Chapter 1 Elliptic functions

1.1 General theorem and properties of elliptic functions

In this chapter we follow reference [5], [6].

1.1.1 History:

The elliptic function is originated from the problem of finding the circumference of the ellipse. The ideal of inverting an elliptic integral to obtain an elliptic function is due to Abel, Jacobi and Gauss. It was realized that "inverse" of certain standard types of such integrals, rather than the integrals themselves. The first properties of the integral ware found by the brothers Jakob and Johann Bernoulli. The first mathematician to study systematically the theory of elliptic integrals was Legendre, he transformed the integrals in numerous types which Legendre called elliptic integrals of the first, second, and third kinds.

1.1.2 Doubly periodic functions:

A function f of a complex variable is called periodic with period 2ω if

$$f(z + 2\omega) = f(z)$$

whenever z and $z + 2\omega$ are in the domain of f. A function f is called doubly periodic function if it has two periods $2\omega_1$ and $2\omega_2$ whose ratio is not purely real. A meromorphic function f in the complex plane which has doubly periods is called an elliptic function.

Note: (In reference [2])

If $2\omega_1$ and $2\omega_2$ are periods whose ratio is real, then it is not double periodic for a nonconstant elliptic function.

Case 1. If $\frac{2\omega_1}{2\omega_2} = \frac{a}{b}$, where a and b are relatively prime integers, then there exists integers m and n such that mb + na = 1.

Let $\omega = 2\omega_1 + 2\omega_2$. Then ω is a period and we have the following

$$\omega = \omega_1(m + n\frac{\omega_1}{\omega_2}) = \omega_1(m + n\frac{a}{b}) = \frac{\omega_1}{b}(mb + na) = \frac{\omega_1}{b},$$

so $\omega_1 = b\omega$ and $\omega_2 = a\omega$. Thus both ω_1 and ω_2 integer multiples of ω.

Case 2. If $\frac{\omega_1}{\omega_2} = \lambda$, λ is an irrational number. Given $\varepsilon > 0$, there are always integers p and q such that

$$|p\lambda - q| = \left|p(\frac{\omega_1}{\omega_2}) - q\right| < \varepsilon \quad \text{or} \quad |2p\omega_1 - 2q\omega_2| < 2\varepsilon|\omega_2|$$

But then $2p\omega_1 - 2q\omega_2$ would be a period of f of arbitrary small modulus, which is impossible.

1.1.3 Period parallelograms:

Suppose that in the plane of the variable z we mark the points $0, 2\omega_1$, $2\omega_2$ and $2\omega_1 + 2\omega_2$, generally, all the points whose complex coordinates are of the form $2n\omega_1 + 2m\omega_2$, where m and n are integers. Consider the points of set $0, 2\omega_1, 2\omega_2$ and $2\omega_1 + 2\omega_2$,and we obtain a parallelogram as the vertices. If no point ω inside or on the boundary of this parallelogram such that f($z + \omega$) = f(z) for all values of z, this parallelogram is called a fundamental period parallelogram for
an elliptic function with periods $2\omega_1$, $2\omega_2$. Such a translated parallelogram, with no zeros or poles on its boundary, will be called a cell.

1.1.4 Properties of the elliptic functions:

1. If an elliptic function f has no poles in some period parallelogram, then f is constant.
2. If an elliptic function f has no zeros in some period parallelogram, then f is a constant function.

3. The contour integral of an elliptic function taking along the boundary of any cell is zero.

4. The sum of the residues of an elliptic function at its poles in any period parallelogram is zero.

5. (Abel's theorem) The number of zeros of an elliptic function in any period parallelogram is equal to the number of poles, each counted with multiplicity.

1.2 Weierstrass elliptic function

1.2.1 Definition:

The Weierstrass elliptic function $\wp(\mathbf{z})$ is one of the famous elliptic function, which is to define by the equation

$$\wp(z) = \frac{1}{z} + \sum_{m,n}{}' \{ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} \quad \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \} \tag{1.1}$$

where ω_1, ω_2 satisfy the condition that the ratio is not purely real; the summation extends over all integer values (positive, negative and zero) of m and n, simultaneous zero values of m and n excepted. For brevity, we write $\Omega_{m,n}$ in place of $2m\omega_1 + 2n\omega_2$.

1.2.2 Properties of $\wp$(z):

1. $\wp(z)$ is an even function with single double pole at $\Omega_{m,n}$ for integers m, n.

2. $\wp(z)$ satisfies the differential equation,

$$[\wp'(z)]^2 = 4\wp^2(z) - g_2\wp(z) - g_3 \tag{1.2}$$

where $g_2 = 60\sum_{m,n}{}' \Omega^{-4}{}_{m,n}$, $g_3 = 140\sum_{m,n}{}' \Omega^{-6}{}_{m,n}$ (called the invariants)

3. (Properties of homogeneity)

$$\wp(\lambda z; \lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(z; \omega_1, \omega_2), \quad \lambda \neq 0 \tag{1.3}$$

$$\wp(\lambda z; \lambda^{-4} g_2, \lambda^{-6} g_3) = \lambda^{-2}\wp(z; g_2, g_3), \quad \lambda \neq 0 \tag{1.4}$$

where $\wp(z;\omega_1,\omega_2)$ denote the function formed with periods $2\omega_1$, $2\omega_2$ and $\wp(z; g_2, g_3)$ denote the function formed with invariants g_2, g_3.

4. (Addition-theorem)

a. If $u + v + w = 0$, then

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0 \tag{1.5}$$

b.

$$\wp(z + y) = \frac{1}{4}\{\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\}^2 - \wp(z) - \wp(y) \tag{1.6}$$

c.

$$\wp(2z) = \frac{1}{4}\{\frac{\wp''(z)}{\wp'(z)}\}^2 - 2\wp(z) \tag{1.7}$$

unless $2z$ is a period. The result is called the duplication formula.

1.2.3 The constants e_1, e_2, e_3:

Let $\wp(z)$ be the Weierstrass elliptic function with periods $2\omega_1$, $2\omega_2$. The value $\wp(\omega_1)$, $\wp(\omega_2)$, $\wp(\omega_3)$ (where $\omega_1 + \omega_2 + \omega_3 = 0$) are all unequal; and, if their value be e_1, e_2, e_3, respectively, then the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$ and $e_1 \neq e_2 \neq e_3$. We have

$$e_1 + e_2 + e_3 = 0$$

$$e_1 \cdot e_2 \cdot e_3 = \frac{g_3}{4}$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4}$$

1.2.4 The Weierstrass-zeta function:

The Weierstrass-zeta $\zeta(z)$ defined by the equation

$$\frac{d\zeta(z)}{z} = -\wp(z)\text{, with } \lim_{z\to 0}\{\zeta(z)-\frac{1}{z}\}=0.$$

Since the series for $\wp(z)-\frac{1}{z^2}$ converges uniformly throughout any domain from which the neighborhoods of the points $\Omega'_{m,n}$ are excluded, we can term-by-term integrate and get

$$\begin{aligned}\zeta(z)-\frac{1}{z} &= -\int_0^z \{\wp(z)-\frac{1}{z^2}\}dz \\ &= \sum_{m,n}{}' \int_0^z \{\frac{1}{(z-\Omega_{m,n})^2}-\frac{1}{\Omega'_{m,n}}\}dz\end{aligned}$$

and so
$$\zeta(z)=\frac{1}{z}+\sum_{m,n}{}' \{\frac{1}{z-\Omega_{m,n}}+\frac{1}{\Omega_{m,n}}+\frac{z}{\Omega_{m,n}{}^2}\}. \tag{1.8}$$

1.2.5 Properties of Weierstrass-zeta function:

1. $\zeta(z)$ is an odd function. It is not a doubly-periodic function, and the residue of $\zeta(z)$ at every pole is 1.

2. If we integrate the equation

$$\wp(z+2\omega_1)=\wp(z)\text{, and } \wp(z+2\omega_2)=\wp(z),$$

we get

$$\begin{aligned}\zeta(z+2\omega_1)&=\zeta(z)+2\eta_1,\\ \zeta(z+2\omega_2)&=\zeta(z)+2\eta_2,\\ \zeta(z+2\omega_3)&=\zeta(z)+2\eta_3\end{aligned}$$

where $2\eta_1$, $2\eta_2$, and $2\eta_3$ are the constants, putting $z=2\omega_1$, $z=2\omega_2$, and $z=2\omega_3$, respectively, and use the face $\zeta(z)$ is an odd function, we have $\eta_1=\zeta(\omega_1)$, $\eta_2=\zeta(\omega_2)$, and $\eta_3=\zeta(\omega_3)$, where $\omega_1+\omega_2+\omega_3=0$. We get that relation $\eta_1+\eta_2+\eta_3=0$. This is the quasi-periodicity.

4. (Properties of homogeneity)

$$\zeta(\lambda z;\lambda\omega_1,\lambda\omega_2)=\lambda^{-1}\zeta(z;\omega_1,\omega_2),\quad \lambda\neq 0$$

5. (Legendre's relation)

$$\eta_1\omega_2 - \eta_2\omega_1 = \eta_2\omega_3 - \eta_3\omega_2 = \eta_3\omega_1 - \eta_1\omega_3 = \frac{\pi\iota}{2}. \tag{1.9}$$

1.2.6 The Weierstrass-sigma function:

The Weierstrass-sigma function defined by the equation

$$\frac{d}{dz}\log\sigma(z) = \zeta(z), \text{ with } \lim_{z\to 0}\frac{\sigma(z)}{z} = 1,$$

On account of the uniformity of convergence of the series of $\zeta(z)$, except near the poles of $\zeta(z)$, we may integrate the series term-by-term.

we get

$$\sigma(z) = z\prod_{m,n}{}' \{(1-\frac{z}{\Omega_{m,n}})\exp(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega^2{}_{m,n}})\}. \tag{1.10}$$

1.2.7 Properties of Weierstrass-sigma function:

1. The product for $\sigma(z)$ converges absolutely and uniformly in any bounded domain.

2. The function $\sigma(z)$ is an odd function with simple zeros at all the points $\Omega_{m,n}$.

3. If we integrate the equation

$$\zeta(z+2\omega_1) = \zeta(z) + 2\eta_1,$$

we get

$$\sigma(z+2\omega_1) = ce^{2\eta_1 z}\sigma(z),$$

where c is the constant, putting $z = -\omega_1$, and then $c = -e^{2\eta_1\omega_1}$.

Consequently,

$$\sigma(z+2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z);$$
$$\sigma(z+2\omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma(z);$$
$$\sigma(z+2\omega_3) = -e^{2\eta_3(z+\omega_3)}\sigma(z),$$

where $\omega_1 + \omega_2 + \omega_3 = 0$, and $\eta_1 + \eta_2 + \eta_3 = 0$. This is the quasi-periodicity. The exponential $-e^{2\eta_r(z+\omega_r)}$ (where $r = 1,2,3$) is called the periodicity factor of the $\sigma(z)$.

4. (Properties of homogeneity)

$$\sigma(\lambda z; \lambda\omega_1, \lambda\omega_2) = \lambda\sigma(z; \omega_1, \omega_2)$$

1.3 The Theta-functions

1.3.1 Theta-function:

Let τ be (constant) complex number whose imaginary part is positive; and write $q = e^{i\pi\tau}$, so that $|q| < 1$.

