國 立 交 通 大 學

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碩 士 論 文

多重穩定性對生物擺動器的延遲、耦合強度和凹凸性之關係

Delay, Coupling Strengths, Concavity and Multistable Synchronization of Biological Oscillators

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中 華 民 國 九 十 七 年 六 月

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摘　要

我們研究了對於兩個 Mirollo-Strogatz-type 的擺動器系統中耦合強度係數、延遲時間和其函數凹凸性之間的互相影響。對於延遲函數是凹向上的擺動器，我們將可以得到一個關於激進耦合與壓抑耦合的完整的相圖。特別地，我們證明出加了延遲時間與激進耦合在一個擺動器上，將會產生了一個多重穩定的同步化。在另外一方面，對於加了延遲時間的壓抑耦合會產生同相同步化的情形是不穩定的。

關鍵詞：生物擺動器、延遲時間、函數凹凸性、壓抑耦合、激進耦合。

中 華 民 國 九 十 七 年 六 月

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ABSTRACT

We study the interaction between coupling strengths, delay and concavity for a pair of two Mirollo-Strogatz-type oscillators. For a pair of two concave oscillators with a nonzero delay, the complete phase diagrams with respect to both inhibitory and excitatory coupling are given. In particular, we prove that the delay and excitatory coupling induce multistable synchronization for such system. On the other hand, the in-phase synchronization is shown to be unstable for inhibitory coupling if the delay is not zero.

Key words: Biological Oscillators, Delay, Concavity, and Inhibitory and Excitatory Coupling.

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光陰似箭，兩年轉眼間就過了。遙想兩年前的我，懷著既期待又害怕的心情入學。非常興奮能考上交通大學，並且期待能跟大家一起學習、相處；但同時也害怕同學高手如雲，自己沒能表現好。

這兩年上過很多老師的課。實變的吳培元老師、常微分方程的李明佳老師、動態系統理論的林松山老師、高等機率的吳慶堂老師、應用數學方法的賴明治老師、動態系統導論的莊重老師、生物資訊的許元春老師，每位都是認真教學、且富有知識與理念的老師。十分感謝以上老師們給予我的指導，上老師的課對我而言都是寶貴的學習經驗。

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CONTENTS

1. **Introduction**

Large assemblies of oscillator units can spontaneously evolve to a state of large scale organization. Synchronization is the best known phenomenon of this kind, where after some transient regime a coherent oscillatory activity of the set of oscillators emerges. This interesting phenomenon is quite common in many different disciplines such as engineering [28], physics [4, 13] and [24], chemistry [14], as well as biology [27]. For example, southeastern fireflies, where thousands of individuals gathered on trees flash in unison. Other examples of biological oscillators are the rhythmic activity of cells of the heart pacemaker [12, 18, 20] and [26], of cells of pancreas [22] and [23], and of neural networks [2, 8, 20, 21] and [25]. In the recent years, this topic has gained increasing attention as synchronous oscillations have been observed in the visual cortex [11, 6, 5], which were related to Gestalt properties of the stimulus. It has been pointed out that synchronous firing activity may be a part of higher brain functions and a method for integrating distributed information in an abstract representation [16, 17]. Abstracting from biophysical details, neurons belong to an important class of oscillators characterized by a pulselike interaction, i.e., where the coupling consists in the transmission of a short pulse from an oscillator to its partners. For an understanding of the general principles underlying synchronization phenomena, it is useful to consider abstract oscillator models which contain various existing models under very general assumptions and can be treated conveniently. We begin with describing the Peskin's model of n integrate-and-fire oscillators. Let the state of the i-th oscillator be denoted by x_i, where x_i is subject to the dynamics $\frac{dx_i}{dt} = -r_i x_i + s_i$, $0 \leq x_i \leq 1$, $i = 1, 2, \cdots, n$ with input $s_i > 0$, a normalized threshold 1 and leakiness $r_i \geq 0$. When $x_i = 1$, the ith oscillator fires and x_i jumps back to zero. As a consequence of the firing of ith oscillator, the activation of any other oscillator j is incremented by the coupling $\epsilon_{j,i}$. Should no confusion arise, we write $\epsilon_{j,i}$ as ϵ_{ji}. This model was later generalized by Mirollo and Strogatz [19]. It was assumed that the state variable x_i evolves according to a map f_i. When x_i reaches the threshold, the oscillator fires and x_i jumps back instantly to zero, and the activation of any other oscillator j is incremented by the positive coupling ϵ_{ij}. Specifically, x_i evolve according to $x_i = f_i(\phi_i)$, where $f_i : [0, 1] \to [0, 1]$ is smooth, and strictly increasing, i.e., $f_i' > 0$ on $(0, 1)$. Here ϕ_i is a phase variable so that (i) $\dfrac{d\phi_i}{dt} = \dfrac{1}{T_i}$, where T_i is the cycle period for oscillator x_i when evolving freely, (ii) $\phi_i = 0$ when the oscillator is at its lowest state $x_i = 0$, and (iii) $\phi_i \equiv 1$ at the end of cycle when the oscillator reaches the threshold $x_i = 1$. Therefore, f_i satisfy $f_i(0) = 0$, $f_i(1) = 1$. These maps f_i are to be called evolution maps. The inverses of f_i are to be denoted by g_i. If $f_i \equiv f$, $g_i \equiv g$, $T_i \equiv T$ and $\epsilon_{ij} \equiv \epsilon$ for all i, j, then the corresponding system is called identical. Otherwise, it is called nonidentical. Moreover, if $f_i'' > 0$(resp., $f_i'' < 0$) for all i, then the system is said to consist of the concave(resp., convex) oscillators. Assuming the identical system of convex oscillators, Mirollo and Strogatz [19] proved rigorously that the globally pulse-coupled oscillators always synchronize with zero phase difference. Their results solved the first conjecture of Peskin. Recently, Chang and Juang [3] prove that for "nearly" identical system of convex oscillators, it will fire in unison. That solves the Peskin's second conjecture. Those results pertain only to excitatory couplings, in realistic applications, however, one is also confronted with inhibitory couplings, which are abundant, e.g., in the central nervous systems and whose importance for synchronization was pointed out recently [15]. Furthermore, in most applications the transmission of a pulse requires a finite propagation time. It is an important question, how synchronization over long distances can be achieved when such temporal delays prevail (like in the visual cortex [7]). That is to say every biological system has to deal with substantial delays that seem, heuristically speaking, to constrain the process of synchronization.

In [9, 10], the study of general mechanisms of synchronization of the system of convex oscillators in cases where delay and also inhibitory couplings are present was given. In particular, they presented a complete mathematical analysis for pairs of two Mirollo-Strogatz-type oscillators for a wide range of delays τ and coupling strengths ϵ. Specifically, they showed that for inhibitory couplings, the presence of delays can lead to stable in-phase synchronization. For excitatory couplings, they showed that no stable in-phase synchronization exists. However, it was shown numerically in [1] that globally coupled oscillators with pulse interaction can synchronize under broader conditions. In particular, they demonstrated even the nonidentical system of concave oscillators can synchronize provided that the concavity of the system is not too large. Such numerical observation is also recently proved in [3].

The purpose of this thesis is to study how the roles of the concavity of the oscillators, the presence of delays and excitatory / inhibitory couplings play out in reaching or not reaching synchronization, and then compare the results with those from the system consisting of convex oscillators [9, 10].

We study the interaction between coupling strengths, delay and concavity for a pair of two Mirollo-Strogatz-type oscillators. For a pair of two concave oscillators with a nonzero delay, the complete phase diagrams with respect to both inhibitory and excitatory coupling are given. In particular, we prove that the delay and excitatory coupling induce multistable synchronization for such system. On the other hand, the in-phase synchronization is shown to be unstable for inhibitory coupling if the delay is not zero.

Summing up the earlier results and ours here, we conclude the following. For system of two identical concave oscillators, it will not reach synchrony without delay. With delay, the system will acquire in-phase synchronization with excitatory coupling on.

2. **Model**

The network consists of N relaxation oscillators, which are caricatures of real pulse-coupled neurons in biological systems [19]. Each oscillator i may be described by a smooth function $f(\phi_i)$, which is concave up and monotonically increasing [$f' > 0,\ f'' > 0,\ f(0) = 0$, f(1)=1]. f plays the role of an amplitude (e.g. the membrane potential) and $\phi_i \in [0, 1]$ is a phase, which in the case of vanishing input from other oscillators corresponds to the normalized time elapsed since the last firing of i. We assume from here on that the speed of each oscillator is one. When f reaches the threshold $f_s := 1$, the oscillator fires and ϕ_i and f are reset to zero. After a time delay $t = \tau,\ \tau \in (0, 0.5)$, the spike reaches all the other oscillators (no self-interaction) and raises (excitatory couplings) or lowers (inhibitory couplings) their amplitudes by an amount $\epsilon = \bar{\epsilon}(N-1)^{-1}$, where $\bar{\epsilon}$ denotes the normalized coupling strength ($\bar{\epsilon} \in (0, 1]$). The coupling to the oscillators j may be represented equivalently by an increase or decrease in phase $\Delta\phi_j$

$$\phi_j + \Delta\phi_j = g(min[f(\phi_j) + \epsilon, 1]) := F_+(\phi_j, \epsilon), \quad \text{where } g = f^{-1} \tag{2.1}$$

$$\phi_j + \Delta\phi_j = g(max[f(\phi_j) + \epsilon, 0]) := F_-(\phi_j, \epsilon), \quad \text{where } g = f^{-1} \tag{2.2}$$

where Eqs. (2.1) and (2.2) refer to excitatory and inhibitory coupling, respectively. We point out that the concavity of f is responsible for the dependence of $\Delta\phi_j$ on ϕ_j, the larger the phase ϕ_j, the smaller the phase shift $\Delta\phi_j$.

