國 立 交 通 大 學

應用數學系

碩 士 論 文

局部聯結之累積和發射的振盪器之同步化

Synchronization in Locally Coupled Integrate-and-Fire Oscillators

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中 華 民 國 九 十 七 年 六 月

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碩 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

June 2008

Hsinchu, Taiwan, Republic of China

中華民國九十七年六月

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摘　　要

這邊考慮的是累積和發射之局部聯結的模型。這個問題是被 Mirollo 和 Strogatz [SIAM J. Appl. Math., 50, 1645-1662(1990)]所提出來的，與 Mirollo 和 Strogatz 提出的假定一樣，每一個震盪器 x_i 都有一個 f_i。在這篇論文考慮的是，凹向下的四個振盪器，另外聯結是近鄰耦合並加上週期邊界條件，最後得出來的結果是幾乎對所有的點都發生同步化。

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ABSTRACT

A model of four integrate-and-fire oscillators locally coupled is studied. The problem was first raised by Mirollo and Strogatz [SIAM J. Appl. Math., 50, 1645-1662(1990)]. We assume, as in Mirollo and Strogatz's model, that each oscillator x_i evolves according to a map f_i. We show in this thesis that the system of four convex oscillators (i.e., $f_i < 0$) that have nearest-neighbor coupling with periodic boundary conditions is firing in unison for almost all initial conditions.

Key words: synchronization, integrate-and-fire, locally coupled.

誌　　謝

感謝莊重教授在這兩年的鼓勵、照顧與指導。莊老師悉心指導，充分教導我關於此領域的各項知識。老師也非常親切，很關心學生，從老師身上也學會很多待人處事的道理。

感謝交大應用數學系的老師，藉由修課常常獲益良多。

感謝學姊郁泉，感謝學長育豪、俊銘，時常耐心的引導我，讓我順利完成學業。

感謝在交大認識的朋友，不論是討論課業還是分享生活點滴，都讓我更喜歡交大。感謝我的室友，因爲你們的陪伴，使我睡覺不在害怕。

最後，感謝爸爸媽媽常常打電話關心我，雖然相隔有點遙遠，但仍常常鼓勵我，讓我可以順利完成學業。

目　　　錄

1. **Introduction**

Large assemblies of oscillator units can spontaneously evolve to a state of large scale organization. Synchronization is the best known phenomenon of this kind, where after some transient regime a coherent oscillatory activity of the set of oscillators emerges. This interesting phenomenon is quite common in many different disciplines such as engineering [54], physics [13, 30] and [48], chemistry [31], as well as biology [53]. For example, southeastern fireflies, where thousands of individuals gathered on trees flash in unison. Other examples of biological oscillators are the rhythmic activity of cells of the heart pacemaker [25, 35, 39] and [51], of cells of pancreas [45] and [46], and of neural networks [11, 18, 39, 41] and [47]. Synchronization of oscillators has been studied in both phase-coupled [3, 4, 5, 6, 14, 15, 16, 17, 26, 27, 29, 32, 33, 34, 37, 40, 49, 50, 52] and [56], where the interaction between the oscillators is smooth and continuous in time, and pulsed-coupled models [1, 7, 9, 10, 19, 20, 21, 23, 24, 28, 31, 36, 42, 43, 44] and [55], where the membrane voltage is discontinuously reset to a fixed value once it reaches a certain threshold. It should be noted that pulse-coupled models are of greater relevance for neuroscience applications since synaptic coupling is often spike mediated.

The purpose of this thesis is to study synchronization in locally coupled integrate-and-fire oscillators. We begin with describing the Peskin's model [39] of n integrate-and-fire oscillators. Let the state of the i-th oscillator be denoted by x_i, where x_i are subject to the dynamics $\frac{dx_i}{dt} = -r_i x_i + I_i$, $0 \leq x_i \leq 1$, $i = 1, 2, \cdots, n$ with input $I_i > 0$, a normalized threshold 1 and leakiness $r_i \geq 0$. When $x_j = 1$, the jth oscillators fires and x_j jumps back to zero. As a consequence of the firing of jth oscillator, the activation of any other oscillator i is incremented by the coupling ϵ_{ij}. If $\epsilon_{ij} \neq 0$ for all $i \neq j$, then the system of n such oscillators is said to be globally coupled. Otherwise, it is said to be locally coupled. This model was later generalized by Mirollo and Strogatz [36]. Specifically, they assumed that the state variable x_i evolve according to a map f_i. When x_i reaches the threshold, the oscillator fires and x_i jumps back instantly to zero, and the activation of any other oscillator j is incremented by the positive coupling ϵ_{ji}. Specifically, x_i evolve according to $x_i = f_i(\phi_i)$, where $f_i : [0, 1] \to [0, 1]$ is smooth, and strictly increasing, i.e., $f_i' > 0$ on $(0, 1)$. Here ϕ_i is a phase variable so that (i) $\frac{d\phi_i}{dt} = \frac{1}{T_i}$, where T_i is the cycle period for oscillator x_i when evolving freely, (ii) $\phi_i = 0$ when the oscillator is at its lowest state $x_i = 0$, and (iii) $\phi_i \equiv 1$ at the end of cycle when the oscillator reaches the threshold $x_i = 1$. Therefore, f_i satisfy $f_i(0) = 0$, $f_i(1) = 1$. These maps f_i are to be called evolution maps. The inverses of f_i are to be denoted by g_i. If $f_i \equiv f$,

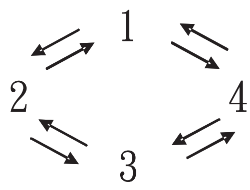

FIGURE 1.1

$g_i \equiv g$, $T_i \equiv T$ and $\epsilon_{ij} \equiv \epsilon$ for all i, j, then the corresponding system is called identical.

For Peskin's model, $f_i(\phi) = \frac{I_i}{r_i}(1 - e^{-r_i T_i \phi})$ and $T_i = \frac{\ln(\frac{I_i}{I_i - r_i})}{r_i}$. Peskin [39] conjectured that, first, for identical oscillators that are globally coupled, the system approaches a state in which all oscillators are firing synchronously for almost all initial conditions and that, second, this remains true even when the oscillators are not quite identical. The first part of the conjecture was essentially proved by Mirollo and Strogatz [36] with convex oscillators (i.e., $f_i'' < 0$). The second part of Peskin's conjecture was verified by Senn and Urbanczik [44] with flat oscillators (i.e., $f_i'' \equiv 0$). The key feature in those proofs rely on the non-concavity of the evolution functions f_i. However, Bottani [8] numerical showed that even concave oscillators (i.e., $f_i'' > 0$) can synchronize provided that the concavity is not too large. Recently Chang and Juang [12] proves the second part of Peskin's conjecture for the system of convex oscillators. Moreover, they also reconfirm Mirollo and Strogatz's observation that convexity of the oscillators indeed plays an important role in achieving synchrony. Specifically, they show that for concave oscillators if they are "nearly" identical, then no synchronization is to occur for initial conditions in a set of positive measure. That is to say, in general, concave oscillators may synchronize for almost all initial conditions only if they are not that identical. Indeed, they further prove that the imbalance between the speeds and/or coupling strengthens of the oscillators induces the synchronization of the system provided that the concavity of the evolution maps is sufficiently small. The last part of their results verifies the numerical observation of Bottani [8].

In the celebrated paper of Mirollo and Strogatz, they also raise an open question. How would the dynamics be affected if one replaces the all-to-all coupling with more local interactions, e.g., between the nearest neighbors or on a ring of d-dimensional chain, or more general graph [2, 22, 38]? Would the system still always end up firing in unison, or would more complex modes of organization become possible? In this thesis, we prove that for the identical system of four convex oscillators being the nearest neighbor coupling with periodic boundary conditions, the system always ends up firing in unison. See Fig1.1 for more explanation of such coupling. The locally coupling rules mean that, for instance, if oscillator 2 reaches the threshold, then its nearest neighbors, oscillator 3 and 1 receive the coupling strength ϵ.

We next describe the dynamics of such system. Without loss of generality, we let the speed $\frac{1}{T}$ of each oscillator be one. Let $\Phi^0 = (\phi_1^0, \phi_2^0, \phi_3^0, \phi_4^0)$ be the initial condition, which denote the phase position of four oscillators labeled 1,2,3,4, respectively. Suppose ϕ_4^0 is the first one reaching the threshold. Then the resulting phase position of the first three oscillators are, respectively,

$$\phi_i^1 = g(f(1 - \phi_4^0 + \phi_i^0) + \epsilon), i = 1, 3$$
$$\phi_2^1 = 1 - \phi_4^0 + \phi_2^0.$$

provided that $f(1 - \phi_4^0 + \phi_i^0) + \epsilon < 1, i = 1, 3$ and $1 - \phi_4^0 + \phi_2^0 < 1$. Suppose, in addition, that $f(1 - \phi_4^0 + \phi_1^0) + \epsilon \geq 1$. That is to say that the first oscillator also reach the threshold after receiving the coupling strength ϵ from the fourth oscillator. Then $\phi_3^1 = g(f(1-\phi_4^0+\phi_3^0)+\epsilon)$ and $\phi_2^1 = g(f(1-\phi_4^0+\phi_2^0)+\epsilon)$. Such chain reaction might continue if the second and/or the third oscillator also reach the threshold due to the earlier chain reaction.