Consider the function $\vartheta_4(z,q)$, defined by the series

$$\vartheta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}$$

It is evident that

$$\vartheta_4(z,q) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz,$$

and that $$\vartheta_4(z+\pi, q) = \vartheta_4(z,q);$$

further $$\vartheta_4(z+\pi\tau, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{2n} e^{2niz}$$

$$= -q^{-1}e^{-2iz} \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{(n+1)^2} e^{2(n+1)iz},$$

and so $$\vartheta_4(z+\pi\tau, q) = -q^{-1}e^{-2zi}\vartheta_4(z,q).$$

In consequence of these results, $\vartheta_4(z,q)$ is called a quasi doubly-periodic function, and accordingly 1 and $-q^{-1}e^{-2zi}$ are called the periodicity factors associated with the periods π and $\pi\tau$.

1.3.2 Four types of theta-functions:

The function $\vartheta_3(z,q)$ is defined by the equation

$$\vartheta_3(z,q) = \vartheta_4(z + \frac{\pi}{2}z, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz$$

Next, $\vartheta_1(z,q)$ is defined in terms of $\vartheta_4(z,q)$ by the equation

$$\vartheta_1(z,q) = -\imath\ e^{zi+\frac{1}{4}\pi i\tau}\vartheta_4(z+\frac{\pi\tau}{2},q)$$

$$= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z\ .$$

Lastly, $\vartheta_2(z,q)$ is defined by the equation

$$\vartheta_2(z,q) = \vartheta_1(z+\frac{\pi}{2},q) = 2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}\cos(2n+1)z\ .$$

Summary,

$$\vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z\ . \tag{1.11}$$

$$\vartheta_2(z,q) = 2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}\cos(2n+1)z\ . \tag{1.12}$$

$$\vartheta_3(z,q) = 1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nz\ . \tag{1.13}$$

$$\vartheta_4(z,q) = 1+\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz\ . \tag{1.14}$$

For brevity, the parameter q will usually not be specified, so that $\vartheta_i(z)$ will be written for $\vartheta_i(z,q)$, for i =1,2,3,4.

1.3.3 The theta-functions as infinite products:

$$\vartheta_1(z,q) = 2q^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2z+q^{4n})\ . \tag{1.15}$$

$$\vartheta_2(z,q) = 2q^{\frac{1}{4}}\cos z\prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n}\cos 2z+q^{4n}) \tag{1.16}$$

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n}\cos 2z+q^{4n-2})\ . \tag{1.17}$$

$$\vartheta_4(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2z+q^{4n-2})\ . \tag{1.18}$$

1.3.4 Properties of the theta-functions:

1. $\vartheta_1(z)$ is an odd function and the other theta function are even functions, because of the trigonometrical series.

2. The zero of $\vartheta_i(z,q)$, for i =1,2,3,4, $m,n\in Z$

$\vartheta_1(z)=0$, where $z=0+m\pi+n\pi\tau$,

$\vartheta_2(z)=0$, where $z=\frac{\pi}{2}+m\pi+n\pi\tau$,

$\vartheta_3(z)=0$, where $z=\frac{\pi}{2}+\frac{\pi\tau}{2}+m\pi+n\pi\tau$,

$\vartheta_4(z)=0$, where $z=\frac{\pi\tau}{2}+m\pi+n\pi\tau$.

3. The identity $\vartheta^4{}_2(0)+\vartheta^4{}_4(0)=\vartheta_3{}^4(0)$. (1.17)

4. The identity $\vartheta_1^{'}(0)=\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$. (1.18)

5. The differential equation satisfied by the theta-functions

$$\frac{\partial^2\vartheta_j(z,\tau)}{\partial z^2}=4i\pi\frac{\partial\vartheta_j(z,\tau)}{\partial\tau},\ \text{for}\ j=1,2,3,4. \tag{1.19}$$

1.4 Jacobian elliptic functions

1.4.1 Definition of Jacobian elliptic functions:

The elliptic functions $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$ are defined as ratios of theta-functions as below:

$$\mathrm{sn}\,u=\frac{\vartheta_3(0)}{\vartheta_2(0)}\frac{\vartheta_1(z)}{\vartheta_4(z)} \tag{1.20}$$

$$\mathrm{cn}\,u=\frac{\vartheta_4(0)}{\vartheta_2(0)}\frac{\vartheta_2(z)}{\vartheta_4(z)} \tag{1.21}$$

$$\mathrm{dn}\,u=\frac{\vartheta_4(0)}{\vartheta_3(0)}\frac{\vartheta_3(z)}{\vartheta_4(z)} \tag{1.22}$$

where $z=\frac{u}{\vartheta_3{}^2(0)}$.

Note:

If the parameter τ is purely imaginary (q real), the elliptic functions are all real for real values of u.

1.4.2 Double periodicity of the Jacobian elliptic functions:

We can use the following identities to find the double periods of the elliptic functions $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$. Writing $N = qe^{2zi}$, then

$$\vartheta_1(z) = -\vartheta_1(z+\pi) = -N\vartheta_1(z+\pi\tau) = N\vartheta_1(z+\pi+\pi\tau)$$

$$\vartheta_2(z) = -\vartheta_2(z+\pi) = N\vartheta_2(z+\pi\tau) = -N\vartheta_2(z+\pi+\pi\tau)$$

$$\vartheta_3(z) = \vartheta_3(z+\pi) = N\vartheta_3(z+\pi\tau) = N\vartheta_3(z+\pi+\pi\tau)$$

$$\vartheta_4(z) = \vartheta_4(z+\pi) = -N\vartheta_4(z+\pi\tau) = N\vartheta_4(z+\pi+\pi\tau)$$

Thus,

$$\mathrm{sn}\,(u + 2\pi\vartheta_3{}^2(0)) = \mathrm{sn}\,u\,, \quad \mathrm{sn}\,(u + \pi\tau\vartheta_3{}^2(0)) = \mathrm{sn}\,u\,,$$

$\mathrm{sn}\,u$ has two periods $2\pi\vartheta_3{}^2(0)$ and $\pi\tau\vartheta_3{}^2(0)$ whose ratio $\frac{1}{2}\tau$ must be the complex (with positive imaginary part). We shall write

$$K = \frac{1}{2}\pi\vartheta_3{}^2(0)\,,\ \ iK^{'} = \frac{1}{2}\pi\tau\vartheta_3{}^2(0) = \tau K\,,$$

then,

$$\mathrm{sn}\,(u + 4K) = \mathrm{sn}\,(u + 2iK^{'}) = \mathrm{sn}\,u\,,$$

The periods of $\mathrm{sn}\,u$ are $4K$ and $2iK^{'}$.
Also,

$$\mathrm{cn}\,(u + 4K) = \mathrm{sn}\,(u + 2K + 2iK^{'}) = \mathrm{cn}\,u\,,$$

$$\mathrm{dn}\,(u + 2K) = \mathrm{dn}\,(u + 4iK^{'}) = \mathrm{dn}\,u\,,$$

$\mathrm{cn}\,u$ has periods $4K$ and $2K + 2iK^{'}$; $\mathrm{dn}\,u$ has periods $2K$ and $4iK^{'}$.

Now, the zeros of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$ are determinable by the theta-functios and the definition of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$.

	Double periods	Zeros
$\mathrm{sn}\,u$	$4K$ and $2iK^{'}$	$\vartheta_3{}^2(0)(0 + m\pi + n\pi\tau) = 2mK + 2inK^{'}$
$\mathrm{cn}\,u$	$4K$ and $2K + 2iK^{'}$	$\vartheta_3{}^2(0)(\frac{\pi}{2} + m\pi + n\pi\tau) = (2m+1)K + 2inK^{'}$
$\mathrm{dn}\,u$	$2K$ and $4iK^{'}$	$\vartheta_3{}^2(0)(\frac{\pi}{2} + \frac{\pi\tau}{2} + m\pi + n\pi\tau) = (2m+1)K + i(2n+1)K^{'}$

Table 1-1 (The periods and zeros of Jacobian elliptic functions)

1.4.3 Properties of Jacobian elliptic functions:

1. (Differential equation)

Let $\kappa = \dfrac{\vartheta_2^{\ 2}(0)}{\vartheta_3^{\ 2}(0)}$ be a parameter, and $\operatorname{sn} u$ satisfy the differential

equation

$$(\frac{dy}{du})^2 = (1-y^2)(1-\kappa^2 y^2)\,, \tag{1.23}$$

The parameter κ is called the modulus; if $\kappa^{'} = \dfrac{\vartheta_4^{\ 2}(0)}{\vartheta_3^{\ 2}(0)}$, so

$\kappa^2 + \kappa^{'2} = 1$, $\kappa^{'}$ is called the complementary modulus. In the numerical express of

κ, and $\kappa^{'}$,

$$\kappa = 4q^{\frac{1}{2}} \prod_{n=1}^{\infty} (\frac{1+q^{2n}}{1+q^{2n-1}})^4 \,, \tag{1.24}$$

$$\kappa^{'} = \prod_{n=1}^{\infty} (\frac{1-q^{2n-1}}{1+q^{2n-1}})^4 \,. \tag{1.25}$$

Note:

If q is real and $0 \le q < 1$, then $0 \le \kappa < 1$, $0 < \kappa^{'} \le 1$.

On the other hand, $\operatorname{cn} u$, and $\operatorname{dn} u$ satisfy the following differential equation, respectively.

$$(\frac{dy}{du})^2 = (1-y^2)(\kappa^{'2} + \kappa^2 y^2)\,,$$

$$(\frac{dy}{du})^2 = (1-y^2)(y^2 - \kappa^{'2})\,.$$

2. The identities

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \tag{1.26}$$

$$\mathrm{dn}^2 u + \kappa^2 \mathrm{sn}^2 u = 1, \tag{1.27}$$

$$\mathrm{dn}^2 u - \kappa^2 \mathrm{cn}^2 u = \kappa'^2. \tag{1.28}$$

3. The derivative of Jacobian elliptic functions.

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn}u = \mathrm{cn}u\,\mathrm{dn}u, \tag{1.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{cn}u = -\mathrm{sn}u\,\mathrm{dn}u, \tag{1.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{dn}u = -\kappa^2\mathrm{sn}u\,\mathrm{cn}u. \tag{1.31}$$

4. The power series expansions of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$.

we can use the face that $\mathrm{sn}0 = 0$, $\mathrm{cn}0 = 1, \mathrm{dn}0 = 1$, and 3. in section1.4.3 to find the Maclaurin series as following,

$$\mathrm{sn}u = u - \frac{1}{3!}(1+\kappa^2)u^3 + \frac{1}{5!}(1+14\kappa^2+\kappa^4)u^5 - \cdots, \tag{1.32}$$

$$\mathrm{cn}u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}(1+4\kappa^2)u^4 - \cdots, \tag{1.33}$$

$$\mathrm{dn}u = 1 - \frac{1}{2!}\kappa^2 u^2 + \frac{1}{4!}(4\kappa^2+\kappa^4)u^4 - \cdots. \tag{1.34}$$

1.4.4 Elliptic integral of the first kind:

The function $\mathrm{sn}\,u$ satisfies the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 = (1-y^2)(1-\kappa^2 y^2),$$

we have the integral representation of $\mathrm{sn}\,u$ is

$$u = \int_0^{y(u)} \frac{1}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}dt, \quad \text{thus} \quad y = \mathrm{sn}(u,\kappa). \tag{1.35}$$

A special case of the integral representation is

$$K = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}} dt \,, \tag{1.36}$$

this is the *complete elliptic integral of the first kind*, if we let $t = \sin\phi$, and we have at once

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(1-\kappa^2 \sin^2\phi)}} d\phi \,. \tag{1.37}$$

1.4.5 The graph of Jacobian sn(u, κ):

a. $\mathrm{sn}(u, \frac{1}{2})$:

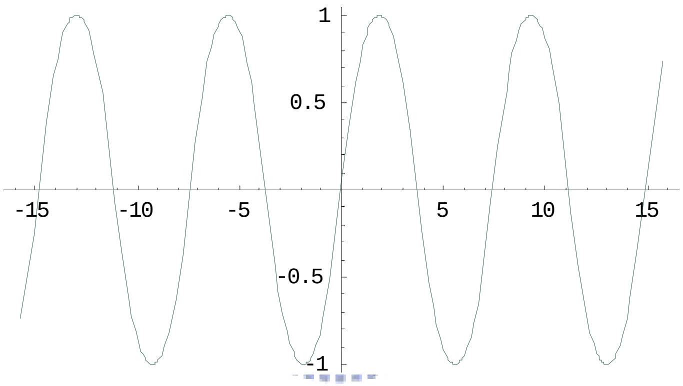


Fig 1-1 (The graph of $\mathrm{sn}(u, \frac{1}{2})$)

b. $\mathrm{sn}(u, \frac{1}{\sqrt{2}})$:

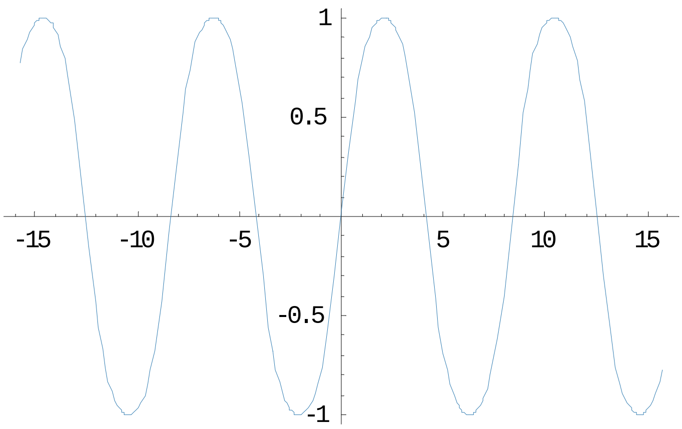


Fig 1-2 (The graph of $\mathrm{sn}(u, \frac{1}{\sqrt{2}})$)

c. $\mathrm{sn}(u,1)$：

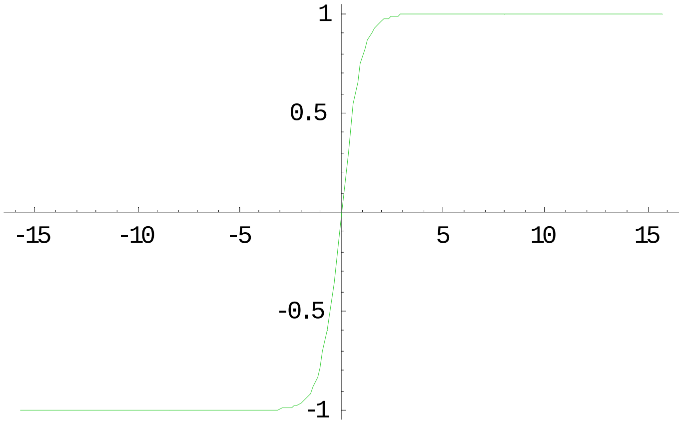


Fig 1-3 (The graph of $\mathrm{sn}(u,1)$)

Chapter 2 The simple pendulum

2.1 Introduction:

Let l be the length of the suspension, g the gravitational acceleration, and m the mass of the bob. Then, if θ is the angle made by the string with the downward vertical and v is the velocity of the bob at any time t, its energy is conserved provided

$$\frac{1}{2}mv^2 + (-mgl\cos\theta) = \text{constant}, \tag{2.1}$$

Kinetic energy + Potential energy = constant.

where the kinetic energy is $\frac{1}{2}mv^2$, and the potential energy is $(-mgl\cos\theta)$. Since $v = l\dot{\theta}$, putting $\omega^2 = \frac{g}{l}$, consequently, the equation (2.1) can be written in the form

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \text{constant} \tag{2.2}$$

2.2 Analysis:

First, let's consider the differential equation of the following

$$\ddot{U}(t) + \sin U(t) = 0, \quad U(0) = 0. \tag{2.3}$$

by the energy method,

$$\dot{U}(t) \times \ddot{U}(t) + \dot{U}(t) \times \sin U(t) = 0. \tag{2.4}$$

integration the both side of (2) we obtation that

$$\frac{1}{2}[\dot{U}(t)]^2 - \cos U(t) = E, \qquad \text{where } E \geq -1,$$

$$\frac{1}{2}[\dot{U}(t)]^2 + [1 - \cos U(t)] = E^*, \qquad \text{where } E^* = E + 1 \geq 0. \tag{2.5}$$

the kinetic energy is $\frac{1}{2}[\dot{U}(t)]^2$ and the potential energy is $1 - \cos U(t)$, we call the E^* to be the total energy.

Before solving the equation (2.5), we analyze the relation between the kinetic energy, the potential energy and the phase portrait. Given the energy $E^* = 1.5$, $E^* = 2$, and $E^* = 3$.

The relation of potential energy(P.E) and the angle(U) as following,

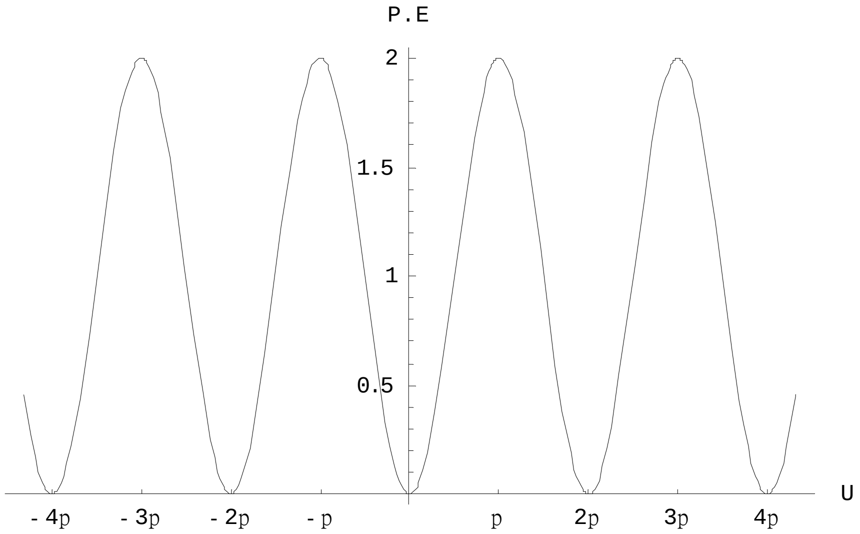


Fig 2-1 (The graph of U-P.E)

The phase portrait $(t,\dot{U})$

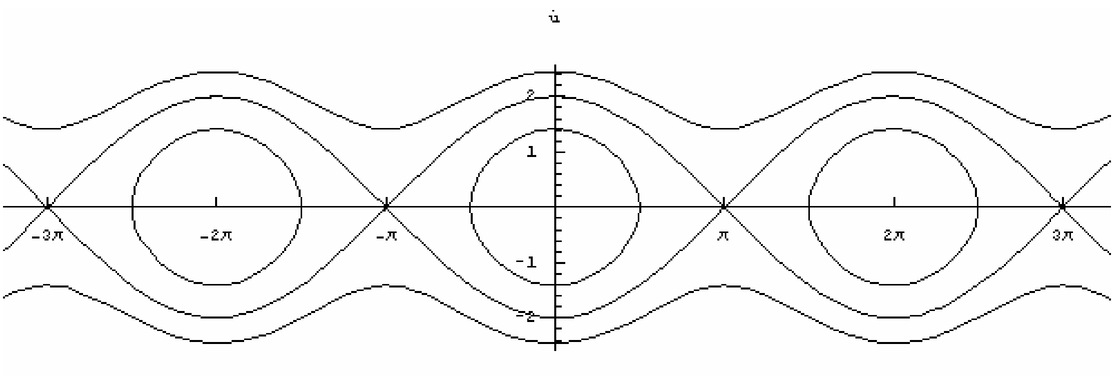


Fig 2-2 (Phase Portrait $(t,\dot{U})$)

2.3 Apply the Jacobian elliptic function to solve the pendulum motion:

Second, we want to use the Jacobian elliptic function and the elliptic integral of first kind to solve (2.3). The integral representation of $U(t)$ is

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2E^* - 2[1-\cos\xi]}} d\xi \ ,$$

$$\Rightarrow \qquad t = \int_0^{U(t)} \frac{1}{\sqrt{2E + 2\cos\xi}} d\xi \ , \text{ since } \ E^* = E + 1 ,$$

(a) When $0 < E^* < 2$, i.e. $-1 < E < 1$, there is a α in $(0,\pi)$, such that $-\cos\alpha = E$, since $\cos x$ in $(0,\pi)$ is an one-to-one mapping, the E can be chosen.

And

$$
\begin{aligned}
t &= \int_0^{U(t)} \frac{1}{\sqrt{2E + 2\cos\xi}} d\xi\,, \\
&= \int_0^{U(t)} \frac{1}{\sqrt{-2\cos\alpha + 2\cos\xi}} d\xi\,, \\
&= \int_0^{U(t)} \frac{1}{\sqrt{-2(1-2\sin^2\frac{\alpha}{2}) + 2(1-2\sin^2\frac{\xi}{2})}} d\xi\,, \\
&= \frac{1}{2}\int_0^{U(t)} \frac{1}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\xi}{2}}} d\xi\,.
\end{aligned}
\tag{2.6}
$$

Let $0 < \kappa = \sin\frac{\alpha}{2} < 1$, and $z = \frac{\sin\frac{\xi}{2}}{\kappa}$, then

$$
t = \int_0^{\frac{1}{\kappa}} \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}} dz\,. \tag{2.7}
$$

According to the Jacobian elliptic function sn(t), we have that

$$
\mathrm{sn}(t) = \frac{1}{\kappa}\sin\frac{U(t)}{2}\,.
$$

that is $U(t) = 2\sin^{-1}(\kappa\cdot\mathrm{sn}(t))$.

The motion is periodic and the period is equal to $4K$. Since the pendulum occupies the highest position $(u = \alpha)$ after a quarter-period.

Because of the following,

$$
\dot{U}^2 = 0 = 2E + 2\cos U = -2\cos\alpha + 2\cos U \quad \Rightarrow \quad U = \alpha\,.
$$

And the period T is

$$
T = 4\int_0^1 \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}} dz = 4K\,, \tag{2.8}
$$

where K is the complete elliptic integral of the first kind.

(b) When $E^* = 2$, ie $E = 1$,

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2 + 2\cos\xi}} d\xi ,$$

$$= \frac{1}{\sqrt{2}} \int_0^{U(t)} \frac{1}{\sqrt{1 + \cos\xi}} d\xi ,$$

$$= \frac{1}{\sqrt{2}} \int_0^{U(t)} \frac{1}{\sqrt{2 - 2\sin^2 \frac{\xi}{2}}} d\xi ,$$

$$= \frac{1}{2} \int_0^{U(t)} \frac{1}{\sqrt{1 + \sin^2 \frac{\xi}{2}}} d\xi . \quad (2.9)$$

Let $\sin\frac{\xi}{2} = x$, then

$$t = \int_0^{\sin\frac{U(t)}{2}} \frac{1}{1 - x^2} dx .$$

Integration gives us that

$$t = \frac{1}{2} \ln \frac{1 + \sin\frac{U(t)}{2}}{1 - \sin\frac{U(t)}{2}} , \quad (2.10)$$

which implies that

$$\sin\frac{U(t)}{2} = \tanh(t) \text{, ie } U(t) = 2\sin^{-1}\tanh(t) . \quad (2.11)$$

This formula shows that as t increasing from 0 to ∞ the angle $U(t)$ increasing from 0 to π, ie, the pendulum always moves in one direction, and the uppermost position, which it never attains, is its limit position.