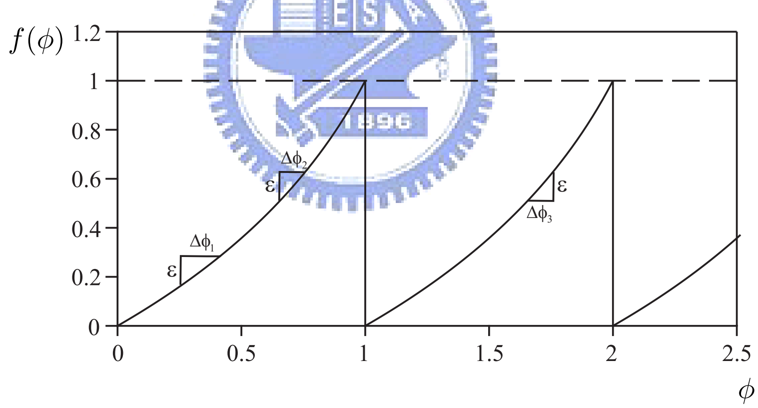


FIGURE 2.1. Function $f(\phi)$ and the dependence of the phase shift on ϕ. With excitatory couplings, an increase of ϵ in the amplitude f corresponds to a shift of phase that is larger when starting with a smaller phase ($\Delta\phi_2 < \Delta\phi_1$). A negative phase shift $\Delta\phi_3$ occurs with inhibitory couplings.

In this work, we consider the identical system of two concave oscillators with the delay. Before we treat a pair of these oscillators in a mathematical analysis, we note some simple properties of the functions F_- and F_+ introduced in Eqs. (2.1) and (2.2) that we will need in the next paragraph:

A1: $F_+(\phi, \epsilon) > \phi$, for $\phi < 1$.

A2: $F_-(\phi, \epsilon) < \phi$, for $\phi > 0$.

A3: $F_+(c + \phi, \epsilon) - F_+(c - \phi, \epsilon) < (c + \phi) - (c - \phi) = 2\phi$, for $c - \phi > 0$ and

$F_+(c+\phi, \epsilon) < 1$.

A4: $F_-(c+\phi, \epsilon) - F_-(c-\phi, \epsilon) > (c+\phi) - (c-\phi) = 2\phi$, for $c+\phi < 1$ and $F_-(c-\phi, \epsilon) > 0$.

A5: $f(\phi_2) - f(\phi_1) < f(\phi_2 + a) - f(\phi_1 + a)$, if $\phi_1 < \phi_2$, $a > 0$, $f' > 0$, $f'' > 0$.

A6: $f(\phi_1' + a') - f(\phi_1' - a') < f(\phi_2' + a') - f(\phi_2' - a')$, for $\phi_1' = \phi_1 + \dfrac{a}{2}$, $\phi_2' = \phi_2 + \dfrac{a}{2}$, $a' = \dfrac{a}{2}$ and $\phi_1 < \phi_2$.

A7: $F_+(\phi_2, \epsilon) - \phi_2 < F_+(\phi_1, \epsilon) - \phi_1$, if $0 < \phi_1 < \phi_2$, $f' > 0$, $f'' > 0$, $F(\phi_2, \epsilon) < 1$.

A8: $F_+(\phi + \tau, \epsilon) < F_+(\phi, \epsilon) + \tau$ follows directly from **A7**.

3. **Mathematical Analysis**

In this section, we will derive phase diagrams that allow one to determine if and how two oscillators synchronize their activities. We hereby consider a system S of two oscillators A and B, both either inhibitorily or excitatorily coupled together with time delay τ.

To study the dynamics of S, we begin with assuming that the oscillator A just reaches the threshold and is reset to $\phi_A = 0$ and $\phi = \phi_B > 0$. We further assume that if $\phi_B < \tau$, then ϕ_B must have fired ϕ_B time earlier. This assumption is not necessary in a mathematical sense, but makes the analysis easier by reducing the number of case distinctions. It should also be noted that since the speed of the oscillators is assumed to be one, the phase ϕ and the time ϕ is interchangeable. As the system S evolves, the phase positions of ϕ_A and ϕ_B at time t are to be denoted by $\phi_A(t)$ and $\phi_B(t)$, respectively. We next define a firemap and a return map for the system S of two oscillators A and B. Let $t = t_{p,i}$ denote the time when oscillator i has just fired its pth time and its phase is reset to zero. In this situation the system S is said to reach a ground state. The firemap and the return map are defined as follows, similarly as in [10]:

(i) Firemap $h(\phi_j(t_{p,i})) = \phi_l(t_{q,k})$, where $j \neq i,\ l \neq k$ and

$$t_{q,k} = \begin{cases} \min\limits_{r\in\mathbb{N},k\in\{A,B\}} \{t_{r,k} | t_{r,k} > t_{p,i} + \tau\} & \text{if } \phi \in I_1, \\ \min\limits_{r\in\mathbb{N},k\in\{A,B\}} \{t_{r,k} | t_{r,k} > t_{p,i}\} & \text{otherwise .} \end{cases}$$

(ii)Return map $R(\phi_j(t_{p,i})) = \phi_j(t_{p+1,i})$ for $i \neq j$.

Suppose the system S just reaches a ground state in oscillator i, then the firemap takes the phase position of the non-ground state oscillator into the phase position of the non-ground state oscillator when the system reaches the immediate next ground state. The return map takes the phase position of the non-ground state oscillator into the phase position of the non-ground state oscillator when the system reaches the next ground state in the same oscillators.

Before we begin the analysis of the different cases or configurations of the dynamics, we want to illustrate our motivation for our choice of intervals in the subspace of initial phase differences ϕ. Let us consider that S is a GS with oscillator A just being reset to $\phi_A = 0$ such that $\phi = \phi_B$. In a first interval I_1, both oscillators have fired, but their spikes did not reach their destination yet. Therefore, the consequences of the two pulses being received have to be evaluated. In a second interval I_2, only the spike of oscillator A did not reach B and has to be taken into account. In a third interval I_3, oscillator B will reach the threshold before the spike of A can be received. These considerations lead to the following definitions for I_1, I_2, and I_3:

$$\begin{aligned} I_1: &\quad \phi_B \in [0, \tau), \\ I_2: &\quad \phi_B \in [\tau, 1-\tau], \\ I_3: &\quad (Ex0,\ In0): \phi_B \in\ (1-\tau, 1]. \end{aligned}$$

On the one hand, the dynamics is very simple if we are looking at domain I_3. After a time $t = 1 - \phi$ (from here on, we identify t with the time elapsed since S has been in a GS), S will be in a GS with $\phi_B = 0$, which leads to a firemap $h(\phi) = 1 - \phi$. Additional case distinctions are not required here, and I_3 will be referenced as region $Ex0$ (for excitatory couplings) or $In0$ (for inhibitory couplings) for reasons of conformity.

On the other hand, the detailed analysis of I_1 and I_2 following in the next section requires one to distinguish between excitatory and inhibitory coupling. In each case, we first heuristically describe the temporal development of S to motivate the mathematical notation of the dynamics that will follow afterwards. For brevity, we denote a spike originating from oscillator A as "spike A" and the oscillator A simply as "A". Each domain will be partitioned into smaller regions, which we denote ExN or InN with successive numbering for excitatory and inhibitory coupling, respectively.

4. Excitatory Couplings, $\epsilon > 0$

4.1. Construction of the FireMap.

Configuration I_1

$Ex1:$ $\epsilon < 1 - f(\tau+\phi)$, $\epsilon < f(1-\phi) - f(\tau-\phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [0, \min\{g(1-\epsilon)-\tau, u_{1_b}, \tau\})$, where u_{1_b} satisfies $\epsilon = f(1-\phi) - f(\tau-\phi)$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi \to F_+(\tau-\phi,\epsilon)$	τ
τ	$F_+(\tau-\phi,\epsilon)+\phi < 1$	$\tau+\phi \to F_+(\tau+\phi,\epsilon) < 1$

$$h(\phi) = 1 - |F_+(\tau+\phi,\epsilon) - F_+(\tau-\phi,\epsilon) - \phi| \tag{4.1}$$

Using relation **A3**, we know that $0 \le F_+(\tau+\phi,\epsilon) - F_+(\tau-\phi,\epsilon) \le 2\phi$ for all $\phi \in Ex1$, and both the left and right equalities hold just as $\phi = 0$. Then the firemap satisfies

$$\begin{aligned} h(\phi) &= 1 - |F_+(\tau+\phi,\epsilon) - F_+(\tau-\phi,\epsilon) - \phi| \\ &\ge 1 - |2\phi - \phi| \\ &= 1 - \phi > 1 - \tau. \end{aligned} \tag{4.2}$$

Moreover, $h(\phi) = 1 - \phi$ just as $\phi = 0$.

$Ex2_a$: $1 - f(\phi+\tau) \le \epsilon < f(1-\phi) - f(\tau-\phi)$, and $\phi \in I_1$ (For fixed ϵ, $\phi \in [g(1-\epsilon)-\tau, \min\{u_{1_b}, \tau\})$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi \to F_+(\tau-\phi,\epsilon)$	τ
τ	$F_+(\tau-\phi,\epsilon)+\phi < 1$	$\tau+\phi \to F_+(\tau+\phi,\epsilon) = 1$

$$h(\phi) = F_+(\tau-\phi,\epsilon) + \phi \tag{4.3}$$

Since $F_+(\tau-\phi,\epsilon) > F_+(\tau+\phi,\epsilon) - 2\phi$ $\forall \phi \in Ex2_a$, and $F_+(\tau+\phi,\epsilon) = 1$,

$$\begin{aligned} h(\phi) &= F_+(\tau-\phi,\epsilon) + \phi \\ &> F_+(\tau+\phi,\epsilon) - 2\phi + \phi \\ &\ge 1 - 2\phi + \phi = 1 - \phi \\ &\ge 1 - \tau. \end{aligned} \tag{4.4}$$

$Ex2_b:$ $f(1-\phi)-f(\tau-\phi)\le\epsilon<1-f(\tau+\phi)$, and $\phi\in I_1$ (For fixed ϵ, $\phi\in[u_{1_b},\min\{g(1-\epsilon)-\tau,\tau\})$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi\to F_+(\tau-\phi,\epsilon)<1$	τ
$\tau-\phi+1-F_+(\tau-\phi,\epsilon)$	$1\to 0$	$\tau+1-F_+(\tau-\phi,\epsilon)$
τ	$F_+(\tau-\phi,\epsilon)+\phi-1\ge 0$	$\tau+\phi\to F_+(\tau+\phi,\epsilon)<1$

$$h(\phi)=F_+(\tau-\phi,\epsilon)-F_+(\tau+\phi,\epsilon)+\phi \tag{4.5}$$

Since $F_+(\tau-\phi,\epsilon)-F_+(\tau+\phi,\epsilon)<0\ \forall\phi\in Ex2_b$, the firemap satisfies

$$\begin{aligned} h(\phi) &= F_+(\tau-\phi,\epsilon)-F_+(\tau+\phi,\epsilon)+\phi \\ &< 0+\phi=\phi. \end{aligned} \tag{4.6}$$

$Ex3:$ $f(1-\phi)-f(\tau-\phi)\le\epsilon<1-f(\tau-\phi)$, $1-f(\tau+\phi)\le\epsilon<1-f(\tau-\phi)$ and $\phi\in I_1$.