Now we define the firing map that describes the changing of the phase of oscillators after one firing (i.e., some oscillators reaching the threshold).

Definition 1.1. *Let* $S = \{\phi = (\phi_1, \phi_2, \phi_3, \phi_4) : 0 = \min_{1\leq i\leq 4}\{\phi_i\} \leq \max_{1\leq i\leq 4}\{\phi_i\} < 1\}$. *Then the firing map* h *is defined as the mapping from* S *to* S *satisfying*

$$h(\phi) = \phi_{new},$$

where ϕ_{new} *is the new phase of oscillators from the original phase* ϕ *after another shortest time that causes some oscillator to reach the threshold and right after the time that the spike from the fired oscillator is achieved to the corresponding oscillators.*

We remark that from above definition, $h(S) \subseteq S$ and thus iterations under h is well-defined. That is, $h_i(\phi_1^{i-1}, \phi_2^{i-1}, \phi_3^{i-1}, \phi_4^{i-1}) =: (\phi_1^i, \phi_2^i, \phi_3^i, \phi_4^i)$ is well-defined for all $i > 0$. In the following, for the reason of clarity, we denote h to be h_i if firing map h is acted on the phase $(\phi_1^{i-1}, \phi_2^{i-1}, \phi_3^{i-1}, \phi_4^{i-1})$.

Definition 1.2. *Define function* $F_m(\phi) = g(\min[f(\phi)+m\epsilon, 1])$ *in the interval* $[0, 1]$, *where* g *is the inverse function of* f.

Lemma 1.1. *Let* $f'' < 0$.

(1) *If* $\phi_1 < \phi_2$, *then* $F_m(\phi_1) \leq F_m(\phi_2)$.
If $\phi_1 < \phi_2$ *and* $F_m(\phi_1) < 1$, *then* $F_m(\phi_1) < F_m(\phi_2)$.

(2) *If* $0 < F_m(\phi) < 1$, *then* $\dfrac{dF_m(\phi)}{d\phi} > 1$.

(3) *If* $\phi_1 < \phi_2$ *and* $F_m(\phi_2) < 1$, *then* $\phi_2 - \phi_1 < F_m(\phi_2) - F_m(\phi_1)$.

(4) *If* $\phi, \delta \geq 0$ *and* $F_m(\phi + \delta) < 1$, *then* $F_m(\phi) + \delta \leq F_m(\phi + \delta)$.
If $\delta > 0$ *and* $F_m(\phi) < 1$, *then* $F_m(\phi - \delta) < F_m(\phi) - \delta$.

Proof.

(1) Since $F_m(\phi_1) < 1, F_m(\phi_1) = g(f(\phi_1) + m\epsilon)$, then

$$\begin{aligned} F_m(\phi_1) &= g(f(\phi_1) + m\epsilon) \\ &< g(\min[f(\phi_2) + m\epsilon, 1]) \\ &= F_m(\phi_2). \end{aligned}$$

(2) Since $0 < F_m(\phi) < 1, F_m(\phi) = g(f(\phi) + m\epsilon)$, then

$$\begin{aligned} \frac{dF_m(\phi)}{d\phi} &= g'(f(\phi) + m\epsilon)f'(\phi) \\ &> g'(f(\phi))f'(\phi) \\ &= 1. \end{aligned}$$

(3) Since $\dfrac{d(F_m(\phi) - \phi)}{d\phi} > 0$, we can get the result.

(4) By (3), we can get the result.

$\square$

Lemma 1.2. *For the Peskin's model,*

$$\frac{dx}{dt} = -rx + I$$

where $I > r > 0$. *The evolution map* f *and its inverse function* g *are given in the following, respectively*

$$f(\phi) = \frac{I}{r} - \frac{I}{r}\left(\frac{I-r}{I}\right)^{\phi},$$

$$g(x) = \frac{\ln\left(\dfrac{I - rx}{I}\right)}{\ln\left(\dfrac{I-r}{I}\right)}.$$

Definition 1.3.
$S_1 = \{(\phi_1, \phi_2, \phi_3, \phi_4)$: *three neurons having initial state zero*$\}$.
$S_2 = \{(\phi_1, \phi_2, \phi_3, \phi_4)$: *two neurons having initial state zero*$\}$.
$S_3 = \{(\phi_1, \phi_2, \phi_3, \phi_4)$: *one neuron having initial state zero*$\}$.

Definition 1.4.

$N_1 = \{\phi \in S_1$: *after iterations, the states never reach synchronization*$\}$.

$N_2 = \{\phi \in S_2$: *after iterations, the states never reach synchronization*$\}$.

$N_3 = \{\phi \in S_3$: *after iterations, the states never reach synchronization*$\}$.

2. Three Neurons Having Initial State Zero

Lemma 2.1.

(1) $F_1(1-\phi) = 1$ *if and only if* $\phi \le 1 - g(1-\epsilon) =: \phi_l$.

(2) $0 \le 2 - \phi - F_1(1-\phi) \le 1$ *for all* $\phi \in [0,1]$.

(3) *If* $F_1(1-\phi) < 1$, *then* $F_2(2-\phi-F_1(1-\phi)) = 1$.

Proof.

(1) $F_1(1-\phi) = 1$ if and only if $f(1-\phi)+\epsilon \ge 1$ if and only if $\phi \le 1 - g(1-\epsilon)$.

(2) $2 - \phi - F_1(1-\phi) = (1-\phi) + (1 - F_1(1-\phi)) \ge 0 + 0 = 0$. On the other hand, $2 - \phi - F_1(1-\phi) \le 1$ if and only if $\phi + F_1(1-\phi) \ge 1$. But $\phi + F_1(1-\phi) \ge \phi + (1-\phi) = 1$. Thus, $2 - \phi - F_1(1-\phi) \le 1$.

(3) Note that if $F_1(1-\phi) < 1$, then

$$\begin{aligned} F_2(2-\phi-F_1(1-\phi)) &= 1 \Leftrightarrow \\ F_2\left(2-\phi-g\left(f(1-\phi)+\epsilon\right)\right) &= 1 \Leftrightarrow \\ f\left(2-\phi-g\left(f(1-\phi)+\epsilon\right)\right) &\ge 1-2\epsilon. \end{aligned}$$

Let $E(\phi) = f\left(2-\phi-g\left(f(1-\phi)+\epsilon\right)\right),\ \forall \phi \in [\phi_l, 1]$. Then $E(\phi)$ is well-defined, and $0 < E(\phi) < 1$ for all $\phi \in [\phi_l, 1]$. Moreover,

$$\begin{aligned} E'(\phi) &= f'\left(2-\phi-g\left(f(1-\phi)+\epsilon\right)\right) \cdot \\ &\qquad \left[-1 + g'\left(f(1-\phi)+\epsilon\right) f'(1-\phi)\right] \\ &> f'\left(2-\phi-g\left(f(1-\phi)+\epsilon\right)\right) \cdot \\ &\qquad \left[-1 + g'\left(f(1-\phi)\right) f'(1-\phi)\right] = 0. \end{aligned}$$

It implies that $E(\phi)$ is increasing and $E(\phi) \ge E(\phi_l) = 1-\epsilon$ for all ϕ in the interval $[\phi_l, 1]$. That is, $f\left(2-\phi-g\left(f(1-\phi)+\epsilon\right)\right) \ge 1-\epsilon > 1-2\epsilon$ for all ϕ in the interval $(\phi_l, 1]$. Thus, the proof is completed.

□

Lemma 2.2. *There is some* ϕ *in* $[\phi_l, 1]$ *such that* $F_2(1-F_1(1-\phi)) < 1$ *if and only if* $\epsilon < \frac{1}{2}$.

Proof. As $\phi \in [\phi_l, 1]$, $F_2(1 - F_1(1 - \phi)) = g(\min[f(1 - g(f(1 - \phi) + \epsilon)) + 2\epsilon, 1])$.

($\Longrightarrow$) Suppose not, i.e., $\epsilon \geq \frac{1}{2}$. Then $f(1 - g(f(1 - \phi) + \epsilon)) + 2\epsilon \geq 1$. It follows that $F_2(1 - F_1(1 - \phi)) = 1$, a contradiction.

($\Longleftarrow$) If $\epsilon < \frac{1}{2}$. Then

$$\begin{aligned}
&F_2(1 - F_1(1 - \phi)) < 1 \\
\Leftrightarrow\ &f(1 - g(f(1 - \phi) + \epsilon)) + 2\epsilon < 1 \\
\Leftrightarrow\ &1 - g(f(1 - \phi) + \epsilon) < g(1 - 2\epsilon) \\
\Leftrightarrow\ &f(1 - g(1 - 2\epsilon)) < f(1 - \phi) + \epsilon \\
\Leftrightarrow\ &f(1 - g(1 - 2\epsilon)) - \epsilon < f(1 - \phi) \qquad (\dagger) \\
\Leftrightarrow\ &g(f(1 - g(1 - 2\epsilon)) - \epsilon) < 1 - \phi \\
\Leftrightarrow\ &\phi < 1 - g(f(1 - g(1 - 2\epsilon)) - \epsilon) =: \phi_r.
\end{aligned}$$

Thus for any ϕ in the interval $[\phi_l, \phi_r)$, $F_2(1 - F_1(1 - \phi)) < 1$. □

Remark 2.1. *(i)* $0 < f(1 - g(1 - 2\epsilon)) - \epsilon < 1$. *(ii)* $\phi_l < \phi_r$.

