

國立交通大學

應用數學系

碩士論文

單擺運動之函數理論

The Function theory of A Pendulum Motion



研究生：龔柏任

指導教授：李榮耀 教授

中華民國九十七年六月

單擺運動之函數理論

The Function theory of A Pendulum Motion

研 究 生：龔柏任 Student：Bo-Renn Gong

指導教授：李榮耀 Advisor：Jong-Eao Lee

國 立 交 通 大 學

應 用 數 學 系

碩 士 論 文



A Thesis

Submitted to Department of Applied Mathematics
College of Science

National Chiao Tung University
in Partial Fulfillment of the Requirements
for the Degree of
Master

in

Applied Mathematics

June 2008

Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 七 年 六 月

單擺運動之函數理論

研究生：龔柏任

指導老師：李榮耀 教授

國立交通大學

應用數學系

摘要

我們研究一個單擺運動。理想的單擺運動 $u(t)$ 是能量守恆的，因此其數學模型

$$\ddot{u}(t) + \sin u(t) = 0, u(0) = 0$$

的運動軌跡被初始總能量決定唯一性，從而，所有的解可以由能量守恆律去分析跟求解。有三種解由初始總能量來區分，即週期解 (the periodic solutions) (時間的週期)，隔間解 (the seperatrices) 和 波動解 (the wavetrains).

在第一部分裡面，從守恆律我們把非線性的 ODE 問題轉換成所謂的反問題 (積分形式)，然後用古典的橢圓函數將解給表示出來。注意到這些反運算的積分裡面有多值的被積分函數，所以數值的量化計算不可能，例如週期解的週期，等...

在第二部份，我們在 genusN 的黎曼空間上發展積分技巧來完成對這些積分數值上的計算。並且給一些例子。

中華民國九十七年六月

The Function Theory of A Pendulum Motion

Student : Bo-Renn Gong

Advisor : Jong-Eao Lee

Department of Applied Mathematics

National Chiao Tung University

Abstract

We study the motions of a pendulum. An ideal pendulum motion $u(t)$ is energy-conservative, so the traces of the motions of its mathematic model

$$\ddot{u}(t) + \sin u(t) = 0, u(0) = 0$$

are uniquely determined by the initial total energies, and, consequently, all solutions are able to be analyzed and solved by the conservation law of energies. There are three kinds of solutions characterized by the initial total energies, namely the periodic solutions (in time), the seperatrices, and the wavetrains.

In part I, from the conservation laws, we transferred the nonlinear ODE problem into the so-called inverse problem (in an integral form), and then expressed the solutions $u(t)$ in terms of classical elliptic functions. Notice that those integrals for the inverse problem have multi-valued integrands, and it is impossible to do numerical computations for quantities such as periods of periodic solution, etc..

In part II, we developed integral techniques on the Riemann surfaces of genus N to carry out the numerical computations for those integrals. Some examples are given.

誌 謝

本篇論文的完成，首先要感謝我的指導老師-李榮耀教授。

老師在這兩年的期間給了我論文上很多寶貴的建議，讓學生能夠更加明白此論文的菁華。老師常提醒我要”置之死地而後生”的做研究，也就是一直用盡腦汁的想問題直到用腦過度的時候才可以去找老師詢問，否則老師是不會馬上給建議的。值得一提的是，老師曾經很嚴肅的跟我說過”不管你研究做的多好，不會做人就是失敗”，這句話我覺得是老師所教給我的最重要的東西，謝謝老師。

在論文口試期間，承蒙李志豪教授，邵錦昌教授與張書銘助理教授給我許多寶貴的意見，讓本篇論文得以更加完善，學生感謝萬分。

此外，我要感謝同研究室的室友們，杜耿松同學、陳子鴻同學，曾世忠同學以及蔡佩純同學。在交大的這兩年有他們陪伴著我度過快樂跟悲傷的日子，謝謝你們。

最後我要感謝我的父母，他們辛苦了大半輩子為了我，在臺南省吃儉用留給在新竹的我，是他們造就了今日的我，身為兒子的我以此論文獻給我最偉大的父母，龔有安、毛素敏。

Contents:

中文摘要.....	i
英文摘要.....	ii
誌謝.....	iii

Contents.....	iv
----------------------	-----------

Chapter 1 Elliptic Functions

1.1 General theorem and properties of elliptic functions.....	1
1.1.1 History.....	1
1.1.2 Doubly periodic functions.....	1
1.1.3 Period parallelograms.....	2
1.1.4 Properties of the elliptic functions.....	2
1.2 Weierstrass elliptic function	
1.2.1 Definition.....	3
1.2.2 Properties of $\wp(z)$	3
1.2.3 The constants e_1, e_2, e_3	4
1.2.4 The Weierstrass-zeta function.....	5
1.2.5 Properties of Weierstrass-zeta function.....	5
1.2.6 The Weierstrass-sigma function.....	6
1.2.7 Properties of Weierstrass-sigma function.....	6
1.3 The Theta-functions	
1.3.1 Theta-function.....	7
1.3.2 Four types of theta-functions.....	7

1.3.3	The theta-functions as infinite products.....	8
1.3.4	Properties of the theta-functions.....	8
1.4	Jacobian elliptic functions	
1.4.1	Definition of Jacobian elliptic functions.....	9
1.4.2	Double periodicity of the Jacobian elliptic functions.....	9
1.4.3	Properties of Jacobian elliptic functions.....	11
1.4.4	Elliptic integral of the first kind.....	12
1.4.5	The graph of Jacobian $\text{sn}(u, \kappa)$	13

Chapter 2 The Simple Pendulum

2.1	Introduction.....	15
2.2	Analysis.....	15
2.3	Apply the Jacobian elliptic function to solve the pendulum motion.	16
2.4	About the period with different total energy.....	19
2.4	Summary.....	20

Chapter 3 Riemann Surface

3.1	Introduction.....	22
3.2	The Riemann surface of $f(z) = \sqrt{\prod_{i=1}^n (z - z_i)}$ with $z_j \in R$	25
3.3	The algebraic and geometric structure for Riemann surface of horizontal cut.....	29
3.4	The Riemann surface of $f(z) = \sqrt{\prod_{i=1}^n (z - z_i)}$ with $z_j \in C$	34
3.5	The algebraic and geometric structure for Riemann surface of vertical cut.....	37

3.6 Application in Riemann surface (Complex Integral).....40
Reference.....50