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi\to F_+(\tau-\phi,\epsilon)<1$	τ
$\tau-\phi+1-F_+(\tau-\phi,\epsilon)$	$1\to 0$	$\tau+1-F_+(\tau-\phi,\epsilon)$
τ	$\phi-1+F_+(\tau-\phi,\epsilon)\ge 0$	$\tau+\phi\to F_+(\tau+\phi,\epsilon)=1$

$$h(\phi)=\phi-1+F_+(\tau-\phi,\epsilon) \tag{4.7}$$

Since $F_+(\tau-\phi,\epsilon)<1$, the firemap satisfies

$$\begin{aligned} h(\phi) &= \phi-1+F_+(\tau-\phi,\epsilon) \\ &< \phi-1+1=\phi. \end{aligned} \tag{4.8}$$

$Ex4:$ $\epsilon\ge 1-f(\tau-\phi)$ and $\phi\in I_1$ (For fixed ϵ, $\phi\in[0,\tau-g(1-\epsilon))$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi\to F_+(\tau-\phi,\epsilon)=1$	τ
τ	ϕ	$\tau+\phi\to F_+(\tau+\phi,\epsilon)=1$

$$h(\phi)=\phi \tag{4.9}$$

The firemap h is with the same initial conditions but A and B exchanged.

Configuration I_2

$Ex5:$ $\epsilon < 1 - f(\tau+\phi)$ and $\phi \in I_2$ (For fixed ϵ, $\phi \in [\tau, g(1-\epsilon)-\tau)$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
τ	τ	$\tau+\phi \to F_+(\tau+\phi,\epsilon) < 1$

$$h(\phi) = 1 - F_+(\tau+\phi,\epsilon) + \tau \tag{4.10}$$

Since $\tau+\phi < F_+(\tau+\phi,\epsilon) < 1$ and $\tau \le \phi$, the firemap satisfies

$$\begin{aligned} h(\phi) &= 1 - F_+(\tau+\phi,\epsilon) + \tau \\ &< 1 - \tau - \phi + \phi = 1 - \tau, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} h(\phi) &= 1 - F_+(\tau+\phi,\epsilon) + \tau \\ &> 1 - 1 + \tau = \tau. \end{aligned} \tag{4.12}$$

$Ex6:$ $\epsilon \ge 1 - f(\phi+\tau)$ and $\phi \in I_2$ (For fixed ϵ, $\phi \in [\max\{\tau, g(1-\epsilon)-\tau\}, 1-\tau]$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
τ	τ	$\tau+\phi \to F_+(\tau+\phi,\epsilon) = 1$

$$h(\phi) = \tau \tag{4.13}$$

Configuration I_3

$Ex0:$ $\phi \in I_3$

time t	ϕ_A	ϕ_B
0	0	ϕ
$1-\phi$	$1-\phi$	$1 \to 0$

$$h(\phi) = 1 - \phi \tag{4.14}$$

4.2. **Dynamics in Cases.**

Configuration I_1

Dynamics in $Ex1 : \epsilon < 1 - f(\tau + \phi)$**,** $\epsilon < f(1 - \phi) - f(\tau - \phi)$ **and** $\phi \in I_1$ **(For fixed** ϵ**,** $\phi \in [0, \min\{g(1 - \epsilon) - \tau, u_{1_b}, \tau\})$**, where** u_{1_b} **satisfies** $\epsilon = f(1 - \phi) - f(\tau - \phi)$**.)** $h(\phi) = 1 - \lvert F_+(\tau + \phi, \epsilon) - F_+(\tau - \phi, \epsilon) - \phi\rvert$

By Eq. (4.2), the firemap $h : Ex1 \mapsto Ex0$. Specifically, when ϕ satisfies $F_+(\tau - \phi, \epsilon) + \phi = F_+(\tau + \phi, \epsilon)$, $h(\phi) = 1$, which means ϕ_A and ϕ_B synchronize. Moreover, the return map R satisfies

$$R(\phi) \;=\; |F_+(\tau + \phi, \epsilon) - F_+(\tau - \phi, \epsilon) - \phi| \leq \phi. \tag{4.15}$$

The equality holds just as $\phi = 0$. Since $\phi \in Ex1$ and $R(\phi) < \phi$, $R(\phi) \in Ex1$.

Proposition 4.1. *For each $\phi \in Ex1$, $R^k(\phi) \to 0$, as $k \to \infty$.*

Proof. Suppose not, i.e., $\exists \phi \in Ex1$, $R^k(\phi) \nrightarrow 0$, as $k \to \infty$. Then $R^k(\phi) \neq 0\ \forall k \in \mathbb{N}$. Otherwise, $R^m(\phi) = 0$ for large enough $m \in \mathbb{N}$. By Eq. (4.15), $R^k(\phi) \in Ex1\ \forall k \in \mathbb{N}$ and $R(\phi) < \phi$. Inductively, $\phi > R(\phi) > \cdots > R^{k-1}(\phi) > R^k(\phi) > 0$. Since $\{R^k(\phi)\}$ is a monotone and bounded sequence in $Ex1$, and $R(\phi)$ is continuous in $Ex1$. $\{R^k(\phi)\}$ is a convergent sequence, i.e., $R^k(\phi) \to \phi^*$, for some $\phi^* \in (0, \tau)$, and such ϕ^* is a fixed point. It is a contradiction since no fixed point in $Ex1\backslash\{0\}$, by Eq. (4.15).

□

Corollary 4.1. *For each $\phi \in Ex1$, ϕ_A and ϕ_B synchronize.*

Dynamics in $Ex2_a : 1 - f(\tau + \phi) \leq \epsilon < f(1 - \phi) - f(\tau - \phi)$ **and** $\phi \in I_1$ **(For fixed** ϵ**,** $\phi \in [g(1 - \epsilon) - \tau, \min\{u_{1_b}, \tau\})$**.)** $h(\phi) = F_+(\tau - \phi, \epsilon) + \phi$

Remark 4.1. $\phi = 0 \notin Ex2_a$.

By Eq. (4.4), the firemap $h : Ex2_a \mapsto Ex0$. Then the return map R satisfies

$$\begin{aligned} R(\phi) &= h(h(\phi)) = 1 - h(\phi) \\ &= 1 - F_+(\tau - \phi, \epsilon) - \phi \\ &= F_+(\tau + \phi, \epsilon) - F_+(\tau - \phi, \epsilon) - \phi \\ &< 2\phi - \phi = \phi. \end{aligned} \tag{4.16}$$

Since $\phi \in Ex2_a$ and $R(\phi) < \phi$, $R(\phi) \in Ex1 \cup Ex2_a$.

Proposition 4.2. *There is no $\phi \in Ex2_a$ such that $R^k(\phi) \in Ex2_a$ for all $k = 0, 1, 2, \cdots$.*

Proof. Suppose not, i.e., $\exists \phi \in Ex2_a$ such that $R^k(\phi) \in Ex2_a$ for all $k = 0, 1, 2 \cdots$. Note that $R(\phi)$ is continuous in $Ex2_a$. Moreover, $R(\phi) < \phi$, $\forall \phi \in Ex2_a$. Inductively $\phi > R(\phi) > \cdots >$

$R^{k-1}(\phi) > R^k(\phi) > 0$. Thus, $R^k(\phi) \to \phi^*$ for some $\phi^* \in Ex2_a$, and such ϕ^* is a fixed point. It makes a contradiction with Eq. (4.16).

□

Corollary 4.2. *For each $\phi \in Ex2_a$, there is a $k \in \mathbb{N}$, depending on ϕ, such that $R^k(\phi) \in Ex1$.*

Dynamics in $Ex2_b : f(1-\phi) - f(\tau-\phi) \leq \epsilon < 1 - f(\tau+\phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [u_{1_b}, \min\{g(1-\epsilon)-\tau, \tau\})$.)

$$h(\phi) = F_+(\tau-\phi, \epsilon) - F_+(\tau+\phi, \epsilon) + \phi$$

Remark 4.2. $\phi = 0 \notin Ex2_b$.

By Eq. (4.6), the firemap $h : Ex2_b \mapsto Ex1 \cup Ex2_b$.

Proposition 4.3. *There is no $\phi \in Ex2_b$ such that $h^k(\phi) \in Ex2_b$ for all $k = 0, 1, 2, \cdots$.*

Proof. Suppose not, i.e., $\exists \phi \in Ex2_b$ such that $h^k(\phi) \in Ex2_b$ for all $k = 0, 1, 2 \cdots$. Note that $h(\phi)$ is continuous in $Ex2_b$. Moreover, $h(\phi) < \phi, \quad \forall \phi \in Ex2_b$. Inductively $\phi > R(\phi) > \cdots > R^{k-1}(\phi) > R^k(\phi) > 0$. Thus, $h^k(\phi) \to \phi^*$ for some $\phi^* \in Ex2_b$, and such ϕ^* is a fixed point. It makes a contradiction with Eq. (4.6).

□

Corollary 4.3. *For each $\phi \in Ex2_b$, there is a $k \in \mathbb{N}$, depending on ϕ, such that $h^k(\phi) \in Ex1$.*

Dynamics in $Ex3 : 1 - f(\tau+\phi) \leq \epsilon < 1 - f(\tau-\phi)$, $f(1-\phi) - f(\tau-\phi) \leq \epsilon < 1 - f(\tau-\phi)$ and $\phi \in I_1$.

$$h(\phi) = \phi - 1 + F_+(\tau-\phi, \epsilon)$$

Remark 4.3. $\phi = 0 \notin Ex3$.

By Eq.(4.8), the firemap $h : Ex3 \mapsto I_1$. More precisely,

(i) If $\epsilon < 1 - f(\tau)$, $h : Ex3 \mapsto Ex1 \cup Ex2_a \cup Ex2_b \cup Ex3$, since the slope of the lower bound of $Ex3$ is negative ($-f'(\tau+\phi) < 0$ and $-f'(1-\phi) + f'(\tau-\phi) < 0$.