Proof. (i) $0 < f(1 - g(1 - 2\epsilon)) - \epsilon$ if and only if $g(1 - 2\epsilon) + g(\epsilon) < 1$. But, since function g is concave up, $g(1 - 2\epsilon) + g(\epsilon) < g(1 - \epsilon) + g(\epsilon) < 1$. Thus, the inequality $0 < f(1 - g(1 - 2\epsilon)) - \epsilon$ indeed holds. On the other hand, the inequality $f(1 - g(1 - 2\epsilon)) - \epsilon < 1$ clearly holds.
(ii) The proof of the inequality can be easily checked by definitions of ϕ_l and ϕ_r and is thus omitted. □

Corollary 2.1. *As* $\epsilon \geq \frac{1}{2}$, $F_2(1 - F_1(1 - \phi)) = 1$ *for all* $\phi \in [\phi_l, 1]$.

$$\text{As } \epsilon < \tfrac{1}{2},\ F_2(1 - F_1(1 - \phi)) \begin{cases} < 1 & \phi \in [\phi_l, \phi_r) \\ = 1 & \phi \in [\phi_r, 1] \end{cases}.$$

Lemma 2.3. *As* $\epsilon < \frac{1}{2}$, *let function* $Q(\phi) = F_2(1 - F_1(1 - \phi))$ *defined in the interval* $[\phi_l, \phi_r]$. *Then function* Q *is continuous in* $[\phi_l, \phi_r]$ *and differentiable in* (ϕ_l, ϕ_r). *Moreover,* $Q'(\phi) > 1$ *for all* $\phi \in (\phi_l, \phi_r)$.

Proof. From the Lemma 2.2, $Q(\phi) = g(f(1 - g(f(1 - \phi) + \epsilon)) + 2\epsilon)$. Hence function Q is clear continuous in $[\phi_l, \phi_r]$ and differentiable in (ϕ_l, ϕ_r). Moreover,

$$\begin{aligned}
Q'(\phi) &= [g'(f(1 - g(f(1 - \phi) + \epsilon)) + 2\epsilon) f'(1 - g(f(1 - \phi) + \epsilon))] \cdot \\
&\quad [g'(f(1 - \phi) + \epsilon) f'(1 - \phi)] \\
&> 1 \cdot 1 = 1.
\end{aligned}$$

for all ϕ in (ϕ_l, ϕ_r). □

Proposition 2.1. *(i) If $\epsilon < \frac{1}{2}$ and $g(1-\epsilon)+g(2\epsilon) \geq 1$, then $Q(\phi)$ has no intersection with the diagonal line in the interval (ϕ_l, ϕ_r). Moreover, $Q(\phi) > \phi$ and $|Q(Q(\phi)) - Q(\phi)| > |Q(\phi) - \phi|$ provided $Q(\phi) \in (\phi_l, \phi_r)$. (ii) If $\epsilon < \frac{1}{2}$ and $g(1-\epsilon) + g(2\epsilon) < 1$, then $Q(\phi)$ has unique one intersection point with the diagonal line, called such point p, in the interval (ϕ_l, ϕ_r). Moreover, $Q(\phi) > \phi$ for all $\phi \in (p, \phi_r)$ and $Q(\phi) < \phi$ for all $\phi \in (\phi_l, p)$, and $|Q(Q(\phi)) - Q(\phi)| > |Q(\phi) - \phi|$ provided $Q(\phi) \in (\phi_l, \phi_r)$.*

Proof. We would just give the proof of (ii) since the similarity of that of (i). If $g(1-\epsilon) + g(2\epsilon) < 1$, then it follows $Q(\phi_l) < \phi_l$. By the intermediate value theorem, there are intersection points in the interval (ϕ_l, ϕ_r) since $Q(\phi_r) = 1$. Furthermore, since $Q'(\phi) > 1$ for all points in the interval (ϕ_l, ϕ_r), by the inequality of the scalar ordinary differential equation, the intersection point is unique.

Let $d(\phi) = Q(\phi) - \phi$. Then $Q'(\phi) = Q'(\phi) - 1 > 0$, and $d(p) = Q(p) - p = 0$. Thus for any $\phi \in (p, \phi_r)$, $d(\phi) > d(p) = 0$, i.e., $Q(\phi) > \phi$. On the other hand, for any given ϕ_1, $\phi_2 \in (p, \phi_r)$ with $\phi_2 > \phi_1$, $d(\phi_2) > d(\phi_1)$. That is $Q(\phi_2) - \phi_2 > Q(\phi_1) - \phi_1$. Substituting ϕ_1 and ϕ_2 with ϕ and $Q(\phi)$, respectively, then we have the conclusion that $|Q(Q(\phi)) - Q(\phi)| > |Q(\phi) - \phi|$. □

Theorem 2.1. *If $\epsilon \geq \frac{1}{2}$, then synchronization occurs.*

Proof. **Case 1:** $\phi \in [0, \phi_l]$.

Time	1	2	3	4
0	0	0	0	ϕ
$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	1
$(1-\phi)^+$	$F_1(1-\phi)$	$(1-\phi)$	$F_1(1-\phi)$	1
$(1-\phi)^+$	1	$F_2(1-\phi)$	1	1
$(1-\phi)^+$	1	1	1	1

Case 2: $\phi \in (\phi_l, 1]$.

Time	1	2	3	4
0	0	0	0	ϕ
$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	1
$(1-\phi)^+$	$F_1(1-\phi)$	$(1-\phi)$	$F_1(1-\phi)$	1
$(1-F_1(1-\phi))$	1	$2-\phi-F_1(1-\phi)$	1	$1-F_1(1-\phi)$
$(1-F_1(1-\phi))^+$	1	$F_2(2-\phi-F_1(1-\phi))$	1	$F_2(1-F_1(1-\phi))$
$(1-F_1(1-\phi))^+$	1	1	1	1

Thus, synchronization occurs. □

Theorem 2.2. *If $\epsilon < \frac{1}{2}$ and $g(1-\epsilon) + g(2\epsilon) \geq 1$, then synchronization occurs.*

Proof. **Case 1:** $\phi \in [0, \phi_l]$.

Time	1	2	3	4
0	0	0	0	ϕ
$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	1
$(1-\phi)^+$	$F_1(1-\phi)$	$(1-\phi)$	$F_1(1-\phi)$	1
$(1-\phi)^+$	1	$F_2(1-\phi)$	1	1
$(1-\phi)^+$	1	1	1	1

Case 2: $\phi \in [\phi_r, 1]$.

Time	1	2	3	4
0	0	0	0	ϕ
$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	1
$(1-\phi)^+$	$F_1(1-\phi)$	$(1-\phi)$	$F_1(1-\phi)$	1
$(1-F_1(1-\phi))$	1	$2-\phi-F_1(1-\phi)$	1	$1-F_1(1-\phi)$
$(1-F_1(1-\phi))^+$	1	$F_2(2-\phi-F_1(1-\phi))$	1	$F_2(1-F_1(1-\phi))$
$(1-F_1(1-\phi))^+$	1	1	1	1

Case 3: $\phi \in (\phi_l, \phi_r)$.

Time	1	2	3	4
0	0	0	0	ϕ
$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	$(1-\phi)$	1
$(1-\phi)^+$	$F_1(1-\phi)$	$(1-\phi)$	$F_1(1-\phi)$	1
$(1-F_1(1-\phi))$	1	$2-\phi-F_1(1-\phi)$	1	$1-F_1(1-\phi)$
$(1-F_1(1-\phi))^+$	1	$F_2(2-\phi-F_1(1-\phi))$	1	$F_2(1-F_1(1-\phi))$
$(1-F_1(1-\phi))^+$	1	1	1	$F_2(1-F_1(1-\phi))$

It can be observed that if ϕ lies in the Case 1 or Case 2, then synchronization occurs. On the other hand, if ϕ lies in the Case 3, then from the Proposition 2.1 after some finite iterations, ϕ would lie in the Case 2 and then synchronization occurs. Hence, synchronization occurs in each case, and the proof is completed. □

Theorem 2.3. *If $\epsilon < \frac{1}{2}$ and $g(1-\epsilon)+g(2\epsilon) < 1$, then synchronization occurs except at $\phi = p$.*

Proof. The proof is similar as in the Theorem 2.2 and is thus omitted. Especially,

Time	1	2	3	4
0	0	0	0	p
$1-p$	$F_1(1-p)$	$(1-p)$	$F_1(1-p)$	0
$1-F_1(1-p)$	0	0	0	p

□

By above, we can get the following Theorem.

Theorem 2.4. *The set N_1 has measure zero in S_1.*

3. Two Neurons Having Initial State Zero

Without loss of generality, the initial states of the system of four neurons having two zeroes can be assumed to have the form $[0, 0, \phi_3^0, \phi_4^0]$ with $0 < \phi_3^0 \le \phi_4^0 < 1$, or $[0, \phi_2^0, 0, \phi_4^0]$ with $0 < \phi_2^0 \le \phi_4^0 < 1$. We discuss, respectively, the two cases in the subsections below.

