$ in Ω.

Proof. Let $w(x) = v(x) - u(x)$. Then

$$w(x) \ge 0 \qquad \text{in } \Omega$$

and

$$\Delta w(x) + c(x) w(x) \le 0 \qquad \text{in } \Omega$$

where

$$c(x) \equiv f'(\tilde{w}(x)) \qquad \text{with } \tilde{w}(x) \in [u(x), v(x)].$$

Let Ω' be any subdomain of Ω with $\bar{\Omega}' \subset \Omega$ and $m' = \max\{0, l'\}$, where $l' = \max\{f'(w) \mid u(x) \le w \le v(x), x \in \bar{\Omega}'\}$. Then, for $x \in \Omega'$, $\Delta w(x) + (c(x) - m')w(x) \le -m'w(x) \le 0$. By maximum principle, either $w \equiv 0$ in Ω' or $w(x) > 0$ for all $x \in \Omega'$. Since $\Omega' \subset \Omega$ is arbitrary, either $w \equiv 0$ in Ω or $w(x) > 0$ for $x \in \Omega$.

The proof is complete.

In the remainder of this paper, it is always assumed that f satisfies the following conditions:

$$\text{(H-0)} \qquad \begin{cases} \text{(i) } f \in C^1((0, \infty)) \cap C^\gamma([0, \infty)) & \text{for some } \gamma \in (0, 1) \\ \text{(ii) } f(u) > 0 & \text{for } u > 0. \end{cases}$$

Then, it is easy to see that the following definition applies.

Definition 2.3. A function $\phi \in C^2(\Omega) \cap C(\bar{\Omega})$ is called a supersolution of semilinear elliptic boundary value problem

$$\begin{aligned} \Delta u(x) + f(u(x)) &= 0 \qquad \text{for } x \in \Omega \\ u(x) &= 0 \qquad \text{for } x \in \partial\Omega \end{aligned} \tag{2.9}$$

if ϕ satisfies

$$\begin{aligned} \Delta \phi + f(\phi(x)) &\le 0 \qquad \text{for } x \in \Omega \\ \phi(x) &\ge 0 \qquad \text{for } x \in \partial\Omega. \end{aligned} \tag{2.10}$$

Similarly, a subsolution $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ of (2.9) satisfies

$$\begin{aligned} \Delta\psi(x) + f(\psi(x)) &\geq 0 \qquad \text{for } x \in \Omega \\ \psi(x) &\leq 0 \qquad \text{for } x \in \partial\Omega. \end{aligned} \tag{2.11}$$

We state the monotone iteration schemes as below.

PROPOSITION 2.4. Let ϕ be a supersolution and ψ a subsolution of (2.9) with $\psi \leq \phi$. Then there exist solution $\underline{u}$ and $\bar{u}$ with

$$\psi \leq \underline{u} \leq \bar{u} \leq \phi$$

where $\underline{u}$ is the minimum solution between ψ and ϕ, and $\bar{u}$ is the maximum solution between ψ and ϕ. $\underline{u}$ may be equal to $\bar{u}$. For the proof see, e.g. Sattinger [15, 16] or Amann [1].

We now give the following theorem which is a special case of Serrin's sweeping principle, see Serrin [17] and Sattinger [16]. For completeness a short proof is given.

THEOREM 2.5. Let $\{\phi_\tau\}$ be a family of supersolutions of (2.9) which is increasing in τ, $0 \leq \tau \leq 1$, i.e.

$$\phi_{\tau_1}(x) \leq \phi_{\tau_2}(x) \qquad \text{for } x \in \Omega$$

if $\tau_1 \leq \tau_2$, and satisfy the following conditions:

(i) $\phi_\tau \in C^2(\Omega) \cap C^1(\bar{\Omega})$, for $\tau \in [0, 1]$,

(ii) $\phi_\tau = 0$ on $\partial\Omega$,

(iii) ϕ_τ is not a solution of (2.9) for $\tau \in (0, 1]$,

(iv) ϕ_τ is continuous in τ, in the sense that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ if $|\tau' - \tau| < \delta$

$$|\phi_\tau(x) - \phi_{\tau'}(x)| \leq \varepsilon t(x) \qquad \text{for } x \in \Omega.$$

Suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of (2.9) with

$$u \leq \phi_1 \qquad \text{in } \Omega.$$

Then either $u \equiv \phi_0$ or $u(x) < \phi_0(x)$ for $x \in \Omega$. A similar result holds for a family of subsolutions $\{\psi_\tau\}$.

Proof. Let $\tau_0 = \inf\{\tau \in [0, 1] \mid \phi_\tau(x) > u(x) \text{ in } \Omega\}$. Then $\phi_{\tau_0}(x) \geq u(x)$ in Ω. By lemma 2.2, either $\phi_{\tau_0} \equiv u$ or $\phi_{\tau_0}(x) > u(x)$ for $x \in \Omega$. We shall prove $\tau_0 = 0$. If $\tau_0 > 0$, then $\varphi_{\tau_0}(x) > u(x)$ for $x \in \Omega$ by (iii).