(ii) If $\epsilon \geq 1 - f(\tau)$, $h : Ex3 \mapsto Ex3 \cup Ex4$, since the slope of the upper bound of $Ex3$ is positive ($f'(\tau-\phi) > 0$).

Proposition 4.4. *(i) If $\epsilon < 1 - f(\tau)$, there is no $\phi \in Ex3$ such that $h^k(\phi) \in Ex3$ for all $k = 0, 1, 2, \cdots$. (ii) If $\epsilon \geq 1 - f(\tau)$, for each $\phi \in Ex3$, $h^k(\phi) \in Ex3$ for all $k \in \mathbb{N}$. Moreover, $h^k(\phi) \to \tau - g(1-\epsilon)$ for all $\phi \in Ex3$.*

Proof.

(i) Suppose not, i.e., $\exists \phi \in Ex3$ such that $h^k(\phi) \in Ex3$, for all $k = 0, 1, 2, \cdots$. Since as $\epsilon < 1 - f(\tau)$, $Ex3 = [g(1-\epsilon), \tau)$, or $Ex3 = [u_{1_b}, \tau)$ and the sequence $\{h^k(\phi)\}$ must satisfy $h^k(\phi) \to \phi^*$, where ϕ^* is a fixed point for h in $Ex3$. It makes a contradiction with Eq. (4.8).

(ii) If $\epsilon \geq 1 - f(\tau)$, $Ex3 = (\tau - g(1-\epsilon), \tau)$, where $\tau - g(1-\epsilon) \geq 0$, and the equality holds just as $\epsilon = 1 - f(\tau)$. Because of

$$\begin{aligned} h(\tau^-) &= \tau - 1 + F_+(\tau - \tau, \epsilon) \\ &= \tau - 1 + g(\epsilon) < \tau, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} h((\tau - g(1-\epsilon))^+) &= \tau - g(1-\epsilon) - 1 + F_+(\tau - \tau + g(1-\epsilon), \epsilon) \\ &= \tau - g(1-\epsilon) - 1 + 1 = \tau - g(1-\epsilon). \end{aligned} \tag{4.18}$$

Also, for any $\phi \in Ex3$,

$$\begin{aligned} h'(\phi) &= 1 + F'_+(\tau - \phi, \epsilon) \\ &= 1 - g'(f(\tau - \phi) + \epsilon) f'(\tau - \phi) \\ &> 1 - g'(f(\tau - \phi)) f'(\tau - \phi) \\ &= 1 - 1 = 0. \end{aligned} \tag{4.19}$$

We obtain $\tau - g(1-\epsilon) = h((\tau - g(1-\epsilon))^+) < h(\phi) < h(\tau^-) < \tau$. Therefore, for each $\phi \in Ex3$, $h^k(\phi) \in Ex3$ for all $k \in \mathbb{N}$. Moreover, by Eqs. (4.8), (4.17), (4.18), (4.19), the sequence $\{h^k(\phi)\}$ is decreasing. Hence, $h^k(\phi) \to \phi^*$. Since there is no fixed point in $Ex3$, $\phi^* \in bd(Ex3)$. It follows that $\phi^* = \tau - g(1-\epsilon)$ which is also the boundary of $Ex4$.

□

Specifically, if $\epsilon = 1 - f(\tau)$, we obtain $\phi^* = 0$, i.e., ϕ_A and ϕ_B synchronize.

Corollary 4.4. *(i) If $\epsilon < 1 - f(\tau)$, for each $\phi \in Ex3$, $h^k(\phi) \in Ex1 \cup Ex2_a \cup Ex2_b$ for some $k \in \mathbb{N}$. (ii) If $\epsilon \geq 1 - f(\tau)$, for each $\phi \in Ex3$, $h^k(\phi) \to \tau - g(1-\epsilon)$ as $k \to \infty$. Specifically, if $\epsilon = 1 - f(\tau)$, for each $\phi \in Ex3$, ϕ_A and ϕ_B synchronize.*

Dynamics in $Ex4 : \epsilon \geq 1 - f(\tau - \phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [0, \tau - g(1-\epsilon))$.)

$$h(\phi) = \phi$$

Thus, $h : Ex4 \mapsto Ex4$. Specifically, if $\phi = 0 \in Ex4$, $h^k(\phi) = 0 \in Ex4$, $\forall k \in \mathbb{N}$.

Corollary 4.5. *For each $\phi \in Ex4$, ϕ is a marginal stable fixed point in $Ex4$.*

Configuration I_2

Dynamics in $Ex5 : \epsilon < 1 - f(\tau + \phi)$ **and** $\phi \in I_2$ **(For fixed** ϵ**,** $\phi \in [\tau, g(1-\epsilon) - \tau)$**.)** $$h(\phi) = 1 - F_+(\tau + \phi, \epsilon) + \tau$$

By Eqs. (4.11), (4.12), we have $\tau < h(\phi) < 1 - \tau$. Hence $h : Ex5 \mapsto I_2$.

Lemma 4.1. *For each $\phi \in Ex5$, $h^k(\phi) \in Ex5$ for $k = 0, 1, 2, \cdots$.*

Proof. For each $\phi \in Ex5$, since $\epsilon < 1 - f(\tau + \phi)$ and $\phi > \tau$, $\epsilon < 1 - f(2\tau)$. Let $\phi_0 = g(1 - \epsilon)$, $\Delta\phi_0 = g(f(\phi_0)+\epsilon)-\epsilon$ and $\Delta 2\tau = g(f(2\tau)+\epsilon)-\epsilon$. Since $g(f(\phi_0)+\epsilon) = 1$ and $f'' > 0$, $\Delta\phi_0 < \Delta 2\tau$, i.e., $F_+(\phi_0, \epsilon) - \phi_0 < F_+(2\tau, \epsilon) - 2\tau$. Clearly, $F_+(\tau + \phi, \epsilon) \geq F_+(2\tau, \epsilon)$ for $\phi \in [\tau, 1 - \tau]$. Then we have

$$\begin{aligned} h(\phi) \quad \leq \quad h(\tau) &= 1 - F_+(2\tau, \epsilon) + \tau \\ &< 1 + \phi_0 - F_+(\phi_0, \epsilon) - 2\tau + \tau \\ &= 1 + g(1 - \epsilon) - g(f(g(1 - \epsilon)) + \epsilon) - \tau \\ &= g(1 - \epsilon) - \tau. \end{aligned}$$

Thus, $h(\phi) \in (\tau, g(1 - \epsilon) - \tau)$, i.e., $h(\phi) \in Ex5$. Inductively, $h^k(\phi) \in Ex5$ for $k = 0, 1, 2, \cdots$. Therefore, the lemma holds.

□

Lemma 4.2. $-1 < h'(\phi) < 0$ *and* $0 < R'(\phi) < 1$ $\forall \phi \in Ex5$.

Proof. From direct computation, $0 > h'(\phi) = -g'(f(\tau + \phi) + \epsilon) \cdot f'(\tau + \phi) > -g'(f(\tau + \phi)) \cdot f'(\tau + \phi) = -1$, since $g'' < 0$ and $g'(f(\tau + \phi)) \cdot f'(\tau + \phi) = 1$. Since $R'(\phi) = h'(h(\phi)) \cdot h'(\phi)$, we obtain $0 < R'(\phi) < 1$.

□

Proposition 4.5. *There exists a unique fixed point for h in $Ex5$, and it is an attractor.*

Proof. Define $A(\phi) = h(\phi) - \phi$. Clearly, $A'(\phi) = h'(\phi) - 1$. From Lemma 4.2, we have $h'(\phi) < 0$, and then $A'(\phi) = h'(\phi) - 1 < 0$. By the definition of $Ex5$, $Ex5 = [\tau, g(1 - \epsilon) - \tau)$. Denote $\delta \equiv g(1 - \epsilon) - \tau$. Now, let's check the values of $A(\delta^-)$ and $A(\tau)$.
(i) If $\phi = \delta$, we figure out that $F_+(\delta + \tau, \epsilon) = 1$. It follows that $h(\delta^-) = \tau$, and

$$A(\delta^-) \quad = \quad h(\delta^-) - \delta = \tau - \delta < 0.$$

(ii) If $\phi = \tau$, we have

$$\begin{aligned} A(\tau) &= h(\tau) - \tau \\ &= 1 - F_+(2\tau, \epsilon) + \tau - \tau \\ &= 1 - F_+(2\tau, \epsilon) > 0. \end{aligned}$$

The last inequality follows directly from the definition of $Ex5$.
Hence h has a unique fixed point $\phi^* \in Ex5$, i.e., $h(\phi^*) = \phi^*$. Since $R(\phi^*) = h^2(\phi^*)$ and $0 < R'(\phi) < 1$, such ϕ^* is also the unique fixed point for R in $Ex5$. Moreover, since

$$\phi^* < R(\phi) < \phi \quad if \quad \phi > \phi^*,$$

and

$$\phi^* > R(\phi) > \phi \quad if \quad \phi < \phi^*,$$

the fixed point ϕ^* for R and hence for h is an attractor.

□

Corollary 4.6. *For each $\phi \in Ex5$, $h^k(\phi)$ converges to the fixed point $\phi^* \in Ex5$, which is an attractor.*

Dynamics in $Ex6 : \epsilon \geq 1 - f(\tau + \phi)$ and $\phi \in I_2$ (For fixed ϵ, $\phi \in [\max\{\tau, g(1-\epsilon) - \tau\}, 1 - \tau]$.)

$$h(\phi) = \tau$$

Thus, $h : Ex6 \mapsto I_2$.

Corollary 4.7. *For each $\phi \in Ex6$, the firemap maps ϕ to the line $\phi = \tau$.*

Configuration I_3

Dynamics in $Ex0 : \phi \in I_3$

$$h(\phi) = 1 - \phi$$

Thus, $h : Ex0 \mapsto I_1$.