(c) When $E^* > 2$, ie $E > 1$,

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2E + 2\cos\xi}} d\xi$$

$$= \int_0^{U(t)} \frac{1}{\sqrt{(2E + 2) - 4\sin^2 \frac{\xi}{2}}} d\xi$$

Let $x = \frac{\xi}{2}$, and $\kappa = \frac{2}{\sqrt{2+2E}} \le 1$, then it become that

$$t = \kappa \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1-\kappa^2 \sin^2 x}} dx .$$

Let $\sin x = y$, then

$$t = \kappa \int_0^{\sin\frac{U(t)}{2}} \frac{1}{\sqrt{1-\kappa^2 y^2}\sqrt{1-y^2}} dy . \qquad (2.11)$$

Therefore,

$$\mathrm{sn}(\frac{t}{\kappa}) = \sin\frac{U(t)}{2} . \qquad . \qquad (2.12)$$

When $E^* > 2$, the Fundamental Theorem of Calculus tells us

$$\frac{dt}{dU} = \frac{1}{\sqrt{2E + 2\cos U(t)}} > 0 . \qquad (2.13)$$

From the integral form $t = \int_0^{U(t)} \frac{1}{\sqrt{2E + 2\cos\xi}} d\xi$ we know that when $t \to \infty$, the angle $U(t) \to \infty$, the pendulum is never stopping.

2.4 About the period with different total energy:

(a) The total energy $0 < E^* < 2$.

We can show that the period is always greater than 2π when the total energy in $0 < E^* < 2$.

Since by (2.8) the period is $T = 4\int_0^1 \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}} dz = 4K$, given an energy E^* in $(0,2)$, there is a correspondence $\kappa = \sqrt{\frac{E^*}{2}}$ in $(0,1)$. Because of that for $z \in (0,1)$, $0 < \frac{1}{\sqrt{1-z^2}} < \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}}$, and we have that

$$\int_0^1 \frac{1}{\sqrt{1-z^2}}dz < \int_0^1 \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}}dz ,$$

then $$2\pi = 4\int_0^1 \frac{1}{\sqrt{1-z^2}}dz < 4\int_0^1 \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}}dz = T .$$

and if $\kappa_1 < \kappa_2$, then $\frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa_1{}^2 z^2}} < \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa_2{}^2 z^2}}$ for $z \in (0,1)$,

thus $T(\kappa_1) < T(\kappa_2)$ (the $T(\kappa)$ denotes the period with parameter κ).

The periodic T is always greater than 2π, and increasing as the total energy increasing from 0 to 2. And the period will tend to ∞ as the total energy E^* tend to 2.

(b) The total energy $E^* = 2$.

In the section 2.3 (b) we know that the $U(t) \to \pi$ as $t \to \infty$, we say the periodic is ∞. It will spend for infinite time returning to the original situation.

(c) The total energy $E^* > 2$.

In the section 2.3 (c) we know that the $U(t) \to \infty$ as $t \to \infty$, and the phase portrait $(t, \dot{U})$ in the section 2.2. It is no periodicity.

2.4 Summary:

The problem of the simple pendulum is a nonlinear second order PDE, if we give a initial energy E^*, then there is a κ to correspond the initial energy E^*, thus there is a Jacobian elliptic function $\mathrm{sn}(u,\kappa)$ to correspond the initial energy E^*. We list two table which is the relation with difference initial energy E^* and some numerical result in the following, and the K is denoted the complete elliptic integral, κ is the modular, and T is denoted the period. The main ideal in section 2.3 can be found in [5] and [7].

	$0 < E^* < 2$	$E^* = 2$	$E^* > 2$
κ	$\sqrt{\frac{E^*}{2}}$	1	$\sqrt{\frac{2}{E^*}}$
Solution	$U(t) = 2\sin^{-1}(\kappa \cdot \mathrm{sn}(t))$	$U(t) = 2\sin^{-1}\tanh(t)$	$U(t) = 2\sin^{-1}\mathrm{sn}(\frac{t}{\kappa})$
Period	$4K$	∞	No periodicity

Table 2-1 (The relation of E^*, κ, $U(t)$, and period)

E^*	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
κ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{2}{\sqrt{5}}$	$\frac{2}{\sqrt{6}}$
T	0	7.4136	8.3439	9.7654	∞	No periodicity	No periodicity

Table 2-2 (E^*, κ, T in numerical value)

Chapter 3 Riemann Surface

In the last chapter, the solution $t = \int_0^{\frac{1}{\kappa}\sin\frac{U(t)}{2}} \frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}} dz$ is as an inverse question. But the function $\frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}}$ is not an analytic function. We need the Riemann surface to structure an analytic function of $\frac{1}{\sqrt{1-z^2}\sqrt{1-\kappa^2 z^2}}$. The construct of the Riemann surface from reference [8]

3.1 Introduction:

Let's consider the function $f(z) = \sqrt{z}$ to define a single-value and analytic function on the Riemann surface.

Assume $z \in C$, and we can use the polar form for z,

$$z = re^{i\theta}, \tag{3.1}$$

$$= re^{i(\theta+2\pi)}. \tag{3.2}$$

To find the $\sqrt{z}$, the (3.1) becomes

$$\sqrt{z} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}},$$

and the (3.1) becomes

$$\sqrt{z} = r^{\frac{1}{2}} e^{i(\frac{\theta+2\pi}{2})} = r^{\frac{1}{2}} e^{i(\frac{\theta}{2}+\pi)} = -r^{\frac{1}{2}} e^{i\frac{\theta}{2}}.$$

Since $r^{\frac{1}{2}} e^{i\frac{\theta}{2}} \neq -r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, we have that $f(z) = \sqrt{z}$ is a multi-valued function at each $z \in C$, and which is not analytic on C. How can we make the function $f(z) = \sqrt{z}$ to become a single-valued function and analytic on C?

Consider the two cuts from 0 to $-\infty$ and let

$$P_1 = \{C \setminus (-\infty, 0] \mid \theta_1 = \arg z \in [-\pi^+, \pi^-)\}$$

and

$$P_2 = \{C \setminus (-\infty, 0] \mid \theta_2 = \arg z \in [\pi^+, 3\pi^-)\}$$ as Fig.3-1 shows.

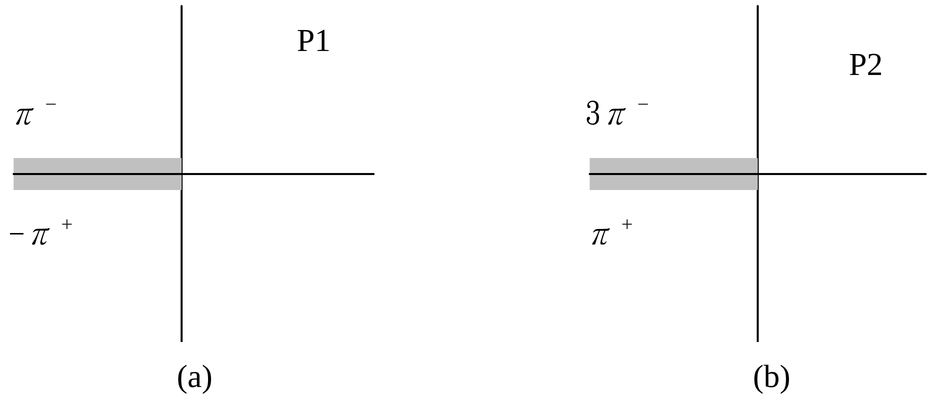


Fig. 3-1 (P1, P2 plane)

We can define two functions $f_1(z)$ and $f_2(z)$ on P1 and P2, respectively, by

$$f_1(z) = \sqrt{z}\,, \quad z \in P_1\,.$$

$$f_2(z) = \sqrt{z}\,, \quad z \in P_2\,.$$

Then $f_1(z) = \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta_1}{2}}$ is a single-valued function defined at each $z \in P_1$ and analytic on P_1.

And $f_2(z) = \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta_2}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\theta_1+2\pi}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\theta_1}{2}} e^{i\pi} = -|z|^{\frac{1}{2}} e^{i\frac{\theta_1}{2}} = -f_1(z)$ is also a single-valued function defined at each $z \in P_2$ and analytic on P_2.

Let $D_1 = \{(-\infty, 0] \mid \arg z = \pi\}$, the figure show as the following.

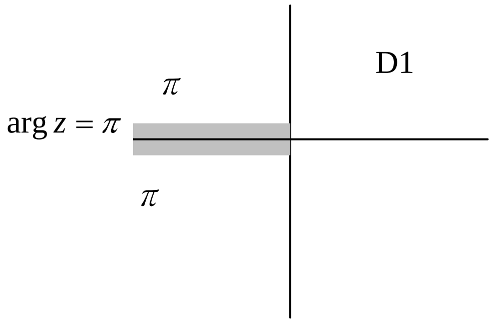


Fig. 3-2 ($D_1 = \{(-\infty,0] \mid \arg z = \pi\}$)

If $z \in P_1$ and $\arg z$ tends to π^- , then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \to |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}}$,

if $z \in P_2$ and $\arg z$ tends to π^+ , then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \to |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}}$,

so $\sqrt{z}$ is continuous cross the cut $(-\infty,0]$ for $z \in D_1$.

Therefore, we can define that

$$f_3(z) = \sqrt{z}\ ,\ z \in D_1,$$

then $f_3(z) = \sqrt{z} = i|z|^{\frac{1}{2}}$ and which is analytic on D_1.

On the other hand, let $D_2 = \{(-\infty,0] \mid \arg z = 3\pi\}$, the figure show as the following.

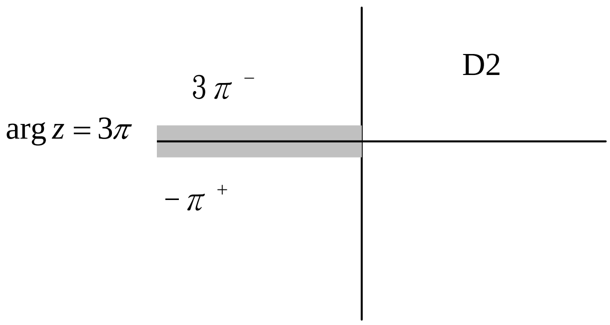


Fig. 3-3 ($D_2 = \{(-\infty,0] \mid \arg z = 3\pi\}$)

If $z \in P_2$ and $\arg z$ tends to $3\pi^-$, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \to |z|^{\frac{1}{2}} e^{i\frac{3\pi}{2}} = -i|z|^{\frac{1}{2}}$,

if $z \in P_1$ and $\arg z$ tends to $-\pi^+$, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \to |z|^{\frac{1}{2}} e^{i(-\frac{\pi}{2})} = -i|z|^{\frac{1}{2}}$,

so $\sqrt{z}$ is continuous cross the cut $(-\infty, 0]$ for $z \in D_2$.

Similarly, we can define that

$$f_4(z) = \sqrt{z}\,, \quad z \in D_2\,,$$

then $f_4(z) = -i|z|^{\frac{1}{2}} = -f_3(z)$ and which is analytic on D_2.

According to the above discussion, we can construct a single-valued function for $\sqrt{z}$, the conclusion as the following:

Let $D = P_1 \cup P_2 \cup (-\infty, 0]$ and the function $F : D \to C$ defined as

$$F(z) = \begin{cases} f_1(z) & , \quad z \in P_1 \\ f_2(z) & , \quad z \in P_2 \\ f_3(z) & , \quad z \in D_1 \\ f_4(z) & , \quad z \in D_2 \end{cases}$$

the function $F(z)$ is really single-valued and analytic on D, satisfying $f_1(z) = -f_2(z)$ and $f_3(z) = -f_4(z)$.