By strong maximum principle

$$\frac{\partial(\phi_{\tau_0} - u)}{\partial n_s}(s) \leq -k < 0$$

for $s \in \partial\Omega$ and some $k > 0$. A similar argument as in proving proposition 2.1(i) implies that

$$\phi_{\tau_0}(x) - u(x) \geq \eta t(x)$$

for $x \in \Omega$ and some $\eta > 0$. By (iv), there exists $0 < \tau' < \tau_0$ such that

$$\phi_{\tau'}(x) - u(x) \geq (\phi_{\tau_0}(x) - \eta t(x)) - u(x) \geq 0 \qquad \text{in } \Omega$$

a contradiction to the definition of τ_0. Hence $\tau_0 = 0$. This completes the proof.

Remark. The conditions in theorem 2.5 are easily verified in our constructions of super- or subsolutions of (1.1). It is clear that some kind of continuity condition of $\{\phi_\tau\}$ with respect to τ has to be satisfied in order that the conclusion of theorem 2.5 holds. (iv) suits our purpose and is easily obtained by applying proposition 2.1(i).

3. (β, β)-TYPE

In the remainder of the paper we shall also assume that f has a power growth at $u \to \infty$, i.e. f satisfies

(H-1) $$\lim_{u \to +\infty} \frac{f(u)}{u^\beta} = 1, \qquad 0 < \beta < 1.$$

In the case of (β, β)-type, we shall prove that for large λ (1.1) has the large solution only. Consider the linear eigenvalue problem

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) \qquad \text{for } x \in \Omega \\ u(x) &= 0 \qquad \text{for } x \in \partial\Omega. \end{aligned} \tag{3.1}$$

Let λ_1 (>0) be the least eigenvalue and $v_1 > 0$ be the corresponding eigenfunction with normalization $\|v_1\|_\infty = 1$. We first prove the following lemma.

LEMMA 3.1. Suppose that f satisfies (H-0), (H-1) and (H-2)

$$f(u) > \sigma u \qquad \text{in } (0, \mu]$$

for some $\sigma > 0$ and $\mu > 0$. Then the solution u_λ of (1.1) satisfies

$$u_\lambda \geq \mu v_1 \qquad \text{if } \lambda \geq \lambda_1/\sigma. \tag{3.2}$$

Furthermore, (1.1) has the minimum positive solution $\underline{u}$ for $\lambda \geq \lambda_1/\sigma$.

Proof. For $\tau \in (0, \mu]$ and $\lambda \geq \lambda_1/\sigma$, we have

$$\begin{aligned} \Delta(\tau v_1(x)) + \lambda f(\tau v_1(x)) &> -\tau\lambda_1 v_1(x) + \lambda\sigma\tau v_1(x) \\ &= \tau v_1(x)(\lambda\sigma - \lambda_1) \\ &\geq 0. \end{aligned}$$

Therefore, $\{\tau v_1\}$, $\tau \in (0, \mu]$, is a family of subsolutions of (1.1) which is strictly increasing in τ and is not a solution for $\tau \in (0, \mu]$. By proposition 2.1(i), $u_\lambda \geq \tau_\lambda v_1$ in Ω for some $\tau_\lambda \in (0, \mu)$. For $\lambda \geq \lambda_1/\sigma$, by theorem 2.5, $u_\lambda \geq \mu v_1$ in Ω. This proves (3.2).

To prove (1.1) has the minimum positive solution, it suffices to construct a supersolution ϕ_λ of (1.1) such that $\phi_\lambda \geq \mu v_1$, in Ω. This can be obtained if f satisfies (H-0) and

(H-1′) $$\lim_{u \to +\infty} \frac{f(u)}{u} = 0.$$

However, in the case of f satisfying (H-0) and (H-1), $\phi_\lambda = \lambda^{1/(1-\beta)}\tau v_0$ is a supersolution for large τ. For details, see the proof of theorem 3.3.

It is known that (1.3) has the unique positive solution in $C^{2+\beta}(\bar{\Omega})$ (unique in the class of $C^2(\Omega) \cap C^1(\bar{\Omega})$), for example, see Aronson and Peletier [3]. We shall give a short proof as below.

PROPOSITION 3.2. There exists a unique positive solution v_β of

$$\begin{aligned} -\Delta u(x) &= u^\beta(x) \qquad \text{for } x \in \Omega \\ u(x) &= 0 \qquad \text{for } x \in \partial\Omega \end{aligned} \tag{1.3}$$

where $0 < \beta < 1$.

Proof. It suffices to prove that (1.3) has a minimum positive solution. In fact, if f satisfies

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{f(u)}{u}\right) < 0 \qquad \text{for } u \in (0, \infty) \tag{3.3}$$

and (2.9) has the minimum positive solution $\underline{u}$. Then (2.9) has the unique positive solution. In fact, if u is a positive solution of (2.9), then

$$\begin{aligned} 0 &= \int_\Omega (u\,\Delta\underline{u} - \underline{u}\,\Delta u)\,\mathrm{d}x \\ &= \int_\Omega (\underline{u}f(u) - uf(\underline{u}))\,\mathrm{d}x \\ &= \int_\Omega \underline{u}u\left(\frac{f(u)}{u} - \frac{f(\underline{u})}{\underline{u}}\right)\mathrm{d}x \\ &\le 0. \end{aligned}$$

This implies that $u = \underline{u}$.