Corollary 4.8. *For each $\phi \in Ex0$, $h(\phi) \in I_1$.*

4.3. **Construction of Phase Diagrams.**

Peskin modeled the pacemaker as a network of N "integrate-and-fire" oscillators, each characterized by a voltagelike state variable x_i, subject to the dynamics

$$\frac{dx_i}{dt} = -rx_i + s, \quad 0 \leq x_i \leq 1, \quad i = 1, 2, \cdots N. \tag{4.20}$$

Mirollo and Strogatz [19] proposed a new model and they assume that x evolves according to $x = f(\phi)$, where $f : [0,1] \to [0,1]$ is smooth, monotonic increasing ($f' > 0$), and concave down ($f'' < 0$). Here $\phi \in [0,1]$ is a phase variable such that (i) $d\phi/dt = 1/T$, where T is a cycle period. (ii) $f(0) = 0$. (iii) $f(1) = 1$. Then they generate the function f and its inverse function g which are given in the following, respectively.

$$\begin{aligned} f(\phi) &= \frac{s}{r} - \frac{s}{r}(\frac{s-r}{s})^{\phi} \\ g(x) &= \frac{\ln(\frac{s-rx}{s})}{\ln(\frac{s-r}{s})} \end{aligned} \tag{4.21}$$

In contrast, we assume there exists a function f which is smooth, monotonic increasing ($f' > 0$), and concave up ($f'' > 0$). So, we switch the function f and g of Eq. (4.21) to be the model of our phase diagrams.

We fixed $s = 1$, and change the variable r, then we have three different kinds of figures in Figure 4.1:

(a) When $r = 0.4$ and $\tau = 0.2$, $f(1-\phi) - f(\tau - \phi) > 1 - f(\tau + \phi)$ $\forall \phi \in I_1$.

(b) When $r = 0.7$ and $\tau = 0.2$, $f(1-\phi) - f(\tau - \phi) < 1 - f(\tau + \phi)$ $\forall \phi \in I_1$.

(c) When $r = 0.72$ and $\tau = 0.35$, $f(1-\phi) - f(\tau - \phi)$ and $1 - f(\tau + \phi)$ have an intersection in I_1.

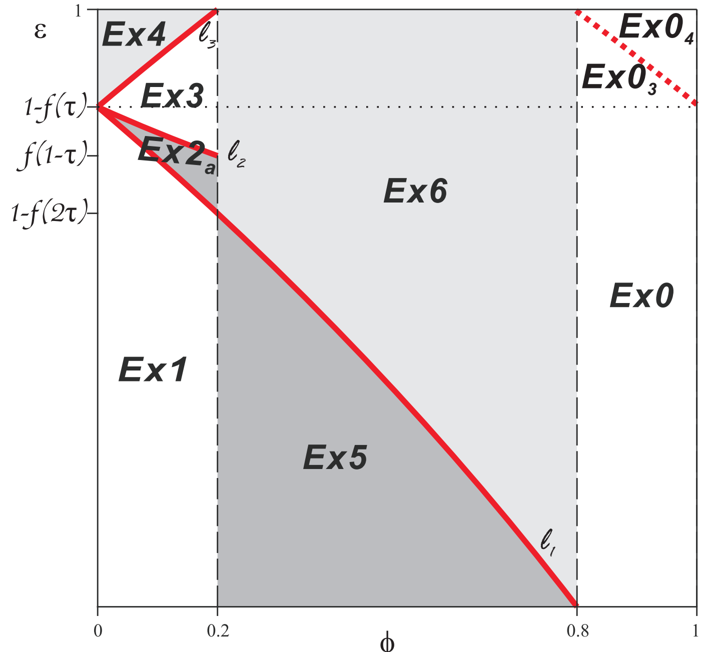
Moreover, we add Figure 4.2 which is a phase diagram we make from Ernst [10] as a contrast. And it's energy function f and inverse function g is exactly the Mirollo-Strogatz-type (Eq. 4.21) oscillators we just mentioned.

And there are some notations for the following figures:

$$\begin{aligned} l_1 &: \epsilon = 1 - f(\tau + \phi) \\ l_2 &: \epsilon = f(1-\phi) - f(\tau - \phi) \\ l_3 &: \epsilon = 1 - f(\tau - \phi) \end{aligned}$$

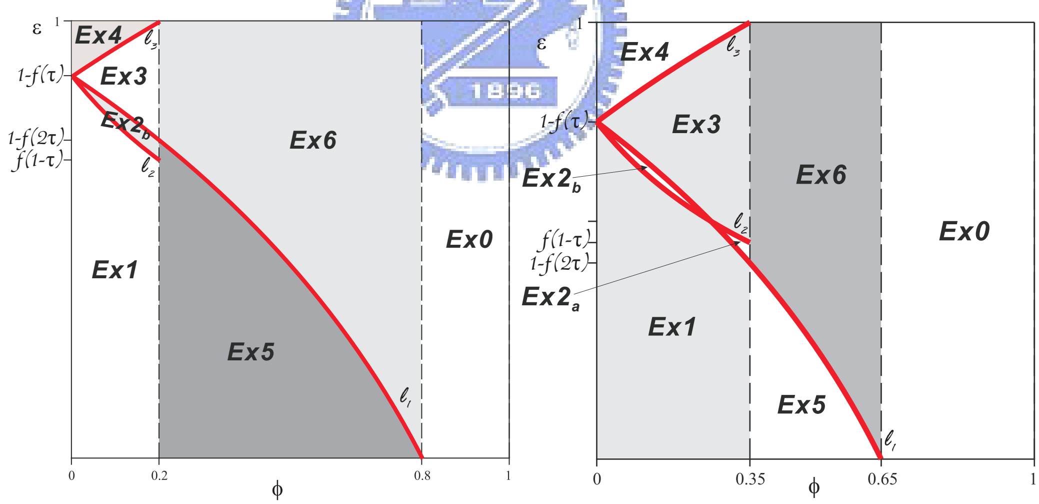
In the following tables, we fixed an arbitrary $\epsilon > 0$, and shift ϕ in [0,1]. Therefore, we clearly see which area (ExN) does the oscillator begin and figure out where it ends.

(I) $f(\phi)=\dfrac{\ln(\frac{s-r\phi}{s})}{\ln(\frac{s-r}{s})}$, $f'>0$, $f''>0$



(a)

FIGURE 4.1. (a) $s=1$, $r=0.4$, $\tau=0.2$. (b) $s=1$, $r=0.7$, $\tau=0.2$. (c) $s=1$, $r=0.72$, $\tau=0.35$.



(b) (c)

ϵ	Dynamics	Multistability
$0<\epsilon<1-f(2\tau)$	$\phi\to 0$, if $\phi\in I_1, I_3$ $\phi\to\phi^*_{Ex5}$, if $\phi\in I_2$	Complete Syn. Lag Syn. with Lag ϕ^*_{Ex5}
$1-f(2\tau)\le\epsilon<1-f(\tau)$	$\phi\to 0$, if $\phi\in I_1, I_3$ $\phi\to\tau$, if $\phi\in I_2$	Complete Syn. Lag Syn. with Lag τ
$1-f(\tau)\le\epsilon\le 1$	$\phi\to\phi$, if $\phi\in Ex4, Ex0_4$ $\phi\to\phi^*_{l3}$, if $\phi\in Ex3, Ex0_3$ $\phi\to\tau$, if $\phi\in I_2$	Lag Syn. with Lag ϕ Lag Syn. with Lag ϕ^*_{l3} Lag Syn. with Lag τ.

[1]

[1] If $\phi\in Ex0_n$, then $h(\phi)\in Exn$ for $n=3,4$.

(II) $f(\phi) = \frac{s}{r} - \frac{s}{r}(\frac{s-r}{s})^{\phi}$**,** $f' > 0$**,** $f'' < 0$

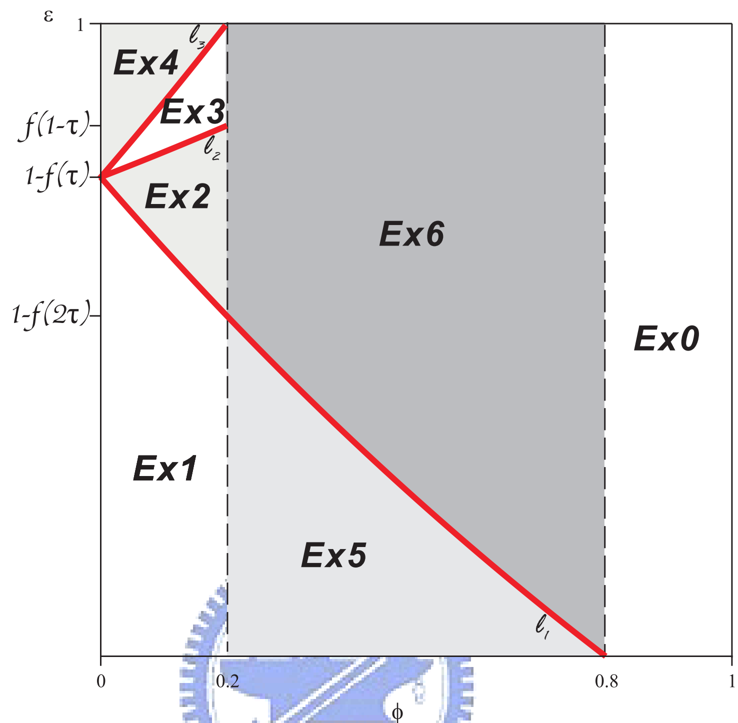


FIGURE 4.2. $s := 1$, $r := 0.4$, $\tau = 0.2$, where $s > r > 0$

ϵ	Dynamics	Multistability
$0 < \epsilon < 1 - f(2\tau)$	$\phi \to \phi^*_{Ex5^*}$, if $\phi \in I_1, Ex5^*, I_3$ $\phi \to$Period$(\tau, h(\tau))$, if $\phi \in I_2 \backslash Ex5^*$	Lag Syn. with Lag $\phi^*_{Ex5^*}$ Lag Syn. with Lag τ
$1 - f(2\tau) \leq \epsilon < 1 - f(\tau)$	$\phi \to \phi^*_{Ex4}$ or τ, if $\phi \in I_1, I_3$ $\phi \to \tau$, if $\phi \in I_2$	Depends on the behavior of $Ex2$ Lag Syn. with Lag τ
$1 - f(\tau) \leq \epsilon \leq 1$	$\phi \to \phi^*_{Ex4}$, if $\phi \in I_1, I_3$ $\phi \to \tau$, if $\phi \in I_2$	Lag Syn. with Lag ϕ^*_{Ex4} Lag Syn. with Lag τ