3.2 The Riemann surface of $f(z) = \sqrt{\prod_{i=1}^{n}(z - z_i)}$ with $z_j \in R$:

Let $f(z) = \sqrt{\prod_{i=1}^{n}(z - z_i)}$, $z_j \in R$ with n distinct real numbers and satisfy that $z_1 > z_2 > \cdots\cdots > z_n$. For example, we can consider that $n = 2$ and the branch point are $z_1 = 1$, and $z_2 = 2$.

If the cut show as in Fig. 3-4

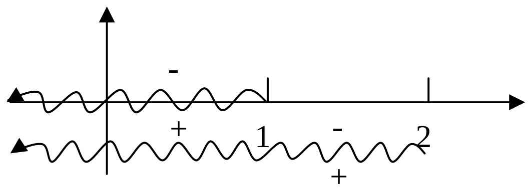


Fig. 3-4 (The cut in two branch point)

(a) If $z \in (-\infty, 1)$, then

$$\arg(z-1) = \begin{cases} -\pi \\ \pi \end{cases},$$

$$\arg(z-2) = \begin{cases} -\pi \\ \pi \end{cases}.$$

Taking $-\pi$: $\sqrt{z-1}\cdot\sqrt{z-2} = |z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}} e^{i(\frac{-2\pi}{2})} = -|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$, (3.3)

Taking π: $\sqrt{z-1}\cdot\sqrt{z-2} = |z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}} e^{i(\frac{2\pi}{2})} = -|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$ (3.4)

Since (3.3) = (3.4), there is no cut in $(-\infty, 1)$.

(b) If $z \in (1,2)$, then

$$\arg(z-1) = 0,$$

$$\arg(z-2) = \begin{cases} -\pi \\ \pi \end{cases}.$$

Taking $-\pi$: $\sqrt{z-1}\cdot\sqrt{z-2} = |z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}} e^{i(\frac{-\pi}{2})} = -i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$, (3.5)

Taking π : $\sqrt{z-1}\cdot\sqrt{z-2} = |z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} = i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$ (3.6)

Since (3.5) $\neq$ (3.6), there is a cut in $(1,2)$

(c) If $z \in (2, \infty)$, then

$$\arg(z-1) = 0$$

$$\arg(z-2) = 0$$

It is clearly, there is no cut in $(2, \infty)$.

Thus, we have a branch cut in (1,2) as Fig 3-5.

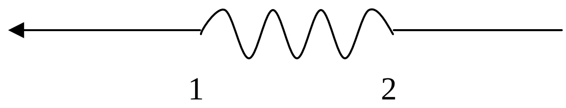


Fig 3-5 Branch cut in (1,2)

The second example considers that $n=3$, and the branch points are $z_1=1$, $z_2=2$, $z_3=3$.

(a) If $z \in (-\infty,1)$, then

$$\arg(z-1)=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(z-2)=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(z-3)=\begin{cases}-\pi\\ \pi\end{cases}.$$

Taking $-\pi$: $\sqrt{z-1}\cdot\sqrt{z-2}\sqrt{z-3}=-i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}|z-3|^{\frac{1}{2}}$, (3.7)

Taking π: $\sqrt{z-1}\cdot\sqrt{z-2}\sqrt{z-3}=i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}|z-3|^{\frac{1}{2}}$. . (3.8)

Since (3.7) $\neq$ (3.8), there is a cut in $(-\infty,1)$.

(b) If $z \in (1,2)$, then

$$\arg(z-1)=0,$$

$$\arg(z-2)=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(z-3)=\begin{cases}-\pi\\ \pi\end{cases}.$$

Taking $-\pi$: $\sqrt{z-1}\cdot\sqrt{z-2}\sqrt{z-3}=-|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}|z-3|^{\frac{1}{2}}$, (3.9)

Taking π: $\sqrt{z-1}\cdot\sqrt{z-2}\sqrt{z-3}=-|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}|z-3|^{\frac{1}{2}}$. (3.10)

Since (3.9) $=$ (3.10), there is no cut in $(1,2)$.

(c) If $z \in (2,3)$, then

$$\arg(z-1)=0,$$

$$\arg(z-2)=0,$$

$$\arg(z-3)=\begin{cases}-\pi \\ \pi\end{cases}.$$

Taking $-\pi$: $\sqrt{z-1}\cdot\sqrt{z-2}\sqrt{z-3}=-i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}|z-3|^{\frac{1}{2}}$. (3.11)

Taking π: $\sqrt{z-1}\cdot\sqrt{z-2}\sqrt{z-3}=i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}|z-3|^{\frac{1}{2}}$ (3.12)

Since (3.11) $\neq$ (3.12), there is a cut in $(2,3)$.

(d) If $z \in (3,\infty)$, then

$$\arg(z-1)=0,$$

$$\arg(z-2)=0,$$

$$\arg(z-3)=0.$$

It is clearly, there is no cut in $(3,\infty)$.

Thus, we have a branch cut in $(-\infty,1)$, $(2,3)$. The graph shows in Fig 3.6

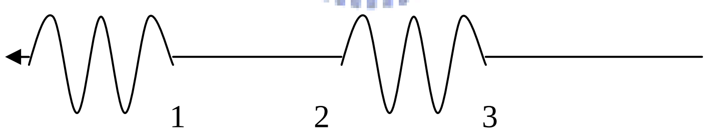
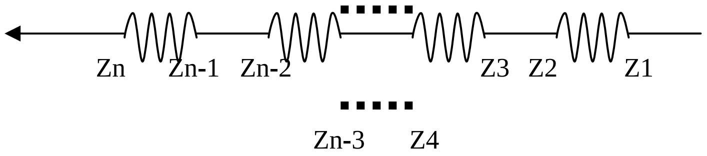


Fig 3.6 Branch cut in $(-\infty,1)$, $(2,3)$

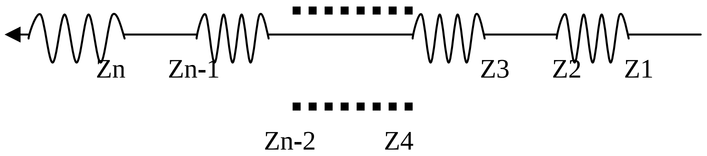
Note:

If we crosses the cut even times in each line section, then it will not change the sign. And if we crosses the cut odd times in each line section, then it will change the sign. It implies that the line section will become a branch cut. That is, given n branch points, if n is even, then the branch cuts are this sections like $[z_n, z_{n-1}]$、$[z_{n-2}, z_{n-3}]$...... and $[z_2, z_1]$. If n is odd, then the branch cuts are this sections like

$(-\infty, z_n]$ 、 $[z_{n-1}, z_{n-2}]$…and $[z_2, z_1]$. The graph shows in Fig 3-7.



n is even.



n is odd.

Fig 3-7

<u>3.3 The algebraic and geometric structure for Riemann surface of horizontal cut:</u>

We will discuss the structure for Riemann surface of $f(z) = \sqrt{\prod_{j=1}^{3}(z - z_j)}$ in horizontal cut.

(a) Algebra structure:

Let $(-\infty, z_3]$ 、 $[z_2, z_1]$ represent the cuts in this Riemann surface and 〝+〞,〝–〞 are defined as following(the initial edge with +, the terminal edge with –) :

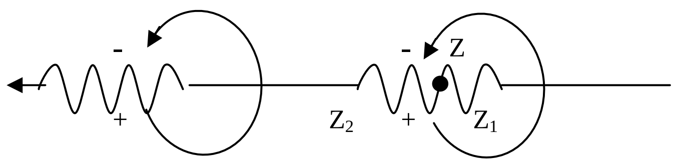


Fig 3-8

(i) If $z \in I^+$ (+ edge of sheet Ⅰ), and $z \in [z_2, z_1]$.

Since $z - z_j > 0 \;\Rightarrow\; \arg(z - z_j) = 0$ for $j = 2,3$.

$$z - z_j < 0 \;\Rightarrow\; \arg(z - z_j) = -\pi \quad \text{for} \quad j = 1.$$

Then

$$f(z) = \sqrt{\prod_{j=1}^{3}(z - z_j)} = \prod_{j=1}^{3}\sqrt{z - z_j}$$

$$= |z - z_1|^{\frac{1}{2}} e^{i(-\frac{\pi}{2})} \cdot \prod_{j=2}^{3}|z - z_j|^{\frac{1}{2}} e^{i\cdot 0}$$

$$= e^{i(-\frac{\pi}{2})} \cdot \prod_{j=1}^{3}|z - z_j|^{\frac{1}{2}} = (-i)\cdot\prod_{j=1}^{3}|z - z_j|^{\frac{1}{2}}. \tag{3.13}$$

(ii) If $z \in I^-$ ($-$ edge of sheet Ⅰ), and $z \in [z_2, z_1]$.

Since $z - z_j > 0 \;\Rightarrow\; \arg(z - z_j) = 0$ for $j = 2,3$.

$$z - z_j < 0 \;\Rightarrow\; \arg(z - z_j) = \pi \quad \text{for} \quad j = 1.$$

Then

$$f(z) = \sqrt{\prod_{j=1}^{3}(z - z_j)} = \prod_{j=1}^{3}\sqrt{z - z_j}$$

$$= |z - z_1|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} \cdot \prod_{j=2}^{3}|z - z_j|^{\frac{1}{2}} e^{i\cdot 0}$$

$$= e^{i(\frac{\pi}{2})} \cdot \prod_{j=1}^{3}|z - z_j|^{\frac{1}{2}} = (i)\cdot\prod_{j=1}^{3}|z - z_j|^{\frac{1}{2}}. \tag{3.14}$$

Because of that $f(z)\,|_{I^-} = -f(z)\,|_{I^+}$, this result implies that

$$f(z)\,|_{II} = -f(z)\,|_{I} \tag{3.15}$$

(b) Geometric structure:

We will discuss the geometric structure for Riemann surface of $f(z)=\sqrt{z}$. Since there is a cut in $(-\infty,0]$, and we obtain one sheet with two edges in each cut by taken of counterclockwise which labeled the edge of lower-cut with + and the edge of upper-cut with –. There are two surface, one is, say sheet Ⅰ with $\arg f(z)\in[-\pi,\pi)$; another is, say sheet Ⅱ with $\arg f(z)\in[\pi,3\pi)$. We must attach the lower edge of sheet Ⅰ to the upper edge of sheet Ⅱ, the upper edge of sheet Ⅰ to the lower edge of sheet Ⅱ. This is without self-intersection. This result of the construction is a Riemann surface whose points are in one-to-one correspondence with the points in the $f(z)-plane$. Especially, this correspondence is continuous if the continuity is defined in the sense suggested by the construction.

We take $n=3$ to discuss the geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ in horizontal cuts, as shown in Fig.3-9. The first is a sphere with two cuts, and become the second like a balloon has two holes. Then we copy this balloon and gum them according to the above method. Finally, it looks like a donut. Because for $n=3$ it have four branch points (∞ is an branch point).