To prove (1.3) has the minimum positive solution. We note that $u^\beta > \lambda_1 u$ for all $u \in (0, \lambda_1^{1/(\beta-1)})$, the existence of minimum solution follows by lemma 3.1. The proof is complete.

We can now prove the following asymptotic theorem for u_λ when λ is large.

THEOREM 3.3. Suppose that f satisfies (H-0), (H-1) and (H-2). Then

$$\lim_{\lambda \to +\infty} u_\lambda(x)/(\lambda^{1/(1-\beta)}v_\beta(x)) = 1 \qquad \text{uniformly on } \bar{\Omega}$$

where u_λ is a solution of (1.1).

Proof. We shall divide the proof into three steps:
Step 1. There exists $\lambda_2 > 0$, if $\lambda > \lambda_2$, then

$$u_\lambda \ge \lambda v_\beta.$$

Step 2. For any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists $\Lambda_\varepsilon = \Lambda(\varepsilon, \varepsilon_1) > 0$, if $\lambda > \Lambda_\varepsilon$, then

$$u_\lambda \ge \lambda^{1/(1-\beta)}(1 - \varepsilon)^{1/(1-\beta)}(1 - \varepsilon_1)^{1/(1-\beta)}v_\beta.$$

Step 3. For any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists $\Lambda'_\varepsilon = \Lambda'(\varepsilon, \varepsilon_1) > 0$, if $\lambda > \Lambda'_\varepsilon$, then

$$u_\lambda \le \lambda^{1/(1-\beta)}(1+\varepsilon)^{1/(1-\beta)}(1+\varepsilon_1)^{1/(1-\beta)}v_\beta.$$

Step 1. We shall first prove that there exists $m > 0$ such that $u_\lambda \ge \lambda m v_0$ if $\lambda \ge \lambda_1/\sigma$. By proposition 2.1(i), there exists $k_1 > 0$ such that $v_1 \ge k_1 v_0$. Hence

$$u_\lambda \ge \mu_1 v_0 \qquad \text{if } \lambda \ge \lambda_1/\sigma$$

where $\mu_1 = \mu k_1 > 0$. Let $\mu' \in (0, \mu_1)$ and $m' = \min\{f(u): u \in [\mu', \infty)\}$. Since f satisfies (H-0) and (H-1), $m' > 0$. Denote by $[\phi \ge U] = \{x \in \Omega \mid \varphi(x) \ge U\}$, etc. If $\lambda \ge \lambda_1/\sigma$, then

$$\begin{aligned} u_\lambda(x) &= \lambda \int_\Omega G(x,y) f(u_\lambda(y))\,dy \\ &\ge \lambda m' \int_{[\mu_1 v_0 \ge \mu']} G(x,y)\,dy \\ &= \lambda m' \left\{ v_0(x) - \int_{[v_0 \ge \mu'/\mu_1]} G(x,y)\,dy \right\} \\ &\ge \lambda m' \{v_0(x) - C_p(\mu'/\mu_1)^{1/p} v_0(x)\} \\ &\ge \lambda m v_0(x) \end{aligned}$$

here (2.5) is used, $m = m'/2$ and μ' is chosen with

$$C_p(\mu'/\mu_1)^{1/p} = \tfrac{1}{2}.$$

Next, we prove that there exists $\lambda_2 \ge \lambda_1/\sigma$, such that $\lambda \ge \lambda_2$ implies $u_\lambda \ge \lambda v_\beta$.

By proposition 2.1(i), there exists $M > 0$ such that

$$\frac{M}{2} v_0 \ge v_\beta.$$

For this M, there exists $U = U(M) > 0$, if $u \ge U$, then $f(u) \ge M$. Therefore

$$\begin{aligned} u_\lambda(x) &\ge \lambda M \int_{[\lambda m v_0 \ge U]} G(x,y)\,dy \\ &\ge \lambda M \{1 - C_p(U/\lambda m)^{1/p}\} v_0(x) \\ &\ge \lambda v_\beta, \end{aligned}$$

here λ_2 is chosen such that $C_p(U/\lambda_2 m)^{1/p} = \frac{1}{2}$.