Definition 3.1.
$S_{2,1} = \{(0, 0, \phi_3^0, \phi_4^0) : 0 < \phi_3^0 \le \phi_4^0 < 1\} \subseteq S_2$.
$S_{2,2} = \{(0, \phi_2^0, 0, \phi_4^0) : 0 < \phi_2^0 \le \phi_4^0 < 1\} \subseteq S_2$.
i.e., $S_2 = S_{2,1} \cup S_{2,1}$.

3.1. Initial data being $[0, 0, \phi_3^0, \phi_4^0]$ with $0 < \phi_3^0 \le \phi_4^0 < 1$.

First, we consider the simplest subcase, where initial data being $[0, 0, x, x]$. It is obvious that the first two neurons, and the last two neurons always stay together, respectively. Thus we can treat $[0, 0, x, x]$ as $[0, x]$ with coupling weights being double. Thus we can derive the conclusions directly below. It is well-known results [36].

Theorem 3.1. *Synchronization must occur for all initial data of the form $[0, 0, x, x]$ where $x \in [0, 1] - \{p\}$ and p satisfies $F_1(1 - F_1(1 - p)) = p$.*

Time	1	2	3	4
0	0	0	x	x
$(1-x)^+$	$F_1(1-x)$	$F_1(1-x)$	0	0
$(1-F_1(1-x))^+$	0	0	$F_1(1-F_1(1-x))$	$F_1(1-F_1(1-x))$

Next, we consider the case where initial data being $[0,0,\phi_3^0,\phi_4^0]$ with $0<\phi_3^0<\phi_4^0<1$. we separate the case into several subcases.

Case 1: $(0,0,\phi_3^0,\phi_4^0)\in S_{2,1},\phi_3^0<\phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,\phi_3^1,0)\to(\phi_1^2,\phi_2^2,0,\phi_4^2)\to(0,0,\phi_3^3,\phi_4^3)$, with $F_1(1-\phi_4^0+\phi_3^0)<1, F_1(1-\phi_3^1+\phi_2^1)<1$, and $F_1(1-\phi_2^2+\phi_4^2)<1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$1-\phi_4^0$	$F_1(1-\phi_4^0+\phi_3^0)$	0
$(1-\phi_3^1)^+$	$1-\phi_3^1+\phi_1^1$	$F_1(1-\phi_3^1+\phi_2^1)$	0	$F_1(1-\phi_3^1)$
$(1-\phi_2^2)^+$	$F_1(1-\phi_2^2+\phi_1^2)=1$	0	$F_1(1-\phi_2^2)$	$1-\phi_2^2+\phi_4^2$
	$\to 0$			$\to F_1(1-\phi_2^2+\phi_4^2)$

Remark 3.1. $F_1(1-\phi_3^1+\phi_2^1)>1-\phi_3^1+\phi_1^1$.

Proof.

$$\begin{aligned}
&F_1(1-\phi_3^1+\phi_2^1)-\phi_1^1\\
&=F_1(2-\phi_4^0-F_1(1-\phi_4^0+\phi_3^0))-F_1(1-\phi_4^0)\\
&>1-F_1(1-\phi_4^0+\phi_3^0)\ \ by\ Lemma 1.1(3)\\
&=1-\phi_3^1.
\end{aligned}$$

□

Remark 3.2. $F_1(1-\phi_2^2+\phi_1^2)=1$.

Proof.

$$\begin{aligned}
&F_1(1-\phi_2^2+\phi_1^2)\\
&=F_1(2+\phi_1^1-\phi_3^1-F_1(1-\phi_3^1+\phi_2^1))\\
&=F_1(2-(\phi_3^1-\phi_2^1)+(\phi_1^1-\phi_2^1)-F_1(1-(\phi_3^1-\phi_2^1)))\\
&\geq F_1(2-(\phi_3^1-\phi_2^1)-F_1(1-(\phi_3^1-\phi_2^1)))\\
&=1.
\end{aligned}$$

The last equality holds since $F_1(1-\phi_3^1+\phi_2^1)<1$ and by Lemma 2.1.(3). □

Definition 3.2. $[(\phi_1^0,\phi_2^0,\phi_3^0,\phi_4^0)\to\cdots\to(\phi_1^{n-1},\phi_2^{n-1},\phi_3^{n-1},\phi_4^{n-1})]\to$ *Repeat*[ ] *denotes the iteration form is* $(\phi_1^0,\phi_2^0,\phi_3^0,\phi_4^0)\to\cdots\to(\phi_1^{n-1},\phi_2^{n-1},\phi_3^{n-1},\phi_4^{n-1})\to(\phi_1^n,\phi_2^n,\phi_3^n,\phi_4^n)\to\cdots\to(\phi_1^{2n-1},\phi_2^{2n-1},\phi_3^{2n-1},\phi_4^{2n-1})\to\cdots\to(\phi_1^{kn},\phi_2^{kn},\phi_3^{kn},\phi_4^{kn})\to$

$(\phi_1^{kn+1}, \phi_2^{kn+1}, \phi_3^{kn+1}, \phi_4^{kn+1}) \to \cdots$. *Moreover,* ϕ_i^{kn+j} *satisfy*

$$\phi_i^{kn+j} \begin{cases} = 0 & \text{if } \phi_i^j = 0 \\ \neq 0 & \text{if } \phi_i^j \neq 0 \end{cases},$$

for all $i = 1, \ldots, 4$, $j = 0, \ldots, n-1$, *and* $k \in \mathbb{N}$.

Definition 3.3. *Let* $B_1 = \{(0,0,\phi_3^0,\phi_4^0) \in S_{2,1} : \phi_3^0 < \phi_4^0$, *and iteration form is* $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (\phi_1^2,\phi_2^2,0,\phi_4^2) \to (0,0,\phi_3^3,\phi_4^3)\}$, *and* $A = \{(0,0,\phi_3^0,\phi_4^0) \in S_{2,1} : \phi_3^0 < \phi_4^0$, *and iteration form is* $[(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (\phi_1^2,\phi_2^2,0,\phi_4^2)] \to$ *Repeat* $[\quad]\}$.

Definition 3.4. $A_i = \{\phi \in S_{2,1},$ *s.t.* $\phi \in B_1, R(\phi) \in B_1, \cdots, R^{i-1}(\phi) \in B_1\}$, *where* $i \in \{0\} \cup \mathbb{N}$ *and* $R = h_3h_2h_1$.

Remark 3.3. *A is the set of initial conditions that live forever in Case1, i.e., without any absorptions in Case1. Then,* $A = \bigcap_{i=1}^{\infty} A_i$.

Theorem 3.2. *The set A has measure zero in* S_2.

Proof. A is measurable since it is a countable intersection of open sets. (In fact, A is closed.)

Consider the return map

$$R = h_3h_2h_1, i.e., R(0,0,\phi_3^0,\phi_4^0) = (0,0,F_1(1-\phi_2^2), F_1(1-\phi_2^2+\phi_4^2)).$$

, map $r(\phi_3^0,\phi_4^0) = (F_1(1-\phi_2^2), F_1(1-\phi_2^2+\phi_4^2))$, and $A' = \{(\phi_3,\phi_4) : (0,0,\phi_3,\phi_4) \in A\}$. Then A is invariant under the map R, and A' is invariant under the map r, i.e.,

$$r(A') \subset A'.$$

r is also one to one, and the Jacobian determinant of r has absolute value greater than one.

Now, suppose $m(A') > 0$, we have $m(r(A')) = \int_{A'} |detJ| dx > \int_{A'} dx = m(A')$. Since $r(A') \subset A', m(r(A')) \leq m(A')$. It is a contradiction, hence, $m(A') = 0$, therefore, $m(A) = 0$. □

Case 2: $(0,0,\phi_3^0,\phi_4^0) \in S_{2,1}, \phi_3^0 < \phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (\phi_1^2,\phi_2^2,0,\phi_4^2) \to (0,0,\phi_3^3,0)$, with $F_1(1-\phi_4^0+\phi_3^0) < 1, F_1(1-\phi_3^1+\phi_2^1) < 1, F_1(1-\phi_2^2+\phi_4^2) = 1$, and $F_2(1-\phi_2^2) < 1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$1-\phi_4^0$	$F_1(1-\phi_4^0+\phi_3^0)$	0
$(1-\phi_3^1)^+$	$1-\phi_3^1+\phi_1^1$	$F_1(1-\phi_3^1+\phi_2^1)$	0	$F_1(1-\phi_3^1)$
$(1-\phi_2^2)^+$	0	0	$F_1(1-\phi_2^2)$	$F_1(1-\phi_2^2+\phi_4^2)=1$
			$\to F_2(1-\phi_2^2)$	$\to 0$

Definition 3.5. *Let* $B_2 = \{(0,0,\phi_3^0,\phi_4^0) \in S_{2,1} : \phi_3^0 < \phi_4^0$*, and iteration form is* $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (\phi_1^2,\phi_2^2,0,\phi_4^2) \to (0,0,\phi_3^3,0)\}$*, and* $B_2' = \{(0,0,\phi_3^0,\phi_4^0) \in S_{2,1} : \phi_3^0 < \phi_4^0$*, and iteration form is* $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (\phi_1^2,\phi_2^2,0,\phi_4^2) \to [(0,0,\phi_3^3,0) \to (\phi_1^4,\phi_2^4,0,\phi_4^4)] \to$ *Repeat*$[\quad]\}$.