5. **Inhibitory Couplings,** $\epsilon < 0$

5.1. **Construction of the FireMap.**

Configuration I_1

$In1:\ |\epsilon| < f(\tau-\phi)$ and $\phi \in I_1$

$\Rightarrow\ \epsilon > -f(\tau-\phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [0, \tau - g(-\epsilon))$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi \to F_-(\tau-\phi,\epsilon) > 0$	τ
τ	$F_-(\tau-\phi,\epsilon)+\phi$	$\tau+\phi \to F_-(\tau+\phi,\epsilon) > 0$

$$h(\phi) = 1 - F_-(\tau+\phi,\epsilon) + F_-(\tau-\phi,\epsilon) + \phi \tag{5.1}$$

Using relation **A4**, $F_-(\tau+\phi,\epsilon) \geq F_-(\tau-\phi,\epsilon) + 2\phi$, we have $\phi_B(\tau) \geq \phi_A(\tau)$. Since $F_-(\tau-\phi,\epsilon) > 0$, the firemap satisfies

$$\begin{aligned} h(\phi) &= 1 - F_-(\tau+\phi,\epsilon) + F_-(\tau-\phi,\epsilon) + \phi \\ &> 1 - (\tau+\phi) + F_-(\tau-\phi,\epsilon) + \phi \\ &= 1 - \tau + F_-(\tau-\phi,\epsilon) \\ &> 1 - \tau. \end{aligned} \tag{5.2}$$

$In2:\ f(\tau-\phi) \leq |\epsilon| < f(\tau+\phi)$ and $\phi \in I_1$

$\Rightarrow -f(\tau-\phi) \geq \epsilon > -f(\tau+\phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [0,\tau)$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi \to F_-(\tau-\phi,\epsilon) = 0$	τ
τ	ϕ	$\tau+\phi \to F_-(\tau+\phi,\epsilon) > 0$

$$h(\phi) = 1 - |F_-(\tau+\phi,\epsilon) - \phi| \tag{5.3}$$

Since $F_-(\tau+\phi,\epsilon) < \tau+\phi$, the firemap satisfies

$$\begin{aligned} h(\phi) &= 1 - |F_-(\tau+\phi,\epsilon) - \phi| \\ &> 1 - |\tau+\phi-\phi| \\ &= 1 - \tau. \end{aligned} \tag{5.4}$$

(i) If $F_-(\tau+\phi,\epsilon) > \phi$,

Define $In2_a : -f(\tau-\phi) \geq \epsilon \geq -f(\tau)$ (For fixed ϵ, $\phi \in [l_{2_a}, \tau]$, where $l_{2_a} = \tau - g(-\epsilon)$.)

$$h(\phi) = 1 - F_-(\tau+\phi,\epsilon) + \phi$$

Define $In2_b : -f(\tau) > \epsilon > f(\phi) - f(\tau+\phi)$ (For fixed ϵ, $\phi \in (l_{2_b}, \tau)$, where l_{2_b} satisfies $\epsilon = f(\phi) - f(\tau+\phi)$.)

$$h(\phi) = 1 - F_-(\tau+\phi,\epsilon) + \phi$$

(ii) If $F_{-}(\tau+\phi,\epsilon)<\phi$,

Define $In2_c : f(\phi)-f(\tau+\phi)>\epsilon>-f(\tau+\phi)$ (For fixed ϵ, $\phi\in[g(-\epsilon)-\tau,\min\{l_{2_b},\tau\})$.)

$$R(\phi)=h(\phi)=1+F_{-}(\tau+\phi,\epsilon)-\phi$$

(iii) If $F_{-}(\tau+\phi,\epsilon)=\phi$,

Define $In2_d : \epsilon=f(\phi)-f(\tau+\phi)$ (For fixed ϵ, $\phi=l_{2_b}$.)

$$h(\phi)=1$$

$In3:$ $f(\tau+\phi)\le|\epsilon|$ and $\phi\in I_1$

$\Rightarrow \epsilon\ge -f(\tau+\phi)$ and $\phi\in I_1$ (For fixed ϵ, $\phi\in[0,\min\{g(-\epsilon)-\tau,\tau\})$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
$\tau-\phi$	$\tau-\phi\rightarrow F_{-}(\tau-\phi,\epsilon)=0$	τ
τ	ϕ	$\tau+\phi\rightarrow F_{-}(\tau+\phi,\epsilon)=0$

$$h(\phi)=R(\phi)=1-\phi \tag{5.5}$$

Configuration I_2

$In4:$ $|\epsilon|<f(\tau+\phi)-f(2\tau)$ and $\phi\in I_2$

$\Rightarrow \epsilon>f(2\tau)-f(\tau+\phi)$ and $\phi\in I_2$ (For fixed ϵ, $\phi\in(l_4,1-\tau]$, where $l_4=g(f(2\tau)-\epsilon)-\tau$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
τ	τ	$\tau+\phi\rightarrow F_{-}(\tau+\phi,\epsilon)>0$

$$h(\phi)=1-F_{-}(\tau+\phi,\epsilon)+\tau \tag{5.6}$$

We assume that $|\epsilon|$ is small enough such that $F_{-}(\tau+\phi,\epsilon)>2\tau$. Since $2\tau<F_{-}(\tau+\phi,\epsilon)<\tau+\phi$, the firemap satisfies

$$\begin{aligned} h(\phi) &= 1-F_{-}(\tau+\phi,\epsilon)+\tau \\ &< 1-2\tau+\tau=1-\tau, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} h(\phi) &= 1-F_{-}(\tau+\phi,\epsilon)+\tau \\ &> 1-\tau-\phi+\tau \\ &= 1-\phi\ge\tau. \end{aligned} \tag{5.8}$$

$In5: \; f(\tau+\phi) - f(2\tau) \le |\epsilon| < f(\tau+\phi)$ and $\phi \in I_2$

$\Rightarrow f(2\tau) - f(\tau+\phi) \ge \epsilon > -f(\tau+\phi)$ and $\phi \in I_2$

time t	ϕ_A	ϕ_B
0	0	ϕ
τ	τ	$\tau+\phi \to F_-(\tau+\phi,\epsilon) > 0$

$$h(\phi) = 1 - |F_-(\tau+\phi,\epsilon) - \tau| \tag{5.9}$$

We assume that $|\epsilon|$ is bigger than that in $In4$ such that $0 < F_-(\tau+\phi,\epsilon) \le 2\tau$, then the firemap satisfies

$$\begin{aligned} h(\phi) &= 1 - |F_-(\tau+\phi,\epsilon) - \tau| \\ &\ge 1 - |2\tau - \tau| \\ &= 1 - \tau. \end{aligned} \tag{5.10}$$

$In6: \; f(\tau+\phi) \le |\epsilon|$ and $\phi \in I_2$

$\Rightarrow -f(\tau+\phi) \ge \epsilon$ and $\phi \in I_2$ (For fixed ϵ, $\phi \in [\tau, g(-\epsilon) - \tau]$.)

time t	ϕ_A	ϕ_B
0	0	ϕ
τ	τ	$\tau+\phi \to F_-(\tau+\phi,\epsilon) = 0$

$$h(\phi) = R(\phi) = 1 - \tau \tag{5.11}$$

Configuration I_3

$In0: \; \phi \in I_3$

time t	ϕ_A	ϕ_B
0	0	ϕ
$1-\phi$	$1-\phi$	$1 \to 0$

$$h(\phi) = 1 - \phi \tag{5.12}$$

5.2. **Dynamics in Cases.**

Configuration I_1

Dynamics in $In1:\ \epsilon > -f(\tau-\phi)$ and $\phi \in I_1$ **(For fixed ϵ, $\phi \in [0, \tau - g(-\epsilon))$.)**
$h(\phi) = 1 - F_-(\tau+\phi,\epsilon) + F_-(\tau-\phi,\epsilon) + \phi$

By Eq. (5.2), thus $h : In1 \mapsto In0$. Then the return map satisfies

$$\begin{aligned} R(\phi) &= h(h(\phi)) = 1 - h(\phi) \\ &= F_-(\tau+\phi,\epsilon) - F_-(\tau-\phi,\epsilon) - \phi \qquad (5.13) \\ &\geq 2\phi - \phi = \phi. \end{aligned}$$

The equality holds just as $\phi = 0$. Since $\phi \in In1$ and $R(\phi) > \phi$, $R(\phi) \in In1 \cup In2$. Specifically, if $\phi = 0 \in In1$, $R^k(\phi) = 0 \in In1$.

Proposition 5.1. *There is no $\phi \in In1\backslash\{0\}$ such that $R^k(\phi) \in In1\backslash\{0\}$ for all $k = 0, 1, 2, \cdots$.*

Proof. Suppose not, i.e., $\exists \phi \in In1\backslash\{0\}$ s.t. $R^k(\phi) \in In1\backslash\{0\}$ for all $k = 0, 1, 2, \cdots$. Note that $R(\phi)$ is continuous in $(0, \tau - g(-\epsilon)]$, since

$$R(\phi) = \begin{cases} F_-(\tau+\phi,\epsilon) - F_-(\tau-\phi,\epsilon) - \phi, & \phi \in In1; \\ F_-(\tau+\phi,\epsilon) - \phi & \phi \in In2_a \vee In2_b. \quad \textbf{(5.14, 5.18)} \end{cases}$$

The return maps of $In2_a$ and $In2_b$ will be proved in Eqs. (5.14), (5.18).

$$\Rightarrow \begin{cases} R((\tau - g(-\epsilon))^-) & = F_-(2\tau - g(-\epsilon)) - (\tau - g(-\epsilon)); \\ R(\tau - g(-\epsilon)) & = F_-(2\tau - g(-\epsilon)) - (\tau - g(-\epsilon)). \end{cases}$$

Moreover, $R(\phi) > \phi$, $\forall \phi \in In1\backslash\{0\}$. Thus, $R^k(\phi) \to \phi^*$ for some $\phi^* \in (0, \tau - g(-\epsilon)]$, and such ϕ^* is a fixed point. That is $R(\phi^*) = \phi^*$. It follows that $\phi^* = \tau - g(-\epsilon)$, or $\epsilon = -f(\tau - \phi^*)$, then $F_-(\tau - \phi^*, \epsilon) = 0$. Hence,

$$\begin{aligned} & R(\phi^*) = \phi^* \\ \Rightarrow\ & F_-(\tau+\phi^*,\epsilon) - F_-(\tau-\phi^*,\epsilon) - \phi^* = \phi^* \\ \Rightarrow\ & F_-(\tau+\phi^*,\epsilon) - \phi^* = \phi^* \\ \Rightarrow\ & F_-(\tau+\phi^*,\epsilon) = 2\phi^* \\ \Rightarrow\ & f(\tau+\phi^*) + \epsilon = f(2\phi^*) \\ \Rightarrow\ & f(\tau+\phi^*) - f(\tau-\phi^*) = f(2\phi^*) - f(0) = f(\phi^*+\phi^*) - f(\phi^*-\phi^*) \end{aligned}$$