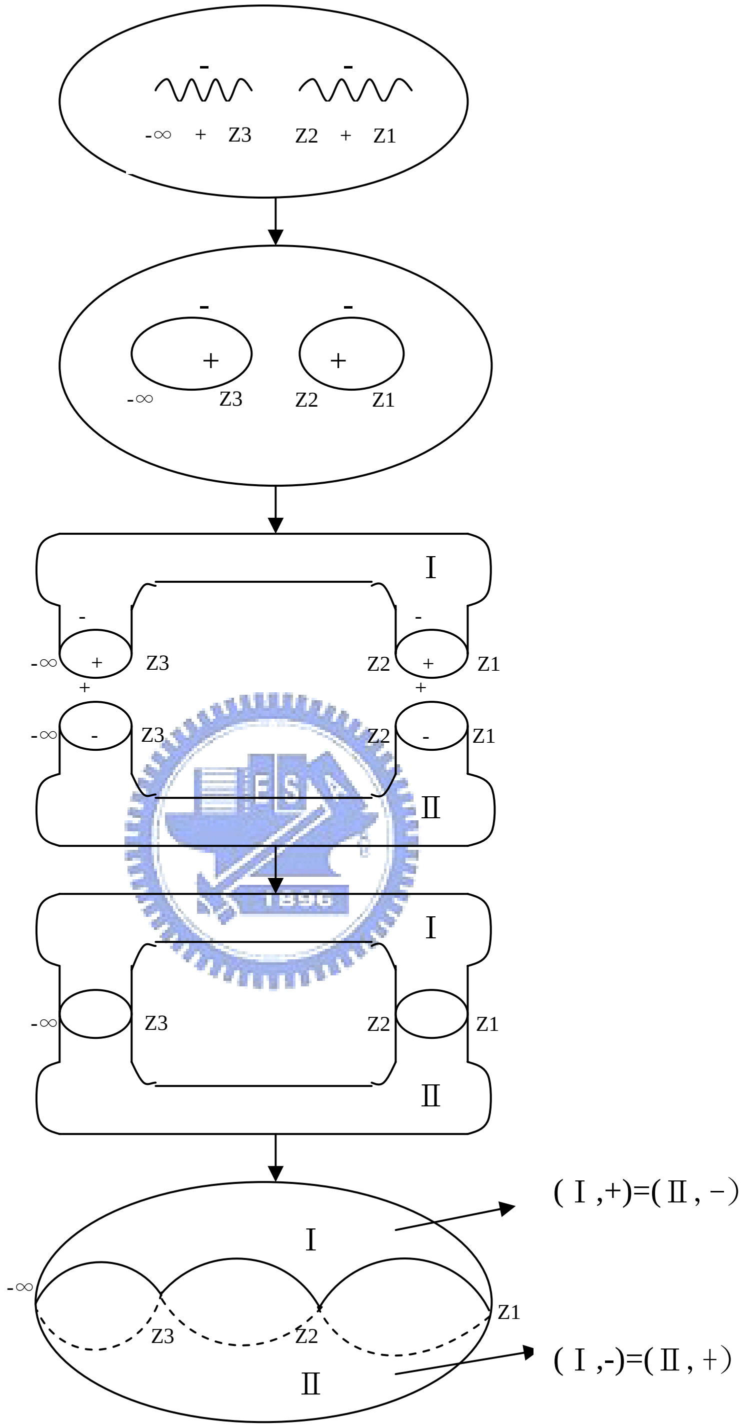


Fig 3-9 Geometric structure

(c) Algebraic structure *v.s* Geometric structure:

We also use $n = 3$ to discuss. Before talking about the relation between algebraic structure and geometric structure, we need to denote something as the following :

(i) If the curve is drawn by solid line :

In algebraic structure, it means the curve is in sheet Ⅰ;

In geometric structure, it means the curve is in the overhead Riemann surface.

(ii) If the curve is drawn by dash line :

In algebraic structure, it means the curve is in sheet Ⅱ;

In geometric structure, it means the curve is in the ventral Riemann surface.

We give some example to show that the curve in algebraic structure and its corresponding in geometric structure in Fig.3-10 to Fig.3-12.

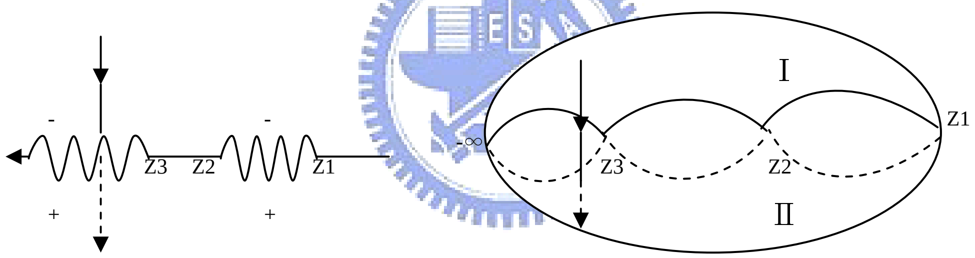


Fig 3-10 Algebraic structure *v.s* Geometric structure

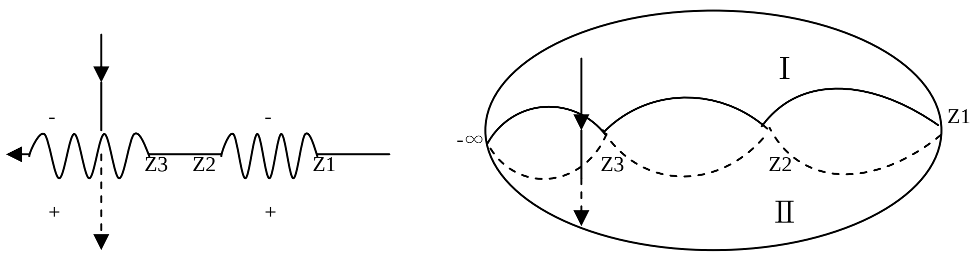


Fig 3-11 Algebraic structure *v.s* Geometric structure

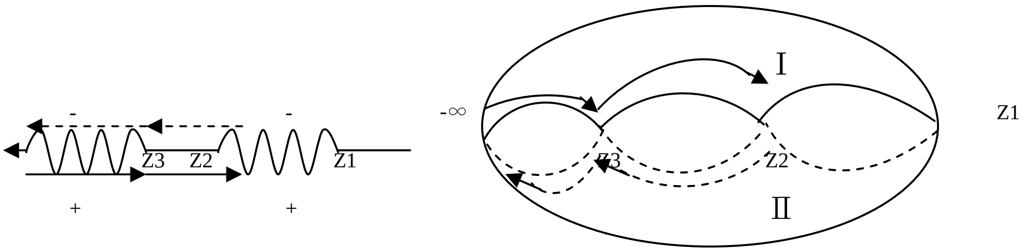


Fig 3-12 Algebraic structure $v.s$ Geometric structure

3.4 The Riemann surface of $f(z)=\sqrt{\prod_{i=1}^{n}(z-z_i)}$ with $z_j \in C$:

In this section, we will discuss the Riemann surface with the vertical cut structure. Similarly, the construction is like the construction of the horizontal cut. We can define that $f(z)=\sqrt{\prod_{i=1}^{n}(z-z_i)}$ and $(z, f(z))$ belong to sheet I iff $\arg f(z) \in [-\frac{3\pi}{2}, \frac{\pi}{2})$,

i.e. $\arg(z-z_j) \in [-\frac{3\pi}{2}, \frac{\pi}{2})$ and $(z, f(z))$ belong to sheet II iff $\arg f(z) \in [\frac{\pi}{2}, \frac{3\pi}{2})$

i.e. $\arg(z-z_j) \in [-\frac{3\pi}{2}, \frac{\pi}{2})$.

About the vertical cut structure analysis method is the same as horizontal cut structure. We can consider that $n=2$ and $z_1 = i$, $z_2 = 2i$.

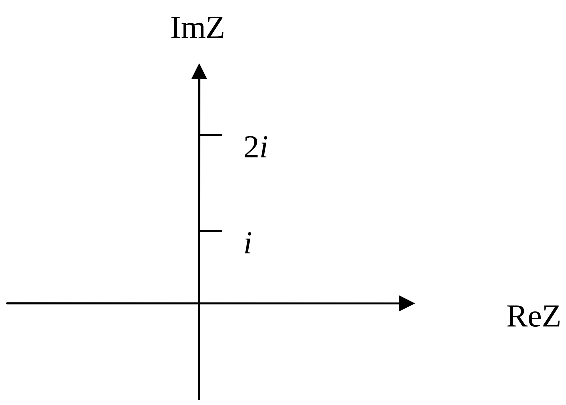


Fig 3-13

(a) If $z \in (\infty, 2i)$, then

$$\arg(z-i) = \begin{cases} -\dfrac{3\pi}{2} \\ \dfrac{\pi}{2} \end{cases},$$

$$\arg(z-2i) = \begin{cases} -\dfrac{3\pi}{2} \\ \dfrac{\pi}{2} \end{cases}.$$

Taking $-\dfrac{3\pi}{2}$: $\sqrt{z-i}\cdot\sqrt{z-2i} = |z-i|^{\frac{1}{2}}|z-2i|^{\frac{1}{2}}e^{i(\frac{-3\pi}{2})} = i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$ (3.13)

Taking $\dfrac{\pi}{2}$: $\sqrt{z-i}\cdot\sqrt{z-2i} = |z-i|^{\frac{1}{2}}|z-2i|^{\frac{1}{2}}e^{i(\frac{\pi}{2})} = i|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$ (3.14)

Since (3.13) = (3.14), there is no cut in $(\infty, 2i)$.

(b) If $z \in (i, 2i)$, then

$$\arg(z-i) = \begin{cases} -\dfrac{3\pi}{2} \\ \dfrac{\pi}{2} \end{cases},$$

$$\arg(z-i) = -\frac{\pi}{2}.$$

Taking $-\dfrac{3\pi}{2}$: $\sqrt{z-i}\cdot\sqrt{z-2i} = |z-i|^{\frac{1}{2}}|z-2i|^{\frac{1}{2}}e^{i(-\pi)} = -|z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$ (3.15)

Taking $\dfrac{\pi}{2}$: $\sqrt{z-i}\cdot\sqrt{z-2i} = |z-i|^{\frac{1}{2}}|z-2i|^{\frac{1}{2}}e^{i(0)} = |z-1|^{\frac{1}{2}}|z-2|^{\frac{1}{2}}$. (3.16)

Since (3.15) $\neq$ (3.16), there is a cut in $(i, 2i)$.

(c) If $z \in (-\infty, i)$, then it is like in (a).

Thus, we have a branch cut in $(i, 2i)$.

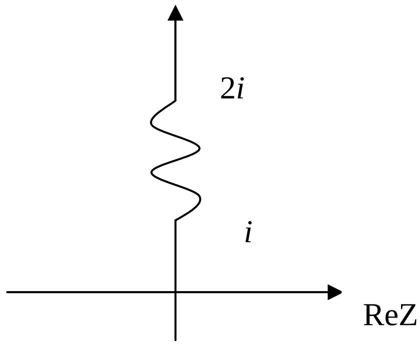


Fig 3-14 Branch cut in $(i, 2i)$

We can use the simpler way to get branch cut. We take $n = 4$ with $z_1 = i$ 、 $z_2 = 2i$ 、 $z_3 = 3i$ and $z_4 = 4i$, that is, $z_1 < z_2 < z_3 < \ldots < z_n$, as shown in Fig.3-15.

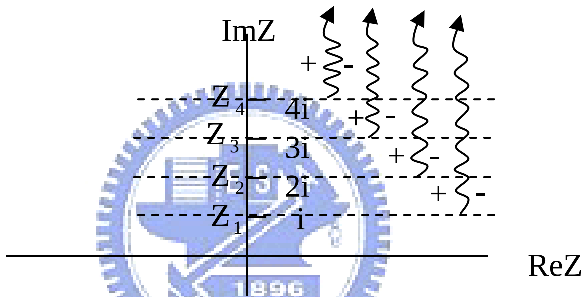


Fig.3-15 Branch cuts

When crossing the cut even times in each line section, it will not change sign. When crossing the cut odd times in each line section will change sign，this implies the line section will form a branch cut. Hence we have the branch cuts in $[z_4, z_3]$ and $[z_2, z_1]$. The cut structure is showed in Fig.3-16.