Theorem 3.3. *The set* B_2' *has measure zero in* S_2.

Proof. Let $C_1 = \{\phi_3^3 : (0,0,\phi_3^3,0) = h_3h_2h_1(\phi)$ for some $\phi \in B_2'\}$, then C_1 has measure zero in $\mathbb{R}$ since the set N_1 has measure zero in S_1. By the fact that function $y = F_2(1-x)$ is diffeomorphism, set $D_1 = \{(\alpha_1,\phi_2^2,\alpha_2) : \phi_2^2$ satisfies $F_2(1-\phi_2^2) = \phi_3^3$ for some $\phi_3^3 \in C_1, 0 \le \alpha_1,\alpha_2 \le 1\}$ has measure zero in $\mathbb{R}^3$, it follows $C_2 = \{(\phi_1^2,\phi_2^2,\phi_4^2) : (\phi_1^2,\phi_2^2,0,\phi_4^2) = h_2h_1(\phi)$ for some $\phi \in B_2'\}$ has measure zero in $\mathbb{R}^3$. Similarly, since h_2 and h_1 is diffeomorphism, set $C_3 = \{(\phi_1^1,\phi_2^1,\phi_3^1) : (\phi_1^1,\phi_2^1,\phi_3^1,0) = h_1(\phi)$ for some $\phi \in B_2'\}$ and then $C_4 = \{(\phi_3^0,\phi_4^0) : (0,0,\phi_3^0,\phi_4^0) \in B_2'\}$ has measure zero in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Hence, B_2' has measure zero in $\mathbb{R}^4$ or S_2. □

Case 3: $(0,0,\phi_3^0,\phi_4^0) \in S_{2,1}, \phi_3^0 < \phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (\phi_1^2,\phi_2^2,0,\phi_4^2) \to (0,0,0,0)$, with $F_1(1-\phi_4^0+\phi_3^0) < 1, F_1(1-\phi_3^1+\phi_2^1) < 1, F_1(1-\phi_2^2+\phi_4^2) = 1$, and $F_2(1-\phi_2^2) = 1$ or $F_1(1-\phi_2^2) = 1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$1-\phi_4^0$	$F_1(1-\phi_4^0+\phi_3^0)$	0
$(1-\phi_3^1)^+$	$1-\phi_3^1+\phi_1^1$	$F_1(1-\phi_3^1+\phi_2^1)$	0	$F_1(1-\phi_3^1)$
$(1-\phi_2^2)^+$	0	0	0	0

Thus, synchronization occurs in this case.

Case 4: $(0,0,\phi_3^0,\phi_4^0) \in S_{2,1}, \phi_3^0 < \phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,\phi_3^1,0) \to (0,0,0,\phi_4^2)$, with $F_1(1-\phi_4^0+\phi_3^0) < 1, F_1(1-\phi_3^1+\phi_2^1) = 1$, and $F_2(1-\phi_3^1) < 1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$1-\phi_4^0$	$F_1(1-\phi_4^0+\phi_3^0)$	0
$(1-\phi_3^1)^+$	$1-\phi_3^1+\phi_1^1$	$F_1(1-\phi_3^1+\phi_2^1)=1$	0	$F_1(1-\phi_3^1)$
	$\to F_1(1-\phi_3^1+\phi_1^1)=1$	$\to 0$		$\to F_1(1-\phi_3^1)$
	$\to 0$			$\to F_2(1-\phi_3^1)$

Remark 3.4. *If $F_1(1-\phi_3^1+\phi_2^1)=1$, then $F_1(1-\phi_3^1+\phi_1^1)=1$.*

Proof. Since $1=F_1(1-\phi_3^1+\phi_2^1)\leq F_1(1-\phi_3^1+\phi_1^1)$, we can get the result. □

Definition 3.6. *Let $B_4=\{(0,0,\phi_3^0,\phi_4^0)\in S_{2,1}:\phi_3^0<\phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,\phi_3^1,0)\to(0,0,0,\phi_4^2)\}$, and $B_4'=\{(0,0,\phi_3^0,\phi_4^0)\in S_{2,1}:$ $\phi_3^0<\phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,\phi_3^1,0)\to[(0,0,0,\phi_4^2)\to$ $(\phi_1^3,\phi_2^3,\phi_3^3,0)]\to$ Repeat[$\,$]$\}$.*

Theorem 3.4. *The set B_4' has measure zero in S_2.*

Proof. Let $C_1=\{\phi_4^2:(0,0,0,\phi_4^2)=h_2h_1(\phi)$ for some $\phi\in B_4'\}$, then C_1 has measure zero in $\mathbb{R}$ since the set N_1 has measure zero in S_1. By the fact that function $y=F_2(1-x)$ is diffeomorphism, set $D_1=\{(\alpha_1,\alpha_2,\phi_3^1):\phi_3^1$ satisfies $F_2(1-\phi_3^1)=\phi_4^2$ for some $\phi_4^2\in C_1, 0\leq\alpha_1,\alpha_2\leq 1\}$ has measure zero in $\mathbb{R}^3$, it follows $C_2=\{(\phi_1^1,\phi_2^1,\phi_3^1):(\phi_1^1,\phi_2^1,\phi_3^1,0)=h_1(\phi)$ for some $\phi\in B_4'\}$ has measure zero in $\mathbb{R}^3$. Similarly, since h_1 is diffeomorphism, set $C_3=\{(\phi_3^0,\phi_4^0):(0,0,\phi_3^0,\phi_4^0)\in B_4'\}$ has measure zero in $\mathbb{R}^2$. Hence, the set B_4' has measure zero in S_2. □

Case 5: $(0,0,\phi_3^0,\phi_4^0)\in S_{2,1},\phi_3^0<\phi_4^0$, and iteration form is $(0,0,\phi_3^0,\phi_4^0)\to$ $(\phi_1^1,\phi_2^1,\phi_3^1,0)\to(0,0,0,0)$, with $F_1(1-\phi_4^0+\phi_3^0)<1, F_1(1-\phi_3^1+\phi_2^1)=1$, and $F_2(1-\phi_3^1)=1$ or $F_1(1-\phi_3^1)=1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$1-\phi_4^0$	$F_1(\phi_3^0+1-\phi_4^0)$	0
$(1-F_1(\phi_3^0+1-\phi_4^0))^+$	0	0	0	0

Thus, synchronization occurs in this case.

Case 6: $(0,0,\phi_3^0,\phi_4^0)\in S_{2,1},\phi_3^0<\phi_4^0$, and the iteration form is $(0,0,\phi_3^0,\phi_4^0)\to$ $(\phi_1^1,\phi_1^1,0,0)\to(0,0,\phi_3^2,\phi_3^2)$, with $F_1(1-\phi_4^0+\phi_3^0)=1, F_1(1-\phi_4^0)<1$, and $F_1(1-\phi_1^1)<1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$F_1(1-\phi_4^0)$	0	0
$(1-F_1(\phi_3^0+1-\phi_4^0))^+$	0	0	$F_1(1-\phi_1^1)$	$F_1(1-\phi_1^1)$

Definition 3.7. *Let* $B_6 = \{(0,0,\phi_3^0,\phi_4^0) \in S_{2,1} : \phi_3^0 < \phi_4^0$, *and the iteration form is* $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_1^1,0,0) \to (0,0,\phi_3^2,\phi_3^2)\}$, *and* $B_6' = \{(0,0,\phi_3^0,\phi_4^0) \in S_{2,1} : \phi_3^0 < \phi_4^0$, *and the iteration form is* $(0,0,\phi_3^0,\phi_4^0) \to [(\phi_1^1,\phi_1^1,0,0) \to (0,0,\phi_3^2,\phi_3^2)] \to$ *Repeat* $[\quad]\}$.

Theorem 3.5. *the set* B_6' *has measure zero in* S_2.

Proof. Let $C_1 = \{\phi_1^1 : (\phi_1^1,\phi_1^1,0,0) = h_1(\phi)$ for some $\phi \in B_6'\}$, since Theorem 3.1. tells us that the set $\{(0,0,x,x)\}$ must synchronization for all initial data, where $x \in [0,1]-\{p\}$ and p satisfies $F_1(1-F_1(1-p)) = p$, then C_1 has measure zero in $\mathbb{R}$. By the fact that function $y = F_1(1-x)$ is diffeomorphism, set $D_1 = \{(\alpha,\phi_4^0) : \phi_4^0$ satisfies $F_1(1-\phi_4^0) = \phi_1^1$ for some $\phi_1^1 \in C_1, 0 \le \alpha \le 1\}$ has measure zero in $\mathbb{R}^2$, it follows $C_2 = \{(\phi_3^0,\phi_4^0) : (0,0,\phi_3^0,\phi_4^0) \in B_6'\}$ has measure zero in $\mathbb{R}^2$. Hence, the set B_6' has measure zero in $\mathbb{R}^4$ or S_2. □

Case 7: $(0,0,\phi_3^0,\phi_4^0) \in S_{2,1}, \phi_3^0 < \phi_4^0$, and the iteration form is $(0,0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,0) \to (0,0,0,0)$, with $F_1(1-\phi_4^0+\phi_3^0) = 1, F_1(1-\phi_4^0) < 1$, and $F_1(1-\phi_1^1) = 1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	$F_1(1-\phi_4^0)$	$F_1(1-\phi_4^0)$	0	0
$(1-F_1(\phi_3^0+1-\phi_4^0))^+$	0	0	0	0

Thus, synchronization occurs in this case.