By relation **A6**, we know $f(\tau+\phi^*) - f(\tau-\phi^*) > f(\phi^*+\phi^*) - f(\phi^*-\phi^*)$, since $\tau > \phi^*$. $\rightarrow\leftarrow$ $\square$

Corollary 5.1. *For each $\phi \in In1\backslash\{0\}$, there is a $k \in \mathbb{N}$, depending on ϕ, such that $R^k(\phi) \in In2$.*

Dynamics in $In2_a : -f(\tau - \phi) \geq \epsilon \geq -f(\tau)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [l_{2_a}, \tau]$, where $l_{2_a} = \tau - g(-\epsilon)$.)

$$h(\phi) = 1 - F_-(\tau + \phi, \epsilon) + \phi$$

By Eq. (5.4), thus $h : In2_a \mapsto In0$. Then the return map satisfies

$$\begin{aligned} R(\phi) &= h(h(\phi)) = 1 - h(\phi) \\ &= F_-(\tau + \phi, \epsilon) - \phi \\ &< \tau + \phi - \phi = \tau. \end{aligned} \tag{5.14}$$

Thus $R(\phi) \in In1 \cup In2$.

Lemma 5.1. *For each fixed ϵ, there exists a fixed point for R in $In2_a$.*

Proof. Note that $R(\phi)$ is continuous in $[\tau - g(-\epsilon), \tau]$.
If $\phi = l_{2_a}$, we know that $\epsilon = -f(\tau - l_{2_a})$. The return map

$$\begin{aligned} R(l_{2_a}) &= F_-(\tau + l_{2_a}, \epsilon) - l_{2_a} \\ &= g(f(\tau + l_{2_a}) - f(\tau - l_{2_a})) - l_{2_a} \\ &> g(f(l_{2_a} + l_{2_a}) - f(l_{2_a} - l_{2_a})) - l_{2_a} \quad \textbf{(A6)} \\ &= 2l_{2_a} - l_{2_a} = l_{2_a}. \end{aligned} \tag{5.15}$$

If $\phi = \tau$, the return map

$$\begin{aligned} R(\tau) &= F_-(2\tau, \epsilon) - \tau \\ &< 2\tau - \tau = \tau. \end{aligned} \tag{5.16}$$

By Eqs. (5.15), (5.16), there exists a fixed point for R in $In2_a$.

□

Since $R(\phi) = \phi$,

$$\begin{aligned} & R(\phi) = \phi \\ \Rightarrow\ & F_-(\tau + \phi, \epsilon) - \phi = \phi \\ \Rightarrow\ & F_-(\tau + \phi, \epsilon) = 2\phi \\ \Rightarrow\ & g(f(\tau + \phi) + \epsilon) = 2\phi \\ \Rightarrow\ & \epsilon = f(2\phi) - f(\tau + \phi). \end{aligned} \tag{5.17}$$

Thus, the fixed point in $In2_a$ must satisfies $\epsilon(\phi) = f(2\phi) - f(\tau + \phi)$.

Proposition 5.2. *For each fixed ϵ, suppose that $\epsilon'(\phi) = 2f'(2\phi) - f'(\tau + \phi) > 0$, then the fixed point for R in $In2_a$ is unique, and it is an attractor.*

Proof. For each fixed ϵ, suppose not, i.e., $\exists \epsilon$, $\phi_1 < \phi_2 \in In2_a$ s.t. $R(\phi_1) = \phi_1$, and $R(\phi_2) = \phi_2$. By Eq. (5.17), we have $\epsilon = f(2\phi_1) - f(\tau + \phi_1) = f(2\phi_2) - f(\tau + \phi_2)$. By the Rolle's Theorem, $\exists \phi' \in (\phi_1, \phi_2)$ s.t. $\epsilon'(\phi') = 0$. This makes contradiction. Therefore, if $2f'(2\phi) - f'(\tau + \phi) > 0$, then the fixed point for R in $In2_a$ is unique.

For any $\phi \in In2_a$,

$$\begin{aligned} R'(\phi) &= F'_-(\tau+\phi,\epsilon) - 1 \\ &= g'(f(\tau+\phi) - |\epsilon|)f'(\tau+\phi) - 1 \\ &> g'(f(\tau+\phi))f'(\tau+\phi) - 1 \\ &> 1 - 1 = 0. \end{aligned}$$

Thus, $l_{2_a} < R(l_{2_a}) < R(\phi) < R(\tau) < \tau$. **Hence, $In2_a$ is a trapping region.**

Let ϕ^* be the fixed point of $R(\phi)$, i.e., $R(\phi^*) = \phi^*$.

(i)As $\phi > \phi^*$,

$$\begin{aligned} & R(\phi) < \phi \quad \text{(Otherwise, there exists another fixed point in } [\phi,\tau].) \\ \Rightarrow \quad & R^k(\phi) < \cdots < R^2(\phi) < R(\phi) < \phi, \quad \forall k = 1,2,\cdots \\ \text{and} \quad & R^k(\phi) > R^k(\phi^*) = \phi^*, \quad \forall k = 1,2,\cdots \quad (\because R \text{ is increasing.}) \\ \Rightarrow \quad & \phi^* < R^k(\phi) < R^{k-1}(\phi) < \cdots < \phi, \quad \forall k = 1,2,\cdots \end{aligned}$$

Thus, the sequence $\{R^k(\phi)\}$ is decreasing, and $R^k(\phi) \to \phi^*$.

(ii)As $\phi < \phi^*$,

Similarly, the sequence $\{R^k(\phi)\}$ is increasing, and $R^k(\phi) \to \phi^*$.

The unique fixed point ϕ^* is an attractor.

□

Corollary 5.2. *For each $\phi \in In2_a$, $R^k(\phi)$ converges to the fixed point ϕ^*. Specifically, if $\phi = 0$, then the fixed point $\phi^* = 0$, i.e., ϕ_A and ϕ_B synchronize.*

Dynamics in $In2_b : -f(\tau) > \epsilon > f(\phi) - f(\tau+\phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [l_{2_b}, \tau)$, where l_{2_b} satisfies $\epsilon = f(\phi) - f(\tau+\phi)$.)

$$h(\phi) = 1 - F_-(\tau+\phi,\epsilon) + \phi$$

Remark 5.1. *$\phi = 0 \notin In2_b$.*

By Eq. (5.4), thus $h : In2_b \mapsto In0$. Then the return map satisfies

$$\begin{aligned} R(\phi) &= h(h(\phi)) = 1 - h(\phi) \\ &= F_-(\tau+\phi,\epsilon) - \phi \qquad (5.18) \\ &< \tau + \phi - \phi = \tau. \end{aligned}$$

Since $-f(\tau) > \epsilon$, $R(\phi) \notin In1 \cup In2_a$. It follows that $R(\phi) \in In2 \backslash In2_a \cup In3$.

Proposition 5.3. *There is no $\phi \in In2_b$ such that $R^k(\phi) \in In2_b$ for all $k = 0, 1, 2, \cdots$.*

Proof. Note that $R(\phi)$ is continuous in $[l_{2_b}, \tau]$. Assume that there exists a fixed point for R in $In2_b$. Since the return map of $In2_b$ is the same as that in $In2_a$. It follows that

$$\begin{aligned} & R(\phi) = \phi \\ \Leftrightarrow\quad & \epsilon = f(2\phi) - f(\tau+\phi) > f(2\phi - 2\phi) - f(\tau + \phi - 2\phi) \quad \textbf{(5.17, A5)} \\ \Leftrightarrow\quad & \epsilon = f(2\phi) - f(\tau+\phi) > -f(\tau - \phi) > -f(\tau) \quad (\text{For } \phi \neq 0 \in In2_b.) \\ \Leftrightarrow\quad & \epsilon > -f(\tau). \end{aligned}$$

This contradicts the region of the ϵ in $In2_b$. Then there is no fixed point for R in $In2_b$.

Since $R(l_{2_b}) = 0$ and $R(\tau) = F_-(2\tau, \epsilon) - \tau < \tau$, it follows that $R(\phi) < \phi$. Otherwise, there exists fixed points in $[l_{2_b}, \tau]$. Thus, ϕ is decreasing in $In2_b$.
Suppose not, i.e., $\exists \phi \in In2_b$, s.t. $R^k(\phi) \in In2_b$, for all $k = 0, 1, 2, \cdots$. Since ϕ is decreasing, $R^k(\phi) \to \phi^*$, for some $\phi^* \in [l_{2_b}, \tau]$. It follows that $\phi^* = l_{2_b}$ is a fixed point in $In2_a$. It makes contradiction.

□

Corollary 5.3. *For each $\phi \in In2_b$, there is a $k \in \mathbb{N}$, depending on ϕ, such that $R^k(\phi) \in In2_c \cup In2_d \cup In3$.*

Dynamics in $In2_c$: $f(\phi) - f(\tau + \phi) > \epsilon > -f(\tau + \phi)$ and $\phi \in I_1$ (For fixed ϵ, $\phi \in [g(-\epsilon) - \tau, \min\{l_{2_b}, \tau\})$.)

$$R(\phi) = h(\phi) = 1 + F_-(\tau + \phi, \epsilon) - \phi$$

Remark 5.2. $\phi = 0 \notin In2_c$.

By Eq. (5.4), thus $R, h : In2_c \mapsto In0$. We iterate the return map and derive

$$\begin{aligned} h(R(\phi)) &= 1 - R(\phi) = 1 - h(\phi) \\ &= \phi - F_-(\tau + \phi, \epsilon) \\ &< \phi. \end{aligned} \tag{5.19}$$

Since $R(\phi) \in In0$, $h(R(\phi)) < \phi$, $h(R(\phi)) \in In2_c \cup In3$.