Step 2. By proposition 2.1(i) and (ii), with $\rho = v_\beta^\beta$,

$$u_\delta(x) \equiv \int_{\Omega_\delta} G(x,y) v_\beta^\beta(y)\,dy \le C_\beta \delta^{1/p} v_\beta(x) \tag{3.4}$$

for $0 < \delta < \delta_0$, and constant C_β depends on p, n, Ω and v_β. We note that (3.4) can be improved to

$$u_\delta(x) \le C'_\beta \delta^{\beta + 1/p} v_\beta(x) \qquad \text{on } \bar{\Omega} \tag{3.5}$$

for $0 < \delta < \delta_0$, and some constant $C'_\beta > 0$. By (H-1), for any $\varepsilon > 0$, there exists $U = U(\varepsilon) > 0$ such that $u \geq U$ implies

$$f(u) \geq (1 - \varepsilon)u^\beta.$$

Therefore,

$$\begin{aligned} u_\lambda(x) &\geq \lambda(1 - \varepsilon) \int_{[\lambda v_\beta \geq U]} G(x, y)(\lambda v_\beta(y))^\beta \, dy \\ &\geq \lambda^{1+\beta}(1 - \varepsilon)v_\beta\{1 - C_\beta(U/\lambda)^{1/p}\} \\ &\geq \lambda^{1+\beta}v_\beta(1 - \varepsilon)(1 - \varepsilon_1) \end{aligned}$$

whenever $\lambda \geq \max\{\lambda_2, \lambda_3\}$, here, $\lambda_3 = (C_\beta/\varepsilon_1)^p U$, i.e. $C_\beta(U/\lambda_3)^{1/p} = \varepsilon_1$. Moreover, after repeating this argument once, we obtain

$$\begin{aligned} u_\lambda(x) &\geq \lambda(1 - \varepsilon) \int_{[\lambda^{1+\beta}(1-\varepsilon)(1-\varepsilon_1)v_\beta \geq U]} G(x, y)\{\lambda^{1+\beta}(1 - \varepsilon)(1 - \varepsilon_1)v_\beta(y)\}^\beta \, dy \\ &\geq \lambda^{1+\beta+\beta^2}(1 - \varepsilon)^{1+\beta}(1 - \varepsilon_1)^\beta \cdot \left\{1 - C_\beta\left(\frac{U}{\lambda^{1+\beta}(1 - \varepsilon)(1 - \varepsilon_1)}\right)^{1/p}\right\} v_\beta(x) \\ &= \lambda^{1+\beta+\beta^2}(1 - \varepsilon)^{1+\beta}(1 - \varepsilon_1)^{1+\beta}v_\beta(x) \end{aligned}$$

whenever $\lambda \geq \max(\lambda_2, \lambda_3, \lambda_4)$, here λ_4 is given in

$$C_\beta\left(\frac{U}{\lambda_4^{1+\beta}(1 - \varepsilon)(1 - \varepsilon_1)}\right)^{1/p} = \varepsilon_1$$

i.e.

$$\lambda_4 = \left\{(C_\beta/\varepsilon_1)^p \frac{U}{(1 - \varepsilon)(1 - \varepsilon_1)}\right\}^{1/(1+\beta)}.$$

Repeating this argument, we can prove that for any $n \geq 2$

$$u_\lambda(x) \geq \lambda^{1+\beta+\cdots+\beta^n}\{(1 - \varepsilon)(1 - \varepsilon_1)\}^{1+\beta+\cdots+\beta^{n-1}} v_\beta(x)$$

whenever $\lambda \geq \max\{\lambda_2, \lambda_3, \ldots, \lambda_{n+2}\}$, with

$$\lambda_{n+2} = \left\{(C_\beta/\varepsilon_1)^p \frac{U}{[(1 - \varepsilon)(1 - \varepsilon_1)]^{1+\beta+\cdots+\beta^{n-2}}}\right\}^{1/(1+\beta+\cdots+\beta^{n-1})}.$$

Since $0 < \beta < 1$, $\lambda_n \to \lambda_\infty$ as $n \to \infty$, where

$$\lambda_\infty = (C_\beta/\varepsilon_1)^{p(1-\beta)} \frac{U^{1-\beta}}{(1 - \varepsilon)(1 - \varepsilon_1)}. \tag{3.6}$$

Let

$$\begin{aligned} \Lambda_\varepsilon &= \max\{\lambda_2, \lambda_3, \ldots\} \\ &= \Lambda(\varepsilon, \varepsilon_1) < +\infty. \end{aligned}$$

If $\lambda \geq \Lambda_\varepsilon$, then

$$u_\lambda(x) \geq \lambda^{1/(1-\beta)}(1 - \varepsilon)^{1/(1-\beta)}(1 - \varepsilon_1)^{1/(1-\beta)}v_\beta(x) \qquad \text{on } \Omega.$$

This completes the proof of step 2.