Case 8: $(0,0,\phi_3^0,\phi_4^0) \in S_{2,1}, \phi_3^0 < \phi_4^0$, and the iteration form is $(0,0,\phi_3^0,\phi_4^0) \to (0,0,0,0)$, with $F_1(1-\phi_4^0) = 1$.

Time	1	2	3	4
0	0	0	ϕ_3^0	ϕ_4^0
$(1-\phi_4^0)^+$	0	0	0	0

Thus, synchronization occurs in this case.

Definition 3.8. $B_{0,1} = B_2' \cup B_4' \cup B_6'$,
$B_{1,1} = \{\phi \in B_1 : R(\phi) \in B_{0,1}\}$,
$B_{2,1} = \{\phi \in B_1 : R(\phi) \in B_{1,1}\}$,
$\vdots$
$B_{k,1} = \{\phi \in B_1 : R(\phi) \in B_{k-1,1}\}$, *where* R *is defined in Theorem 3.2.*.

Lemma 3.1. *The set $B_{k,1}, \forall k$ has measure zero in S_2.*

Proof. Since $B_{0,1} = B'_2 \cup B'_4 \cup B'_6$, the set $B_{0,1}$ has measure zero.

Now, let $(0,0,\phi_3^0,\phi_4^0) \in B_{1,1}$, by Theorem 3.2., or Case 1, we have

$$R(0,0,\phi_3^0,\phi_4^0) = (0,0,F_1(1-\phi_2^2), F_1(1-\phi_2^2+\phi_4^2)).$$

Since $R: B_1 \to R(B_1)$ is diffeomorphism, and $B_{0,1}$ has measure zero, the set $B_{1,1}$ has measure zero. By same way, the set $(B_{2,1}) = 0$ has measure zero in S_2 since $B_{1,1}$ has measure zero and R is diffeomorphism.

By induction, for any $B_{k,1}$, we have $B_{k,1} = \{\phi \in B_1 : R(\phi) \in B_{k-1,1}\}$ has measure zero in S_2. □

Definition 3.9. *Let $N_{2,1}$ =$\{\phi \in S_{2,1}$: after iterations, the states never reach synchronization.$\}$, i.e.,*
$N_{2,1} = A \cup B_{0,1} \cup B_{1,1} \cup B_{2,1} \cup \cdots = A \cup \bigcup_{k\geq 0} B_{k,1}$.

By above, we can get the following Theorem.

Theorem 3.6. *The set $N_{2,1}$ has measure zero in S_2.*

3.2. **Initial data being $[0,\phi_2^0,0,\phi_4^0]$ with $0 < \phi_2^0 \leq \phi_4^0 < 1$.** We consider the case where initial data being $[0,\phi_2^0,0,\phi_4^0]$ with $0 < \phi_2^0 \leq \phi_4^0 < 1$. Similar, separate the case into several subcases, and by same way, we get the following Theorem.

Theorem 3.7. *The set $N_{2,2} = \{\phi \in S_{2,2}$: after iterations, the states never reach synchronization.$\}$ has measure zero in S_2.*

Theorem 3.8. *The set N_2 has measure zero in S_2.*

Proof. Since $N_2 = N_{2,1} \cup N_{2,2}$, we can get the result. □

4. One Neuron Having Initial State Zero

Without loss of generality, the initial states of the system of four neurons having one zero can be assumed to have the form $[0,\phi_2^0,\phi_3^0,\phi_4^0]$ with $0 < \phi_2^0 \leq \phi_4^0 \leq \phi_3^0 < 1, 0 < \phi_2^0 \leq \phi_3^0 \leq \phi_4^0 < 1$, or $0 < \phi_3^0 \leq \phi_2^0 \leq \phi_4^0 < 1$. We discuss, respectively, the three cases below.

Remark 4.1. *Consider the initial state $[0,\phi_2^0,\phi_3^0,\phi_4^0]$ with $0 < \phi_2^0 \leq \phi_4^0 \leq \phi_3^0 < 1$.*

(1) *If $\phi_2^0 = \phi_3^0 = \phi_4^0$, and no absorption occurs, then the iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,0,0,0)$, we can regard it as the caes "3 neurons having initial state zero".*

(2) *If* $\phi_3^0 = \phi_4^0 \neq \phi_2^0$*, and no absorption occurs, then the iteration form is* $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,0)$*, we can regard it as the caes " 2 neurons having initial state zero".*

(3) *If* $\phi_2^0 = \phi_4^0 \neq \phi_3^0$*, and no absorption occurs, then the iteration form is* $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_2^1) \to (\phi_1^1,0,\phi_3^1,0)$*, we can regard it as the caes " 2 neurons having initial state zero".*

Hence, we just consider the initial state $[0,\phi_2^0,\phi_3^0,\phi_4^0]$ *with* $0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1, 0 < \phi_2^0 < \phi_3^0 < \phi_4^0 < 1$ *, or* $0 < \phi_3^0 < \phi_2^0 < \phi_4^0 < 1$.

Definition 4.1.
$S_{3,1} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1\} \subseteq S_3$.
$S_{3,2} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) : 0 < \phi_3^0 < \phi_2^0 < \phi_4^0 < 1\} \subseteq S_3$.
$S_{3,3} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) : 0 < \phi_2^0 < \phi_3^0 < \phi_4^0 < 1\} \subseteq S_3$.
i.e., $S_3 = S_{3,1} \cup S_{3,2} \cup S_{3,3}$.

4.1. **Initial data being $[0,\phi_2^0,\phi_3^0,\phi_4^0]$ with $0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$.** First, we consider the case where initial data being $[0,\phi_2^0,\phi_3^0,\phi_4^0]$ with $0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$. we separate the case into several subcases.

Case 1: $(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (\phi_1^3,0,\phi_3^3,\phi_4^3) \to (0,\phi_2^4,\phi_3^4,\phi_4^4)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, \phi_1^3 < 1$, and $\phi_3^4 > \phi_4^4$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	$F_1(1+\phi_1^2-\phi_2^2)$	0	$F_1(1+\phi_3^2-\phi_2^2)$	$1-\phi_2^2$
$(1-\phi_1^3)^+$	0	$F_1(1-\phi_1^3)$	$1+\phi_3^3-\phi_1^3$	$F_1(1+\phi_4^3-\phi_1^3)$

Remark 4.2. $\phi_2^2 < 1$*, and* $\phi_3^4 < 1$.

Definition 4.2. *Let* $B_1 = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$*, and iteration form is* $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (\phi_1^3,0,\phi_3^3,\phi_4^3) \to (0,\phi_2^4,\phi_3^4,\phi_4^4)\}$*, and* $A = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$*, and iteration form is* $[(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (\phi_1^3,0,\phi_3^3,\phi_4^3)] \to$ *Repeat* $[\quad]\}$.

Theorem 4.1. *The set* A *has measure zero in* S_3.

Proof. The proof is similar as that in Theorem 3.2., thus the proof is omitted. □

Case 2: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4) \to (\phi_1^5, \phi_2^5, 0, 0)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, \phi_1^3 < 1$, $0 < \phi_3^4 < \phi_4^4 < 1$, and $F_1(1 + \phi_2^4 - \phi_4^4) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	$F_1(1+\phi_1^2-\phi_2^2)$	0	$F_1(1+\phi_3^2-\phi_2^2)$	$1-\phi_2^2$
$(1-\phi_1^3)^+$	0	$F_1(1-\phi_1^3)$	$1+\phi_3^3-\phi_1^3$	$F_1(1+\phi_4^3-\phi_1^3)$
$(1-\phi_4^4)^+$	$F_1(1-\phi_4^4)$	$1+\phi_2^4-\phi_4^4$	$F_1(1+\phi_3^4-\phi_4^4)=1$	0
		$\to F_1(1+\phi_2^4-\phi_4^4)$	$\to 0$	

Remark 4.3. $F_1(1 + \phi_3^4 - \phi_4^4) = 1$.

Proof.

$$\begin{aligned} F_1(1+\phi_3^4-\phi_4^4) &= F_1(2+\phi_3^3-\phi_1^3-F_1(1+\phi_4^3-\phi_1^3)) \\ &= F_1(2-(\phi_1^3-\phi_4^3)+(\phi_3^3-\phi_4^3)-F_1(1-(\phi_1^3-\phi_4^3))) \\ &\geq F_1(2-(\phi_1^3-\phi_4^3)-F_1(1-(\phi_1^3-\phi_4^3)) \\ &= 1. \end{aligned}$$

The last equality holds since $F_1(1-(\phi_1^3-\phi_4^3)) < 1$ and by Lemma 2.1.(3). □

Remark 4.4. $\phi_2^2 < 1$.

Definition 4.3. *Let* $B_2 = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4) \to (\phi_1^5, \phi_2^5, 0, 0)\}$, *and* $B_2' = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4) \to [(\phi_1^5, \phi_2^5, 0, 0) \to (\phi_1^6, 0, \phi_3^6, \phi_4^6) \to (0, \phi_2^7, \phi_3^7, \phi_4^7)] \to$

Repeat [ ]}.