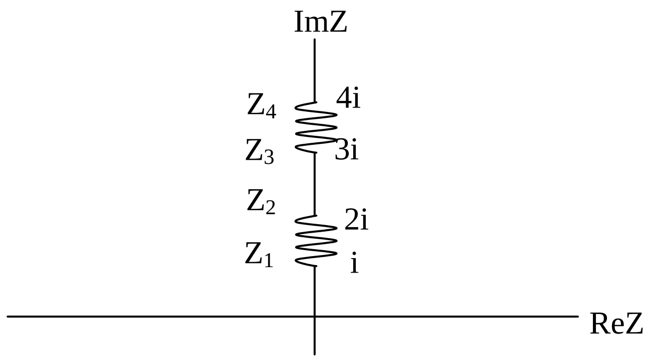


Fig.3-16 Branch cuts in $[z_4, z_3]$ and $[z_2, z_1]$

3.5 The algebraic and geometric structure for Riemann surface of vertical cut:

For simplicity, we use $n = 4$ to discuss the structure for Riemann surface of $f(z) = \sqrt{\prod_{j=1}^{4}(z - z_j)}$ in vertical cut. In the cut structure, we still depend on the countclockwise to take 〝+〞、〝–〞 sign. The definition of solid-line and dash-line are the same as horizontal cut case.

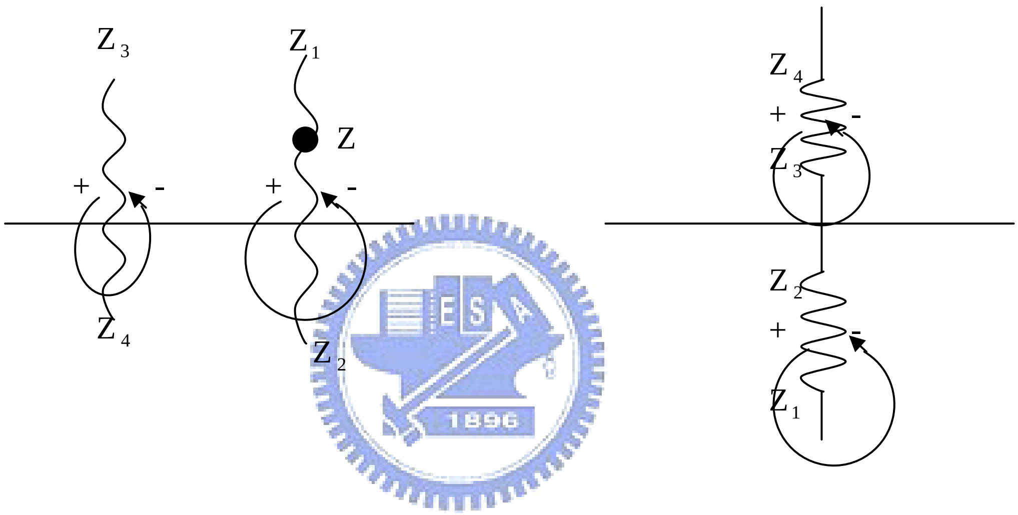


Fig.3-17

(a) Algebra structure:

Let $[z_4, z_3]$、$[z_2, z_1]$ represent the cuts in this Riemann surface and 〝+〞，〝–〞 are defined as following(the initial edge with +, the terminal edge with –) :

(i) If $z \in I^+$ (+ edge of sheet Ⅰ), and $z \in [z_2, z_1]$.

Since $\arg(z - z_1) = -\frac{\pi}{2}$ and $\arg(z - z_2) = -\frac{3\pi}{2}$. $\arg(z - z_j) \in (-\pi, \frac{\pi}{2})$ for $j = 3,4$.

Then

$$f(z)=\sqrt{\prod_{j=1}^{4}(z-z_j)} \quad =\prod_{j=1}^{4}\sqrt{z-z_j}$$

$$=\left|z-z_2\right|^{\frac{1}{2}}e^{i(-\frac{3\pi}{4})}\cdot\prod_{j=1,3,4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\frac{\arg(z-z_j)}{2}}$$

$$=(-\frac{\sqrt{2}}{2}i)\left|z-z_2\right|^{\frac{1}{2}}\cdot\prod_{j=1,3,4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\frac{\arg(z-z_j)}{2}}. \qquad (3.17)$$

(ii) If $z\in I^{-}$ (– edge of sheet Ⅰ), and $z\in[z_2,z_1]$.

Since $\arg(z-z_1)=-\frac{\pi}{2}$ and $\arg(z-z_2)=\frac{\pi}{2}$. $\arg(z-z_j)\in(-\pi,\frac{\pi}{2})$ for $j=3,4$.

Then

$$f(z)=\sqrt{\prod_{j=1}^{4}(z-z_j)}=\prod_{j=1}^{4}\sqrt{z-z_j}$$

$$=\left|z-z_2\right|^{\frac{1}{2}}e^{i(\frac{\pi}{4})}\cdot\prod_{j=1,3,4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\frac{\arg(z-z_j)}{2}} \qquad (3.18)$$

$$=(\frac{\sqrt{2}}{2}i)\left|z-z_2\right|^{\frac{1}{2}}\cdot\prod_{j=1,3,4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\frac{\arg(z-z_j)}{2}} \qquad (3.19)$$

Because of that $f(z)\mid_{I^-} = -f(z)\mid_{I^+}$, this result implies that

$$f(z)\mid_{II} = -f(z)\mid_{I}$$

(b) Geometric structure:

The construct a geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ is the same as horizontal cuts. The graph show in Fig 3-18

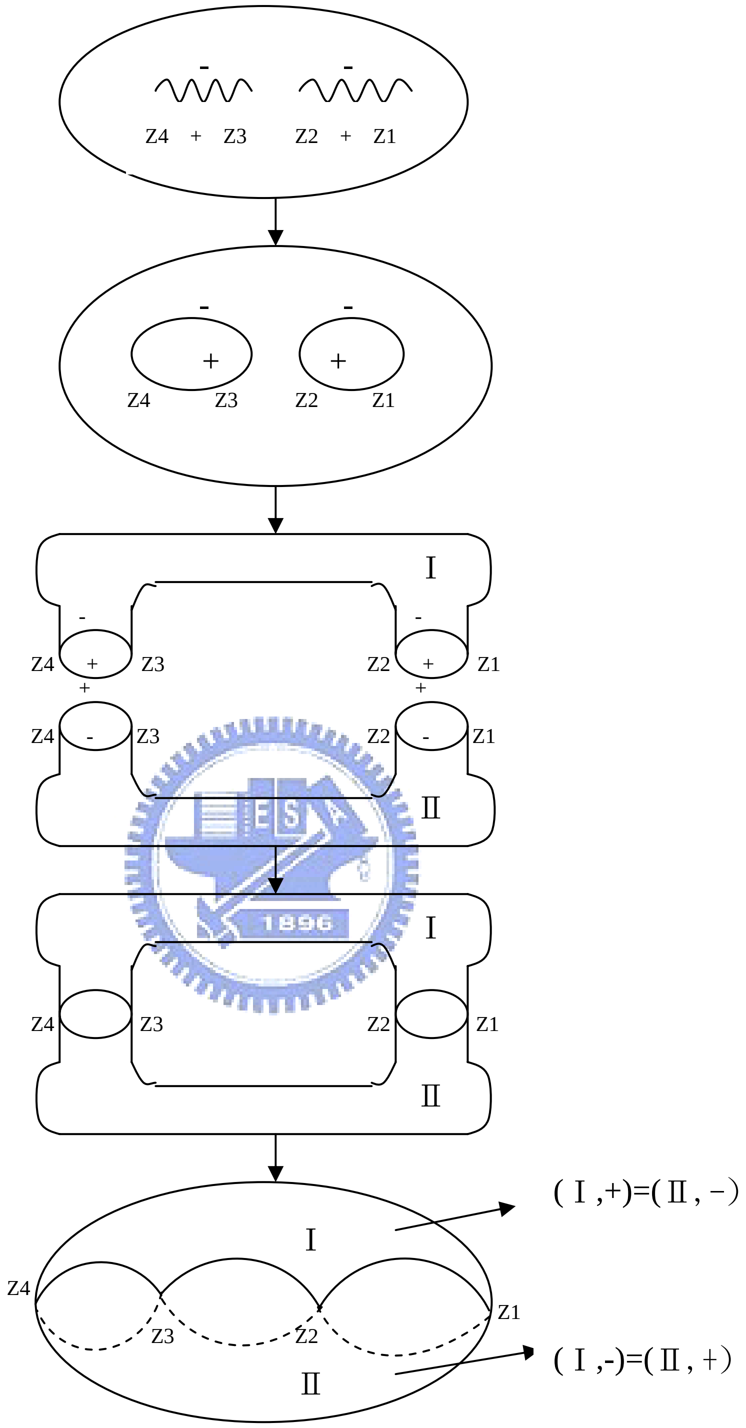


Fig 3-18 Geometric structure

3.6 Application in Riemann surface (Complex Integral):

We will give some examples about the horizontal cut.

Example1.

Evaluate that $\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}dz$, where γ is a circle of radius $\frac{3}{2}$, 2 , and 3 at center $\frac{3}{2}$.

(i) γ is a circle of radius $\frac{3}{2}$ at center $\frac{3}{2}$.

(a) Path integral

Let $z = \frac{3}{2} + \frac{3}{2}e^{i\theta}$, $dz = \frac{3}{2}ie^{i\theta}d\theta$, then

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}dz = \int_{-\pi}^{\pi} \frac{3}{2}ie^{i\theta}\sqrt{\frac{3}{2}+\frac{3}{2}e^{i\theta}-1}\sqrt{\frac{3}{2}+\frac{3}{2}e^{i\theta}-2}d\theta = -\frac{\pi}{4}i \approx -0.785398i$$

(b) Riemann surface

Integral path	1 to 2 angle	1 to 2 value	2 to 1 angle	2 to 1 value
$\sqrt{z-1}$	0	1	0	1
$\sqrt{z-2}$	$-\pi$	$-i$	π	i
Sheet	I^{+}		I^{-}	
Total $\sqrt{z-1}\sqrt{z-2}$	$-i$		i	

Table 3-1

The integral becomes as following,

$$\int_{I^{+}} \sqrt{z-1}\sqrt{z-2}dz = -i\int_{0}^{1}\sqrt{r}\sqrt{r-1}dr \approx -0.392699i\ .$$

$$\int_{I^{-}} \sqrt{z-1}\sqrt{z-2}dz = i\int_{1}^{0}\sqrt{r}\sqrt{r-1}dr \approx -0.392699i\ .$$

By Riemann surface theory we have that,

$$\int_{I^+} \sqrt{z-1}\sqrt{z-2}dz = \int_{I^-} \sqrt{z-1}\sqrt{z-2}dz$$

Therefore, the integral $\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}dz = 2\int_{I^+} \sqrt{z-1}\sqrt{z-2}dz \approx -0.785398i$.

(ii) γ is a circle of radius 2 at center $\frac{3}{2}$.

(a) Path integral

Let $z = \frac{3}{2} + 2e^{i\theta}$, $dz = 2ie^{i\theta}d\theta$, then

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}dz = \int_{-\pi}^{\pi} 2ie^{i\theta}\sqrt{\frac{3}{2}+2e^{i\theta}-1}\sqrt{\frac{3}{2}+2e^{i\theta}-2}d\theta = -\frac{\pi}{4}i \approx -0.785398i.$$

(b) Riemann surface

According to the above (i) and Table 3-1, we have

$$\int_{I^+} \sqrt{z-1}\sqrt{z-2}dz \approx -0.392699i,$$

$$\int_{I^-} \sqrt{z-1}\sqrt{z-2}dz \approx -0.392699i.$$

Therefore, the integral $\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}dz = 2\int_{I^+} \sqrt{z-1}\sqrt{z-2}dz \approx -0.785398i$.