Step 3. It is easy to see that (H-0) and (H-1) implies that

$$f(u) \le A + 2u^{\beta} \qquad \text{for } u \ge 0 \tag{3.7}$$

where $A \ge 0$ is a constant. We first prove that $\{\lambda^{1/(1-\beta)}\tau v_0\}$ is a family of supersolutions of (1.1) if τ is large. In fact

$$\begin{aligned}\Delta(\lambda^{1/(1-\beta)}\tau v_0) + \lambda f(\lambda^{1/(1-\beta)}\tau v_0) &\le -\lambda^{1/(1-\beta)}\tau + \lambda\{A + 2(\lambda^{1/(1-\beta)}\tau v_0)^{\beta}\}\\ &= -\lambda^{1/(1-\beta)}\{\tau - 2\tau^{\beta}v_0^{\beta} - \lambda^{-\beta/(1-\beta)}A\}\\ &\le 0\end{aligned}$$

if

$$\tau \ge 2\|v_0\|_{\infty}^{\beta}\tau^{\beta} + \lambda^{-\beta/(1-\beta)}A. \tag{3.8}$$

Furthermore, it is easy to verify that (3.8) is true if

$$\tau \ge \max\{\lambda^{-1/(1-\beta)}A^{1/\beta}, (2\|v_0\|_{\infty}^{\beta} + 1)^{1/(1-\beta)}\}. \tag{3.9}$$

For $\lambda \ge 1$, we can choose

$$\tau \ge M'_{\beta} \equiv \max\{A^{1/\beta}, (2\|v_0\|_{\infty}^{\beta} + 1)^{1/(1-\beta)}\}. \tag{3.10}$$

Therefore, $\lambda^{1/1-\beta}\tau v_0$, $\tau \ge M'_{\beta}$, is not a solution of (1.1) if τ is large enough. Hence, by Serrin's sweeping principle, we have

$$u_{\lambda} \le \lambda^{1/(1-\beta)}M'_{\beta}v_0$$

for $\lambda \ge \lambda'_2$, for some $\lambda'_2 > 0$. Next, by an argument similar to that in step 2, we can prove that for any $\varepsilon > 0$, $\varepsilon_1 > 0$, there exists $\Lambda'_{\varepsilon} = \Lambda'(\varepsilon, \varepsilon_1) < +\infty$, such that $\lambda \ge \Lambda'_{\varepsilon}$, implies

$$u_{\lambda}(x) \le \lambda^{1/(1-\beta)}(1 + \varepsilon)^{1/(1-\beta)}(1 + \varepsilon_1)^{1/(1-\beta)}v_{\beta}(x) \qquad \text{on } \bar{\Omega}.$$

The proof is complete.

During the proof, we have obtained the following corollary.

COROLLARY 3.4. Suppose that f satisfies (H-0) and (H-1). If $\{u_{\lambda}\}$ is a family of solutions of (1.1) with $u_{\lambda} \ge \mu v_0$ for some $\mu > 0$ and $\lambda \ge \Lambda \ge 0$, then

$$\lim_{\lambda \to +\infty} u_{\lambda}/(\lambda^{1/(1-\beta)}v_{\beta}) = 1 \qquad \text{uniformly on } \bar{\Omega}.$$

To prove an uniqueness result for positive solutions of (1.1), we need the following lemma.

LEMMA 3.5. For any $w \in W_0^{1,2}(\Omega)$ $(=H_0^1(\Omega))$

$$\int_{\Omega} |\nabla w|^2 \ge \int_{\Omega} v_{\beta}^{\beta-1}w^2 \tag{3.11}$$

and

$$\int_{\Omega} |\nabla w|^2 \ge C_{\beta}\int_{\Omega} v_{\beta}^{-2}w^2 \tag{3.12}$$

where $C_{\beta} > 0$ is a constant.

Proof. Consider the linear eigenvalue problem

$$-\Delta w(x) = \mu v_\beta^{\beta-1}(x) w(x) \text{ in } \Omega \qquad w(x) = 0 \text{ on } \partial\Omega.$$

Then $\mu_1 = 1$ is an eigenvalue with positive eigenfunction v_β. Hence 1 is the first eigenvalue. A variational principle which characterizes the first eigenvalue gives (3.11). For (3.12), see Brezis and Turner [6, lemma 2.1] and Lions and Magenes [11, p. 76].

We can now prove the following uniqueness theorem.

THEOREM 3.6. Suppose that f satisfies (H-0), (H-1) and (H-2)

(H-3) $$\limsup_{u\to 0^+} |f'(u)|u^2 < +\infty$$

and

(H-4) $$\limsup_{u\to +\infty} \frac{f'(u)u}{f(u)} < 1.$$

Then there exists $\lambda^* \geq 0$ such that (1.1) has an unique positive solution for $\lambda > \lambda^*$.

Proof. Let $\underline{u}_\lambda$ be the minimum solution and u_λ be any positive solution of (1.1). Let $w_\lambda \equiv u_\lambda - \underline{u}_\lambda$. Then

$$-\Delta w_\lambda = \lambda f'(\theta_\lambda) w_\lambda \tag{3.13}$$

where $\theta_\lambda(x) \in (\underline{u}_\lambda(x), u_\lambda(x))$.