Theorem 4.2. *The set* B_2' *has measure zero in* S_3.

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 3: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4) \to (\phi_1^5, 0, 0, 0)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, \phi_1^3 < 1, 0 < \phi_3^4 < \phi_4^4 < 1, F_1(1+\phi_2^4-\phi_4^4) = 1$, and $F_2(1-\phi_4^4) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	$F_1(1+\phi_1^2-\phi_2^2)$	0	$F_1(1+\phi_3^2-\phi_2^2)$	$1-\phi_2^2$
$(1-\phi_1^3)^+$	0	$F_1(1-\phi_1^3)$	$1+\phi_3^3-\phi_1^3$	$F_1(1+\phi_4^3-\phi_1^3)$
$(1-\phi_4^4)^+$	$F_2(1-\phi_4^4)$	0	0	0

Remark 4.5. $\phi_2^2 < 1$.

Definition 4.4. *Let* $B_3 = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4), \to (\phi_1^5, 0, 0, 0)\}$, *and* $B_3' = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4) \to [(\phi_1^5, 0, 0, 0) \to (0, \phi_2^6, \phi_3^6, \phi_4^6)] \to$ *Repeat* [ ]$\}$.

Theorem 4.3. *The set* B_3' *has measure zero in* S_3.

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 4: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, \phi_3^4, \phi_4^4) \to (0, 0, 0, 0)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, \phi_1^3 < 1, 0 < \phi_3^4 < \phi_4^4 < 1, F_1(1+\phi_2^4-\phi_4^4) = 1$, and $F_2(1-\phi_4^4) = 1$ or $F_1(1-\phi_4^4) = 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	$F_1(1+\phi_1^2-\phi_2^2)$	0	$F_1(1+\phi_3^2-\phi_2^2)$	$1-\phi_2^2$
$(1-\phi_1^3)^+$	0	$F_1(1-\phi_1^3)$	$1+\phi_3^3-\phi_1^3$	$F_1(1+\phi_4^3-\phi_1^3)$
$(1-\phi_4^4)^+$	0	0	0	0

Remark 4.6. $\phi_2^2 < 1$.

Thus, synchronization occurs in this case.

Case 5: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, 0, 0)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, \phi_1^3 < 1, 1 + \phi_3^3 - \phi_1^3 < F_1(1 + \phi_4^3 - \phi_1^3) = 1$, and $F_2(1 - \phi_1^3) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	$F_1(1+\phi_1^2-\phi_2^2)$	0	$F_1(1+\phi_3^2-\phi_2^2)$	$1-\phi_2^2$
$(1-\phi_1^3)^+$	0	$F_1(1-\phi_1^3)$	$1+\phi_3^3-\phi_1^3$	$F_1(1+\phi_4^3-\phi_1^3)=1$
		$\to F_1(1-\phi_1^3)$	$\to F_1(1+\phi_3^3-\phi_1^3)=1$	$\to 0$
		$\to F_2(1-\phi_1^3)$	$\to 0$	

Remark 4.7. *If* $F_1(1 + \phi_4^3 - \phi_1^3) = 1$, *then* $F_1(1 + \phi_3^3 - \phi_1^3) = 1$.

Proof. Since $\phi_4^3 < \phi_3^3$, we can get the result. □

Remark 4.8. $\phi_2^2 < 1$.

Definition 4.5. *Let* $B_5 = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, \phi_2^4, 0, 0)\}$, *and* $B_5' = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to [(0, \phi_2^4, 0, 0) \to (\phi_1^5, 0, \phi_3^5, \phi_4^5)] \to$ *Repeat* $[\ \ ]\}$.

Theorem 4.4. *The set* B_5' *has measure zero in* S_3.

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 6: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (\phi_1^3, 0, \phi_3^3, \phi_4^3) \to (0, 0, 0, 0)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, \phi_1^3 < 1, 1 + \phi_3^3 - \phi_1^3 < F_1(1 + \phi_4^3 - \phi_1^3) = 1$, and $F_2(1 - \phi_1^3) = 1$ or $F_1(1 - \phi_1^3) = 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	$F_1(1+\phi_1^2-\phi_2^2)$	0	$F_1(1+\phi_3^2-\phi_2^2)$	$1-\phi_2^2$
$(1-\phi_1^3)^+$	0	0	0	0

Remark 4.9. $\phi_2^2 < 1$.

Thus, synchronization occurs in this case.

Case 7: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (0, 0, \phi_3^3, \phi_4^3)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, F_1(1 + \phi_1^2 - \phi_2^2) = 1$, and $F_1(1 + \phi_3^2 - \phi_2^2) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1 - \phi_3^0)^+$	$1 - \phi_3^0$	$F_1(1 + \phi_2^0 - \phi_3^0)$	0	$F_1(1 + \phi_4^0 - \phi_3^0)$
$(1 - \phi_1^4)^+$	$F_1(1 + \phi_1^1 - \phi_4^1)$	$1 + \phi_2^1 - \phi_4^1$	$F_1(1 - \phi_4^1)$	0
$(1 - \phi_2^2)^+$	$F_1(1 + \phi_1^2 - \phi_2^2) = 1$	0	$F_1(1 + \phi_3^2 - \phi_2^2)$	$1 - \phi_2^2$
	$\to 0$			$\to F_1(1 - \phi_2^2)$

Remark 4.10. $\phi_2^2 < 1$.

Definition 4.6. *Let* $B_7 = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (0, 0, \phi_3^3, \phi_4^3)\}$, *and* $B_7' = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to [(0, 0, \phi_3^3, \phi_4^3) \to (\phi_1^4, \phi_2^4, 0, \phi_4^4) \to (\phi_1^5, \phi_2^5, \phi_3^5, 0)] \to$ *Repeat* $[\quad]\}$.

Theorem 4.5. *The set* B_7' *has measure zero in* S_3.

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 8: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (0, 0, 0, \phi_4^3)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, F_1(1 + \phi_3^2 - \phi_2^2) = 1$, and $F_2(1 - \phi_2^2) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1 - \phi_3^0)^+$	$1 - \phi_3^0$	$F_1(1 + \phi_2^0 - \phi_3^0)$	0	$F_1(1 + \phi_4^0 - \phi_3^0)$
$(1 - \phi_1^4)^+$	$F_1(1 + \phi_1^1 - \phi_4^1)$	$1 + \phi_2^1 - \phi_4^1$	$F_1(1 - \phi_4^1)$	0
$(1 - \phi_2^2)^+$	0	0	0	$F_2(1 - \phi_2^2)$

Remark 4.11. $\phi_2^2 < 1$.

Definition 4.7. *Let* $B_8 = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to (0, 0, 0, \phi_4^3)\}$, *and* $B_8' = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, *and iteration form is* $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, \phi_2^1, 0, \phi_4^1) \to (\phi_1^2, \phi_2^2, \phi_3^2, 0) \to [(0, 0, 0, \phi_4^3) \to (\phi_1^4, \phi_2^4, \phi_3^4, 0)] \to$ *Repeat* $[\quad]\}$.

Theorem 4.6. *The set B'_8 has measure zero in S_3.*

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 9: $(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (0,0,0,0)$, with $\phi_4^1 < 1, \phi_1^2 < \phi_2^2, F_1(1+\phi_3^2-\phi_2^2) = 1$, and $F_2(1-\phi_2^2) = 1$ or $F_1(1-\phi_2^2) = 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_1^4)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_2^2)^+$	0	0	0	0

Remark 4.12. $\phi_2^2 < 1$.

Thus, synchronization occurs in this case.

Case 10: $(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is$(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (0,0,\phi_3^3,\phi_4^3)$, with $\phi_4^1 < 1, 1 > \phi_1^2 > \phi_2^2$, and $F_1(1+\phi_3^2-\phi_1^2) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_1^2)^+$	0	$F_1(1+\phi_2^2-\phi_1^2)=1$	$1+\phi_3^2-\phi_1^2$	$F_1(1-\phi_1^2)$
		$\to 0$	$\to F_1(1+\phi_3^2-\phi_1^2)$	

Remark 4.13. $F_1(1+\phi_2^2-\phi_1^2) = 1$.

Proof.

$$\begin{aligned} F_1(1+\phi_2^2-\phi_1^2) &= F_1(2+\phi_2^1-\phi_4^1-F_1(1+\phi_1^1-\phi_4^1)) \\ &= F_1(2-(\phi_4^1-\phi_1^1)+(\phi_2^1-\phi_1^1)-F_1(1-(\phi_4^1-\phi_1^1))) \\ &\geq F_1(2-(\phi_4^1-\phi_1^1)-F_1(1-(\phi_4^1-\phi_1^1))) \\ &= 1. \end{aligned}$$

The last equality holds since $F_1(1-(\phi_4^1-\phi_1^1)) < 1$ and by Lemma 2.1.(3). □

Definition 4.8. *Let $B_{10} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (0,0,\phi_3^3,\phi_4^3)\}$, and $B'_{10} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is*

$(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to [(0,0,\phi_3^3,\phi_4^3) \to (\phi_1^4,\phi_2^4,0,\phi_4^4) \to (\phi_1^5,\phi_2^5,\phi_3^5,0)] \to$ *Repeat*[ ]}.