(iii) γ is a circle of radius 3 at center $\frac{3}{2}$.

(a) Path integral

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}dz \approx -0.785398i.$$

(b) Riemann surface

$$\int_{I^+} \sqrt{z-1}\sqrt{z-2}dz \approx -0.392699i,$$

$$\int_{I^-} \sqrt{z-1}\sqrt{z-2}dz \approx -0.392699i.$$

This result does not surprise us. Because the path integral will remain the cut $[1,2]$, others be cancelled by analytic.

Example2.

Evaluate that $\oint_\gamma \frac{1}{\sqrt{z-1}\sqrt{z-2}}dz$, where γ is a circle of radius $\frac{3}{2}$, 2, 3 at center $\frac{3}{2}$.

(i) γ is a circle of radius $\frac{3}{2}$ at center $\frac{3}{2}$.

(a) Path integral

Let $z=\frac{3}{2}+\frac{3}{2}e^{i\theta}$, $dz=\frac{3}{2}ie^{i\theta}d\theta$, then

$$\int_\gamma \frac{1}{\sqrt{z-1}\sqrt{z-2}}dz=\int_{-\pi}^{\pi}\frac{3}{2}ie^{i\theta}\frac{1}{\sqrt{\frac{3}{2}+\frac{3}{2}e^{i\theta}-1}\sqrt{\frac{3}{2}+\frac{3}{2}e^{i\theta}-2}}d\theta=2i\pi.$$

(b) Riemann surface

	1 to 2		2 to 1	
Integral path	angle	value	angle	value
$\sqrt{z-1}$	0	1	0	1
$\sqrt{z-2}$	$-\pi$	$-i$	π	i
Sheet	I^+		I^-	
Total $\frac{1}{\sqrt{z-1}\sqrt{z-2}}$	i		$-i$	

Table 3-2

The integral becomes as following,

$$\int_{I^+}\frac{1}{\sqrt{z-1}\sqrt{z-2}}dz=i\int_0^1\frac{1}{\sqrt{r}\sqrt{1-r}}dr=i\pi$$

$$\int_{I^-}\frac{1}{\sqrt{z-1}\sqrt{z-2}}dz=-i\int_1^0\frac{1}{\sqrt{r}\sqrt{1-r}}dr=i\pi$$

By Riemann surface theory we have that,

$$\int_{I^+} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = \int_{I^-} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz$$

$$\int_{\gamma} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = 2\int_{I^+} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = 2i\pi\,.$$

(ii) γ is a circle of radius 2 at center $\frac{3}{2}$.

(a) Path integral

Let $z = \frac{3}{2} + 2e^{i\theta}$, $dz = 2ie^{i\theta}d\theta$, then

$$\int_{\gamma} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = \int_{-\pi}^{\pi} 2ie^{i\theta}\,\frac{1}{\sqrt{\frac{3}{2}+2e^{i\theta}-1}\sqrt{\frac{3}{2}+2e^{i\theta}-2}}\,d\theta = 2i\pi$$

(b) Riemann surface

According to the above (i) and Table 3-2, we have

$$\int_{I^+} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = i\int_{0}^{1} \frac{1}{\sqrt{r}\sqrt{1-r}}\,dr = i\pi$$

$$\int_{I^-} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = -i\int_{1}^{0} \frac{1}{\sqrt{r}\sqrt{1-r}}\,dr = i\pi$$

By Riemann surface theory

$$\int_{\gamma} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = 2\int_{I^+} \frac{1}{\sqrt{z-1}\sqrt{z-2}}\,dz = 2i\pi\,.$$

(iii) γ is a circle of radius 3 at center $\frac{3}{2}$.

It must be the same as (i) and (ii). The result is also the same as (i) and (ii).

We can say that every closed simple curve γ whose region contain the cut $[z_1, z_2]$ where , z_1 and z_2 in R, then the integral will be

$$\int_{\gamma} \frac{1}{\sqrt{z-z_1}\sqrt{z-z_2}}\,dz = 2\int_{I^+} \frac{1}{\sqrt{z-z_1}\sqrt{z-z_2}}\,dz\,,$$

and
$$\int_{I^+} \frac{1}{\sqrt{z-z_1}\sqrt{z-z_2}}\,dz = \int_{I^-} \frac{1}{\sqrt{z-z_1}\sqrt{z-z_2}}\,dz$$

Example 3

Evaluate that $\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz$, where γ is a circle of radius $\frac{5}{2}$, 3 at center $\frac{5}{2}$.

(i) γ is a circle of radius $\frac{5}{2}$ at center $\frac{5}{2}$.

(a) Path integral

Let $z = \frac{5}{2} + \frac{5}{2}e^{i\theta}$, $dz = \frac{5}{2}ie^{i\theta}d\theta$, then

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz$$

$$= \int_{-\pi}^{\pi} \frac{5}{2}ie^{i\theta}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-1}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-2}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-3}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-4}d\theta$$

$$\approx -9.4369\times10^{-16} + 5.21805\times10^{-15}i$$

(b) Riemann surface

For simple we let $f(z) = \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}$

	1 to 2		2 to 1	
Integral path	angle	value	angle	value
$\sqrt{z-1}$	0	1	0	1
$\sqrt{z-2}$	$-\pi$	$-i$	π	i
$\sqrt{z-3}$	$-\pi$	$-i$	π	i
$\sqrt{z-4}$	$-\pi$	$-i$	π	i
Sheet	I^+		I^-	
Total $f(z)$	i		$-i$	

Table 3-3

	3 to 4		4 to 3	
Integral path	angle	value	angle	value
$\sqrt{z-1}$	0	1	0	1
$\sqrt{z-2}$	0	1	0	1
$\sqrt{z-3}$	0	1	0	1
$\sqrt{z-4}$	$-\pi$	$-i$	π	i
Sheet	I^{+}		I^{-}	
Total $f(z)$	$-i$		i	

Table 3-4

The integral becomes as following,

$$\int_{I^{+}} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz$$

$$= i\int_0^1 \sqrt{r}\sqrt{1-r}\sqrt{2-r}\sqrt{3-r}dr + (-i)\int_0^1 \sqrt{r}\sqrt{1-r}\sqrt{r+1}\sqrt{r+2}dr$$

$$\approx 0.76002i - 0.76002i$$

$$\approx 0$$

$$\int_{I^{-}} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz$$

$$= -i\int_1^0 \sqrt{r}\sqrt{1-r}\sqrt{2-r}\sqrt{3-r}dr + i\int_1^0 \sqrt{r}\sqrt{1-r}\sqrt{r+1}\sqrt{r+2}dr$$

$$\approx 0.76002i - 0.76002i$$

$$\approx 0$$

By Riemann surface theory we also have that,

$$\int_{I^{+}} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz = \int_{I^{-}} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz \approx 0$$

And

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz = 2\int_{I^{+}} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz \approx 0$$

But the numerical value is $-9.4369\times10^{-16} + 5.21805\times10^{-15} i \neq 0$

Why do we gain this result? Does it contradict to our Riemann surface theory?

Let's to look at the value $-9.4369\times10^{-16}+5.21805\times10^{-15}i$, the real part and the imaginary part multiply a small order term, respectively. Because the computer has some error, the value $\oint_{\gamma}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz$ is always equal to zero.

Why does this $\int_{I^+}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz=\int_{I^-}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz=0$?

Because $f(z)=\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}$ in our integral look on as the radius function, it is symmetric! We can try another $f(z)=\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}$ with the same path.

$$\oint_{\gamma}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz$$

$$=\int_{-\pi}^{\pi}\frac{5}{2}ie^{i\theta}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-1}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-2}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-3}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-3.5}d\theta$$

$$\approx1.77636\times10^{-15}+1.03084i$$

$\approx1.03084i$ (the reason is discussed in above)

And

$$\int_{I^+}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz$$

$$=i\int_0^1\sqrt{1-r}\sqrt{r}\sqrt{2-r}\sqrt{2.5-r}dr+(-i)\int_0^{0.5}\sqrt{1.5-r}\sqrt{r}\sqrt{r+1}\sqrt{r+2}dr$$

$$\approx0.68002i-0.164603i$$

$$\approx0.515418i$$

$$\int_{I^-}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz$$

$$=-i\int_1^0\sqrt{1-r}\sqrt{r}\sqrt{2-r}\sqrt{2.5-r}dr+i\int_{0.5}^0\sqrt{1.5-r}\sqrt{r}\sqrt{r+1}\sqrt{r+2}dr$$

$$\approx0.68002i-0.164603i$$

$$\approx0.515418i$$

Again confirms that,

$$\int_{I^+} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz = \int_{I^-} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz$$

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz \approx 1.03084i$$

$$2\int_{I^+} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz \approx 5.15418i \times 2 \approx 1.03084i$$

This time the integral $\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz$ is not zero.

(ii) γ is a circle of radius 3 at center $\frac{5}{2}$.

(a) Path integral

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz$$

$$= \int_{-\pi}^{\pi} 3ie^{i\theta} \sqrt{\frac{5}{2}+3e^{i\theta}-1}\sqrt{\frac{5}{2}+3e^{i\theta}-2}\sqrt{\frac{5}{2}+3e^{i\theta}-3}\sqrt{\frac{5}{2}+3e^{i\theta}-4}d\theta$$

$$\approx 2.22045\times 10^{-16} + 1.70974\times 10^{-14} i$$

$$\approx 0$$

Try again that $\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz$ with γ is a circle of radius 3 at center $\frac{5}{2}$.

$$\oint_{\gamma} \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-3.5}dz \approx 3.55271\times 10^{-15} + 1.03084i \approx 1.03084i .$$

Example 4

Evaluate that $\oint_\gamma \frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$, where γ is a circle of radius $\frac{5}{2}$, at center $\frac{5}{2}$.

(i) γ is a circle of radius $\frac{5}{2}$ at center $\frac{5}{2}$.

(a) Path integral

Let $z=\frac{5}{2}+\frac{5}{2}e^{i\theta}$, $dz=\frac{5}{2}ie^{i\theta}d\theta$, then

$$\oint_\gamma \frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$=\int_{-\pi}^{\pi}\frac{5}{2}ie^{i\theta}\frac{1}{\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-1}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-2}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-3}\sqrt{\frac{5}{2}+\frac{5}{2}e^{i\theta}-4}}d\theta$$

$\approx 2.77556\times 10^{-17}+1.40513\times 10^{-16}\approx 0$.

(b) Riemann surface

For simple we let $f(z)=\frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}$ and by Table 3-3, Table 3-4 we have that,

$$\int_{I^+}\frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$=i\int_0^1\frac{1}{\sqrt{r}\sqrt{1-r}\sqrt{2-r}\sqrt{3-r}}dr+(-i)\int_0^1\frac{1}{\sqrt{r}\sqrt{1-r}\sqrt{r+1}\sqrt{r+2}}dr$$

$\approx 1.68575i-1.68575i$

≈ 0

$$\int_{I^-}\frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$=-i\int_1^0\frac{1}{\sqrt{r}\sqrt{1-r}\sqrt{2-r}\sqrt{3-r}}dr+i\int_1^0\frac{1}{\sqrt{r}\sqrt{1-r}\sqrt{r+1}\sqrt{r+2}}dr$$

$\approx 1.68575i-1.68575i$

≈ 0

By Riemann surface theory we also have that,

$$\int_{I^+} \frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}\,dz = \int_{I^-} \frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}\,dz \approx 0$$

And

$$\oint_{\gamma} \frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}\,dz = 2\int_{I^+} \frac{1}{\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}\,dz \approx 0$$