For any $\varepsilon > 0$, which is sufficiently small, let $U = U(\varepsilon)$ be chosen such that $u \geq U$ imply

$$(1-\varepsilon)u^\beta \leq f(u) \leq (1+\varepsilon)u^\beta$$

$$\frac{f'(u)u}{f(u)} \leq \eta \equiv \left(\limsup_{u\to+\infty} \frac{f'(u)u}{f(u)}\right) + \varepsilon < 1$$

and

$$|f'(u)| \leq Au^{-2} + B \qquad \text{for } u \in (0, U]$$

where $A = A(\varepsilon)$ and $B = B(\varepsilon) \geq 0$. By theorem 3.3, there exists $\Lambda_\varepsilon \geq 0$ such that $\lambda \geq \Lambda_\varepsilon$ implies

$$\lambda^{1/(1-\beta)}(1-\varepsilon)v_\beta \leq \underline{u}_\lambda \leq \theta_\lambda \leq u_\lambda \leq \lambda^{1/(1-\beta)}(1+\varepsilon)v_\beta.$$

From (3.13), we have

$$\int_\Omega |\nabla w_\lambda|^2 = \lambda \int_{[\theta_\lambda \geq U]} f'(\theta_\lambda) w_\lambda^2 + \lambda \int_{[\theta_\lambda \leq U]} f'(\theta_\lambda) w_\lambda^2. \tag{3.14}$$

For the first term of right-hand side of (3.14), if $\lambda \geq \Lambda_\varepsilon$, we have

$$\begin{aligned}
\lambda \int_{[\theta_\lambda \geq U]} f'(\theta_\lambda) w_\lambda^2 &\leq \lambda\eta(1+\varepsilon) \int_{[\theta_\lambda \geq U]} \theta_\lambda^{\beta-1} w_\lambda^2 \\
&\leq \lambda\eta(1+\varepsilon) \int_\Omega \{\lambda^{1/(1-\beta)}(1-\varepsilon)v_\beta\}^{\beta-1} w_\lambda^2 \\
&= \eta(1+\varepsilon)(1-\varepsilon)^{\beta-1} \int_\Omega v_\beta^{\beta-1} w_\lambda^2 \\
&\leq \eta(1+\varepsilon)(1-\varepsilon)^{\beta-1} \int_\Omega |\nabla w_\lambda|^2
\end{aligned}$$

by (3.11). For the second term of right-hand side of (3.14), we have

$$\begin{aligned}\lambda\int_{[\theta_\lambda\le U]} f'(\theta_\lambda)w_\lambda^2 &\le \lambda\int_{[\theta_\lambda\le U]}(A\theta_\lambda^{-2}+B)w_\lambda^2\\ &\le \lambda\int_{[\lambda^{1/(1-\beta)}(1-\varepsilon)v_\beta\le U]} A\{\lambda^{1/(1-\beta)}(1-\varepsilon)v_\beta\}^{-2}w_\lambda^2\\ &\quad+\lambda B\int_{[v_\beta\le U(1-\varepsilon)^{-1}\lambda^{-1/(1-\beta)}]} v_\beta^2\{v_\beta^{-2}w_\lambda^2\}\\ &\le \lambda^{(-1-\beta)/(1-\beta)}(1-\varepsilon)^{-2}(A+BU^2)\int_\Omega v_\beta^{-2}w_\lambda^2\\ &\le \lambda^{(-1-\beta)/(1-\beta)}(1-\varepsilon)^{-2}(A+BU^2)C_\beta^{-1}\int_\Omega|\nabla w_\lambda|^2\end{aligned}$$

by (3.12). Hence, for $\lambda\ge\Lambda_\varepsilon$

$$\{1-\eta(1+\varepsilon)(1-\varepsilon)^{\beta-1}-\lambda^{(-\beta-1)/(1-\beta)}(1-\varepsilon)^{-2}(A+BU^2)C_\beta^{-1}\}\int_\Omega|\nabla w_\lambda|^2\le 0. \quad (3.15)$$

By choosing sufficiently small $\varepsilon>0$ and large $\Lambda'_\varepsilon\ge\Lambda_\varepsilon$, then $\lambda\ge\Lambda'_\varepsilon$ implies $\{\cdots\}>0$ in (3.15). Hence, for $\lambda\ge\Lambda'_\varepsilon$

$$\int_\Omega|\nabla w_\lambda|^2=0 \qquad \text{which implies } w_\lambda\equiv 0.$$

The proof is complete.

4. (α,β)-TYPE

In this section we shall study the multiplicity problem of (1.1) in the case of (α,β)-type, i.e. we consider $f(u)\sim u^\alpha$ for $u\sim 0^+$, $1<\alpha$ when $n=1,2$ and $1<\alpha<(n+2)/(n-2)$ when $n\ge 3$, and $f(u)\sim u^\beta$ for $u\sim+\infty$, $0<\beta<1$. For (α,β)-type, (1.1) has no positive solution if λ is too small. More precisely, we have the following lemma.

LEMMA 4.1. Suppose that f satisfies

$$0<f(u)\le Mu \qquad \text{for } u>0 \quad (4.1)$$

for some constant $M>0$. Then (1.1) has no positive solution if $\lambda<\lambda_1/M$.