Theorem 4.7. *The set B'_{10} has measure zero in S_3.*

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 11: $(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{3,1} with 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is$(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (0,0,0,\phi_4^3)$, with $\phi_4^1 < 1, 1 > \phi_1^2 > \phi_2^2, F_1(1+\phi_3^2-\phi_1^2) = 1$, and $F_2(1-\phi_1^2) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_1^2)^+$	0	0	0	$F_2(1-\phi_1^2)$

Definition 4.9. *Let $B_{11} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (0,0,0,\phi_4^3)\}$, and $B'_{11} = \{(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{2,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to [(0,0,0,\phi_4^3) \to (\phi_1^4,\phi_2^4,\phi_3^4,0)] \to$ Repeat[ ]}.*

Theorem 4.8. *The set B'_{11} has measure zero in S_3.*

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 12: $(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{3,1}$ with $0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is$(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (\phi_1^2,\phi_2^2,\phi_3^2,0) \to (0,0,0,0)$, with $\phi_4^1 < 1, 1 > \phi_1^2 > \phi_2^2, F_1(1+\phi_3^2-\phi_1^2) = 1$, and $F_2(1-\phi_1^2) = 1$ or $F_1(1-\phi_1^2) = 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
$(1-\phi_1^2)^+$	0	0	0	0

Thus, synchronization occurs in this case.

Case 13: $(0,\phi_2^0,\phi_3^0,\phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is$(0,\phi_2^0,\phi_3^0,\phi_4^0) \to (\phi_1^1,\phi_2^1,0,\phi_4^1) \to (0,0,\phi_3^2,0)$, with $\phi_4^1 < 1, 1 = F_1(1+\phi_1^1-\phi_4^1) > 1+\phi_2^1-\phi_4^1$, and $F_2(1-\phi_4^1) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	$F_1(1+\phi_1^1-\phi_4^1)=1$	$1+\phi_2^1-\phi_4^1$	$F_1(1-\phi_4^1)$	0
	$\to 0$	$\to F_1(1+\phi_2^1-\phi_4^1)=1$	$\to F_1(1-\phi_4^1)$	
		$\to 0$	$\to F_2(1-\phi_4^1)$	

Remark 4.14. *If $F_1(1+\phi_1^1-\phi_4^1)=1$, then $F_1(1+\phi_2^1-\phi_4^1)=1$.*

Proof. Since $\phi_1^1<\phi_2^1$, we can get the result. □

Definition 4.10. *Let $B_{13}=\{(0,\phi_2^0,\phi_3^0,\phi_4^0)\in S_{2,1}:0<\phi_2^0<\phi_4^0<\phi_3^0<1,$ and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,0,\phi_4^1)\to(0,0,\phi_3^2,0)\}$, and $B'_{13}=\{(0,\phi_2^0,\phi_3^0,\phi_4^0)\in S_{2,1}:0<\phi_2^0<\phi_4^0<\phi_3^0<1,$ and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,0,\phi_4^1)\to[(0,0,\phi_3^2,0)\to(\phi_1^3,\phi_2^3,0,\phi_4^3)]\to$ Repeat[ ]$\}$.*

Theorem 4.9. *The set B'_{13} has measure zero in S_3.*

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 14: $(0,\phi_2^0,\phi_3^0,\phi_4^0)\in S_{3,1},0<\phi_2^0<\phi_4^0<\phi_3^0<1$, and iteration form is$(0,\phi_2^0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,0,\phi_4^1)\to(0,0,0,0)$, with $\phi_4^1<1, 1=F_1(1+\phi_1^1-\phi_4^1)>1+\phi_2^1-\phi_4^1$, and $F_2(1-\phi_4^1)=1$ or $F_1(1-\phi_4^1)=1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)$
$(1-\phi_4^1)^+$	0	0	0	0

Thus, synchronization occurs in this case.

Case 15: $(0,\phi_2^0,\phi_3^0,\phi_4^0)\in S_{3,1},0<\phi_2^0<\phi_4^0<\phi_3^0<1$, and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,0,0)$, with $F_1(1+\phi_4^0-\phi_3^0)=1$, and$F_1(1+\phi_2^0-\phi_3^0)<1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$1-\phi_3^0$	$F_1(1+\phi_2^0-\phi_3^0)$	0	$F_1(1+\phi_4^0-\phi_3^0)=1$
	$\to F_1(1-\phi_3^0)$			$\to 0$

Definition 4.11. *Let $B_{15}=\{(0,\phi_2^0,\phi_3^0,\phi_4^0)\in S_{3,1}:0<\phi_2^0<\phi_4^0<\phi_3^0<1,$ and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0)\to(\phi_1^1,\phi_2^1,0,0)\}$, and $B'_{15}=\{(0,\phi_2^0,\phi_3^0,\phi_4^0)\in S_{3,1}:0<\phi_2^0<\phi_4^0<\phi_3^0<1,$ and iteration form is $(0,\phi_2^0,\phi_3^0,\phi_4^0)\to[(\phi_1^1,\phi_2^1,0,0)\to(\phi_1^2,0,\phi_3^2,\phi_4^2)\to(0,\phi_2^3,\phi_3^3,\phi_4^3)]\to$ Repeat [ ]$\}$.*

Theorem 4.10. *The set B'_{15} has measure zero in S_3.*

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 16: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, 0, 0, 0)$, with $F_1(1+\phi_2^0-\phi_3^0) = 1$, and $F_2(1-\phi_3^0) < 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	$F_2(1-\phi_3^0)$	0	0	0

Definition 4.12. *Let $B_{16} = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (\phi_1^1, 0, 0, 0)\}$, and $B'_{16} = \{(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1} : 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to [(\phi_1^1, 0, 0, 0) \to (0, \phi_2^2, \phi_3^2, \phi_4^2)] \to$ Repeat [ ]$\}$.*

Theorem 4.11. *The set B'_{16} has measure zero in S_3.*

Proof. The proof is similar as that in Theorem 3.3., thus the proof is omitted. □

Case 17: $(0, \phi_2^0, \phi_3^0, \phi_4^0) \in S_{3,1}, 0 < \phi_2^0 < \phi_4^0 < \phi_3^0 < 1$, and iteration form is $(0, \phi_2^0, \phi_3^0, \phi_4^0) \to (0, 0, 0, 0)$, with $F_1(1+\phi_2^0-\phi_3^0) = 1$, and $F_2(1-\phi_3^0) = 1$ or $F_1(1-\phi_3^0) = 1$.

Time	1	2	3	4
0	0	ϕ_2^0	ϕ_3^0	ϕ_4^0
$(1-\phi_3^0)^+$	0	0	0	0

Thus, synchronization occurs in this case.

Definition 4.13. $B_{0,1} = B'_2 \cup B'_3 \cup B'_5 \cup B'_7 \cup B'_8 \cup B'_{10} \cup B'_{11} \cup B'_{13} \cup B'_{15} \cup B'_{16}$.
$B_{1,1} = \{\phi \in B_1 : R(\phi) \in B_{0,1}\}$,
$B_{2,1} = \{\phi \in B_1 : R(\phi) \in B_{1,1}\}$,
$\vdots$
$B_{k,1} = \{\phi \in B_1 : R(\phi) \in B_{k-1,1}\}$,
where R is return map defined in Case1, i.e., $R = h_4h_3h_2h_1$.

Lemma 4.1. *The set $B_{k,1}, \forall k$ has measure zero S_3.*

Proof. The proof is similar as that in Lemma 3.1., thus the proof is omitted. □

Definition 4.14. *Let $N_{3,1}$ =$\{\phi \in S_{3,1}$: after iterations, the states never reach synchronization$\}$, i.e.,*
$N_{3,1} = A \cup B_{0,1} \cup B_{1,1} \cup B_{2,1} \cup \cdots = A \cup \bigcup_{k\geq 0} B_{k,1}$.

By above, we can get the following Theorem.

Theorem 4.12. *The set $N_{3,1}$ has measure zero in S_3.*

By same way, we get the following Theorem.

Theorem 4.13. *The set $N_{3,2} = \{\phi \in S_{3,2}$: after iterations, the states never reach synchronization.$\}$ has measure zreo in S_3 , and the set $N_{3,3} = \{\phi \in S_{3,3}$: after iterations, the states never reach synchronization.$\}$ has measure zreo in S_3.*

Theorem 4.14. *The set N_3 has measure zero in S_3.*

Proof. Since $N_3 = N_{3,1} \cup N_{3,2} \cup N_{3,2}$, we can get the result. □

5. **Concluding Remarks**

Theorem 5.1. *The system of four convex oscillators (i.e., $f_i'' < 0$) that have nearest-neighbor coupling with periodic boundary conditions is firing in unison for almost all initial conditions.*

Proof. From Theorem 4.14., we can get the result. □

For globally coupled oscillators or the case under study in this thesis, if the firing order of the oscillators i and j is reversed, then the firing oscillator would bring the other oscillator to the threshold that is infinitesimal apart. And these two oscillators will stay firing in unison in the future, which in turn makes the absorption process easier to deal with. However, for $n > 4$, with same locally coupling rule as discussed here, such nice property no longer holds true. It is certainly interesting to give a complete analysis for such case.

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