Proof. Multiplying (1.1) by v_1 and integrating over Ω, gives

$$-\int_\Omega v_1\,\Delta u_\lambda=\lambda\int_\Omega f(u_\lambda)v_1.$$

Hence

$$\begin{aligned}\lambda_1\int_\Omega u_\lambda v_1&=\lambda\int_\Omega f(u_\lambda)v_1\\ &=\lambda M\int_\Omega u_\lambda v_1\end{aligned}$$

by (4.1). Since $\int_\Omega u_\lambda v_1>0$, the lemma follows.

In the remainder of this section, we always assume that f satisfies

$$\text{(H-2')} \qquad \lim_{u \to 0^+} \frac{f(u)}{u^\alpha} = 1$$

where $\alpha > 1$. (H-2') implies that there exists $\mu > 0$ such that $f(u) \le 2u^\alpha$ for $u \in [0, \mu]$. Therefore for any solution u of (1.1)

$$u_\lambda \le \lambda^{1/(1-\beta)} M_\beta v_0$$

where

$$M_\beta = \max\left\{\left(\frac{\lambda_1}{M}\right)^{-1/(1-\beta)} A^{1/\beta},\ (2\|v_0\|_\infty^\beta + 1)^{1/(1-\beta)}\right\}$$

is given in (3.9) with $\lambda = \lambda_1/M$. We now give an existence theorem for type (α, β) problem.

Theorem 4.2. Suppose that f satisfies (H-0), (H-1) and (H-2'), with
(i) $\alpha > 1$ if $n = 1, 2$,
(ii) $1 < \alpha < (n+2)/(n-2)$, if $n \ge 3$.
Then there exists $\lambda_* \ge \lambda_1/M$ such that
(i) when $\lambda < \lambda_*$, (1.1) has no positive solution,
(ii) when $\lambda > \lambda_*$, (1.1) has at least one positive solution.

Proof. The existence of positive solution of (1.1) for large λ can be proved by a variational method, see for example, Ambrosetti and Rabinowitz [2] and Nirenberg [13]. It will also be proved in theorem 4.4 by constructing an appropriate subsolution of (1.1). Let

$$\lambda_* = \inf\{\lambda > 0 : (1.1) \text{ has positive solution at } \lambda\}.$$

Then $\lambda_* \ge \lambda_1/M$ by lemma 4.1. Once (1.1) has a positive solution u_λ at λ, then u_λ is a subsolution of (1.1) for $\lambda' > \lambda$. In fact

$$\Delta u_\lambda + \lambda f(u_\lambda) = 0$$

implies

$$\Delta u_\lambda + \lambda' f(u_\lambda) = (\lambda' - \lambda) f(u_\lambda) > 0$$

if $\lambda' > \lambda$. As in theorem 3.3, $\lambda^{1/(1-\beta)} \tau v_0$ is a supersolution if τ is large enough. Hence (1.1) has solution for $\lambda > \lambda_*$. The proof is complete.

We need the following well-known results, see e.g. [2, 6, 12].

Proposition 4.3. The equations

$$\begin{aligned} -\Delta u &= u^\alpha && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

have a positive solution if
(i) $\alpha > 1$ for $n = 1, 2$, and
(ii) $1 < \alpha < (n+2)/(n-2)$, for $n \ge 3$.
Denote by v_α (any) solution of (4.2). We can prove an uniqueness result for large solution of (1.1) of type (α, β).

THEOREM 4.4. Suppose that f satisfies (H-0), (H-1), (H-2′), (H-3) and (H-4). Then there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, (1.1) has an unique positive solution u_λ with $u_\lambda \geq v_0$. Moreover, for these u_λ

$$\lim_{\lambda \to +\infty} u_\lambda/(\lambda^{1/(1-\beta)} v_\beta) = 1 \qquad \text{uniformly on } \bar{\Omega}.$$

Proof. By corollary 3.4 and theorem 3.6, the solution u_λ of (1.1) is unique if $u_\lambda \geq v_0$ for $\lambda \geq \lambda^*$ and some $\lambda^* > 0$. It remains to prove that there exists positive solution u_λ of (1.1) satisfying $u_\lambda \geq v_0$ if $\lambda \geq \Lambda$ for some $\Lambda > 0$. In the following, we shall show that $\lambda^{1/(1-\beta)} C v_\alpha$ is a subsolution of (1.1) by an appropriately chosen C and large λ (depends on C). (H-1) and (H-2′) imply that there exist U_1 and U_2, $0 < U_1 < U_2 < \infty$, such that

$$f(u) \geq \tfrac{1}{2} u^\alpha \qquad \text{if } u \in (0, U_1)$$

and

$$f(u) \geq \tfrac{1}{2} u^\beta \qquad \text{if } u \in (U_2, \infty).$$

Let

$$m \equiv \min\{f(u) \mid U_1 \leq u \leq U_2\} > 0.$$

(i) For $x \in \Omega$ with $\lambda^{1/(1-\beta)} C v_\alpha(x) \geq U_2$

$$\begin{aligned} \Delta(\lambda^{1/(1-\beta)} C v_\alpha(x)) + \lambda f(\lambda^{1/(1-\beta)} C v_\alpha(x)) &\geq -\lambda^{1/(1-\beta)} C v_\alpha^\alpha(x) + \tfrac{1}{2}\lambda(\lambda^{1/(1-\beta)} C v_\alpha(x))^\beta \\ &= \lambda^{1/(1-\beta)} C^\beta v_\alpha^\beta(x)\{\tfrac{1}{2} - C^{1-\beta} v_\alpha^{\alpha-\beta}\} \\ &\geq 0 \end{aligned}$$

if $\frac{1}{2} \geq C^{1-\beta} \|v_\alpha\|_\infty^{\alpha-\beta}$. Let

$$C = \left(\frac{1}{2\|v_\alpha\|_\infty^{\alpha-\beta}}\right)^{1/(1-\beta)}.$$

(ii) For $x \in \Omega$ with $U_1 \leq \lambda^{1/(1-\beta)} C v_\alpha(x) \leq U_2$,

$$\begin{aligned} \Delta(\lambda^{1/(1-\beta)} C v_\alpha(x)) + \lambda f(\lambda^{1/(1-\beta)} C v_\alpha(x)) &\geq -\lambda^{1/(1-\beta)} C v_\alpha^\alpha(x) + \lambda m \\ &\geq -U_2 v_\alpha^{\alpha-1}(x) + \lambda m \\ &\geq 0 \end{aligned}$$

if $\lambda \geq (1/m) U_2 \|v_\alpha\|_\infty^{\alpha-1}$.

(iii) For $x \in \Omega$ with $\lambda^{1/(1-\beta)} C v_\alpha(x) \leq U_1$

$$\begin{aligned} \Delta(\lambda^{1/(1-\beta)} C v_\alpha(x)) + \lambda f(\lambda^{1/(1-\beta)} C v_\alpha(x)) &\geq -\lambda^{1/(1-\beta)} C v_\alpha^\alpha(x) + \tfrac{1}{2}\lambda(\lambda^{1/(1-\beta)} C v_\alpha(x))^\alpha \\ &= \lambda^{1/(1-\beta)} C v_\alpha^\alpha(x)(\tfrac{1}{2}\lambda^{(\alpha-\beta)/(1-\beta)} C^{\alpha-1} - 1) \\ &\geq 0 \end{aligned}$$

if $\lambda \geq (2C^{1-\alpha})^{(1-\beta)/(\alpha-\beta)}$. Therefore, $\lambda^{1/(1-\beta)} C v_\alpha$ is a subsolution when

$$\lambda \geq \|v_\alpha\|_\infty^{\alpha-1} \cdot \max(2, U_2/m).$$

This completes the proof.

Acknowledgement—This work was partially supported by the National Science Council of the Republic of China.

REFERENCES

1. Amann H., On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana Univ. math. Jl.* **21**, 125–146 (1971).
2. Ambrosetti A. & Rabinowitz P., Dual variational methods in critical point theory and applications, *J. funct. Analysis* **14**, 349–381 (1973).
3. Aronson D. G. & Peletier L. A., Large time behavior of solutions of the porous medium equation in bounded domains, *J. diff. Eqns* **39**, 378–412 (1981).
4. Berestycki H. & Lions P. L., Some applications of the method of super and subsolutions, in Bifurcation and nonlinear eigenvalue problems, *Lecture Notes in Mathematics* **782**, 16–41. Springer, Berlin (1980).
5. Brezis H. & Nirenberg L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Communs pure appl. Math.* **36**, 437–447 (1983).
6. Brezis H. & Turner R. E. L., On a class of superlinear elliptic problems, *Communs partial diff. Eqns* **2**, 601–604 (1977).
7. Crandall M. & Rabinowitz P., Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Archs ration Math. Analysis* **58**, 207–218 (1975).
8. Dancer E. N., Uniqueness for elliptic equation when a parameter is large, *Nonlinear Analysis* **8**, 835–836 (1984).
9. Joseph D. D. & Lundrgen T. S., Quasilinear Dirichlet problems driven by positive source, *Archs ration Mech. Analysis* **49**, 241–269 (1973).
10. Lin S. S., Some uniqueness results for positone problems when a parameter is large, *Chinese J. Math.* **13**, 67–81 (1985).
11. Lions J. L. & Magenes E., *Problems aux Limites non Homogénes et Applications*, Vol. I. Dunod, Paris (1968).
12. Lions P. L., On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24**, 441–467 (1982).
13. Nirenberg L., Variational and topological methods in nonlinear problems, *Bull. Am. math. Soc.* **4**, 267–302 (1981).
14. Protter M. H. & Weinberger H. F., *Maximum Principles in Differential equations*. Prentice-Hall, NJ (1967).
15. Sattinger D. H., Monotone methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. math. Jl.* **21**, 979–1000 (1972).
16. Sattinger D. H., Topics in stability and bifurcation theory, *Lecture Notes in Mathematics* **309**. Springer, Berlin (1973).
17. Serrin J., Nonlinear elliptic equation of second order (unpublished).
18. Shivaji R., Uniqueness results for a class of positone problems, *Nonlinear Analysis* **7**, 223–230 (1983).