Note that $h(R(\phi))$ is continuous in $In2_c$, and $h(R(g(-\epsilon) - \tau)) = g(-\epsilon) - \tau$. Clearly, we know that $h'(R(\phi)) = 1 - F'_-(\tau + \phi, \epsilon) < 0$. For each $\phi \in In2_c$, since $g(-\epsilon) - \tau < \phi$, we have

$$\begin{aligned} & h(R(g(-\epsilon) - \tau)) > h(R(\phi)) \\ \Rightarrow\quad & g(-\epsilon) - \tau > h(R(\phi)) \\ \Rightarrow\quad & h(R(\phi)) \notin In2_c, \quad h(R(\phi)) \in In3. \end{aligned}$$

Corollary 5.4. *For each $\phi \in In2_c$, $h(R(\phi)) \in In3$.*

Dynamics in $In2_d : \epsilon = f(\phi) - f(\tau+\phi)$ **and** $\phi \in I_1$ **(For fixed** ϵ**,** $\phi = l_{2_b}$**.)**

$$h(\phi) = 1$$

Corollary 5.5. *For each $\phi \in In2_d$, ϕ_A and ϕ_B synchronize.*

Dynamics in $In3 : \epsilon \le -f(\tau+\phi)$ **and** $\phi \in I_1$ **(For fixed** ϵ**,** $\phi \in [0, \min\{g(-\epsilon)-\tau, \tau\})$**.)**

$$h(\phi) = R(\phi) = 1 - \phi$$

Thus, $h, R : In3 \mapsto In0$. Specifically, if $\phi = 0$, ϕ_A and ϕ_B synchronize.

Corollary 5.6. *For each $\phi \in In3$, ϕ and $h(\phi)$ is a Period.*

Configuration I_2

Dynamics in $In4 : \epsilon > f(2\tau) - f(\tau+\phi)$ **and** $\phi \in I_2$ **(For fixed** ϵ**,** $\phi \in (l_4, 1-\tau]$**, where** $l_4 = g(f(2\tau)-\epsilon) - \tau$**.)**

$$h(\phi) = 1 - F_-(\tau+\phi, \epsilon) + \tau$$

By Eqs. (5.7), (5.8), thus $\tau < h(\phi) \le 1-\tau$ such that $h : In4 \mapsto In4 \cup In5 \cup In6$. Note that h is continuous in $[l_4, 1-\tau]$. Since,

$$h(\phi) = \begin{cases} 1 - F_-(\tau+\phi,\epsilon) + \tau, & \phi \in In4; \\ 1 - |F_-(\tau+\phi,\epsilon) - \tau| & \phi \in In5. \ \textbf{(5.9)} \end{cases} \Rightarrow \begin{array}{ll} h(l_4^+) & = 1-\tau; \\ h(l_4) & = 1-\tau \end{array} \tag{5.20}$$

Lemma 5.2. *There exists a unique fixed point for h in $In4$.*

Proof. From Eq. (5.20), we know that $h(l_4) = 1 - \tau > l_4$. It follows that

$$\begin{aligned} & h(\phi) = 1 - F_-(\tau+\phi,\epsilon) + \tau \\ \Rightarrow\ & h'(\phi) = -F_-'(\tau+\phi,\epsilon) < -1 \qquad (5.21) \\ \Rightarrow\ & \int_{l_4}^{1-\tau} h'(\phi)d\phi < \int_{l_4}^{1-\tau} -1 d\phi \\ \Rightarrow\ & h(1-\tau) - h(l_4) < -[(1-\tau) - l_4] \\ \Rightarrow\ & h(1-\tau) < h(l_4) - [(1-\tau) - l_4] = l_4 < 1-\tau. \qquad (5.22) \end{aligned}$$

Since $h(l_4) > l_4$ and $h(1-\tau) < 1-\tau$, $h(\phi)$ has fixed points.

Now, we claim that the fixed point for h in $In4$ is unique.
Suppose not, i.e., $\exists \phi_1 < \phi_2 \in In4$ s.t. $h(\phi_1) = \phi_1$ and $h(\phi_2) = \phi_2$. By Mean Value Theorem,

$$\begin{aligned} & h'(\phi') = \frac{h(\phi_2) - h(\phi_1)}{\phi_2 - \phi_1} \quad \text{for some } \phi' \in (\phi_1, \phi_2) \\ \Rightarrow\ & h'(\phi') = 1 \quad \text{for some } \phi' \in (\phi_1, \phi_2) \end{aligned}$$

This is a contradiction with Eq. (5.21). Thus, there exists a unique fixed point ϕ^* for h in $In4$. □

Since there exists a fixed point for h,

$$\begin{aligned} & h(\phi) = \phi \\ \Rightarrow \quad & 1 - F_{-}(\tau+\phi, \epsilon) + \tau = \phi \\ \Rightarrow \quad & 1 + \tau - \phi = g(f(\tau+\phi) + \epsilon) \\ \Rightarrow \quad & \epsilon = f(1+\tau-\phi) - f(\tau+\phi). \end{aligned} \tag{5.23}$$

Thus, the fixed point in $In4$ must satisfy $\epsilon(\phi) = f(1+\tau-\phi) - f(\tau+\phi)$.

Proposition 5.4. *Except for $\phi = \phi^*$, there is no $\phi \in In4$ such that $h^k(\phi) \in In4$ for all $k = 0, 1, 2, \cdots$. i.e., The unique fixed point ϕ^* for h in $In4$ is a repellor.*

Proof. Suppose not, i.e., $\exists \phi \in In4$ s.t. $h^k(\phi) \in In4$, for all $k = 0, 1, 2, \cdots$. Let

$$\begin{aligned} & S_1 := \{\phi : h^1(\phi) \in In4\} && \Rightarrow \quad S_1 \text{ is an interval, } S_1 \neq \emptyset. \ (\because \phi^* \in S_1 \text{ and } h'(\phi) < -1) \\ & S_2 := \{\phi : h^2(\phi) \in In4\} && \Rightarrow \quad S_2 \text{ is an interval, } S_2 \neq \emptyset. \ (\because \phi^* \in S_2 \subseteq S_1 \text{ and } \frac{d}{d\phi}h^2(\phi) > 1) \\ & \quad \vdots && \qquad\qquad \vdots \\ \Rightarrow \quad & S := \{\phi : h^k(\phi) \in In4 \ \forall k\} && = \quad \bigcap_{k=1}^{\infty} S_k \text{ is an interval (noempty).} \end{aligned}$$

By Eq. (5.21), $R'(\phi) = h'(h(\phi))h'(\phi) > 1$. Let $H(\phi) = R(\phi) - \phi$, then we have $H'(\phi) = R'(\phi) - 1 > 0$.

1°

(*i*) $\quad If \ \phi < \phi^*$,

$$H(\phi) < H(\phi^*) = 0 \quad \Rightarrow \quad R(\phi) < \phi.$$

(*ii*) $\quad If \ \phi > \phi^*$,

$$H(\phi) > H(\phi^*) = 0 \quad \Rightarrow \quad \phi < R(\phi).$$

Then (i) $R^k(\phi) < \cdots < R^2(\phi) < R(\phi) < \phi < \phi^*, \quad \forall k \in \mathbb{N}$.

and (ii) $R^k(\phi) > \cdots > R^2(\phi) > R(\phi) > \phi > \phi^*, \quad \forall k \in \mathbb{N}$.

2°

(*i*) $\quad If \ \phi < \phi^*$,

$$\begin{aligned} & H(R(\phi)) < H(\phi) \\ \Rightarrow \quad & R(R(\phi)) - R(\phi) < R(\phi) - \phi \\ \Rightarrow \quad & |R(\phi) - \phi| < |R^2(\phi) - R(\phi)| \end{aligned}$$

(*ii*) $\quad If \ \phi > \phi^*$,

$$\begin{aligned} & H(\phi) < H(R(\phi)) \\ \Rightarrow \quad & |R(\phi) - \phi| < |R^2(\phi) - R(\phi)| \end{aligned}$$

For fixed ϵ, $|R^k(\phi) - \phi| = |R^k(\phi) - R^{k-1}(\phi)| + \cdots + |R(\phi) - \phi|$. By the inequality, $|R^k(\phi) - \phi| > k|R(\phi) - \phi|$, for $k \geq 2$.

Thus, for each point $\phi \in S$ such that $R^{\infty}(\phi) \notin S$. It contradicts the definition of S.

In conclusion, the unique fixed point ϕ^* for h in $In4$ is a repellor, and $h^k(\phi) \in In5 \cup In6$ for some $k \in \mathbb{N}$.

□

Corollary 5.7. *For each $\phi \in In4$, there exists a unique fixed point ϕ^* which is a repellor. Also there is a $k \in \mathbb{N}$, depending on ϕ, such that $h^k(\phi) \in In5 \cup In6$.*

Dynamics in $In5 : f(2\tau) - f(\tau+\phi) \geq \epsilon > -f(\tau+\phi)$ **and** $\phi \in I_2$

$$h(\phi) = 1 - |F_-(\tau+\phi,\epsilon) - \tau|$$

By Eq. (5.10), thus $h : In5 \mapsto In0 \cup l_{1-\tau}$, where $l_{1-\tau}$ is the line $\phi = 1-\tau$. The firemap $h(\phi) = 1-\tau \in l_{1-\tau}$ just as $\phi = l_4$. Also, if $\phi_A(\tau) > \phi_B(\tau)$, $R(\phi) = h(\phi)$.

Furthermore,

$$\begin{aligned} & h(\phi) = 1 \\ \Leftrightarrow \quad & |F_-(\tau+\phi,\epsilon) - \tau| = 0 \\ \Leftrightarrow \quad & g(f(\tau+\phi)+\epsilon) = \tau \\ \Leftrightarrow \quad & \epsilon = f(\tau) - f(\tau+\phi). \end{aligned}$$

Therefore, ϕ_A and ϕ_B synchronize just as $\epsilon(\phi) = f(\tau) - f(\tau+\phi)$.

Corollary 5.8. *For each $\phi \in In5$, $h : In5 \mapsto In0 \cup l_{1-\tau}$. Specifically, when $\epsilon(\phi) = f(\tau) - f(\tau+\phi)$, ϕ_A and ϕ_B synchronize.*

Dynamics in $In6 : -f(\tau+\phi) \geq \epsilon$ **and** $\phi \in I_2$ **(For fixed** ϵ**,** $\phi \in [\tau, g(-\epsilon)-\tau]$**.)**

$$h(\phi) = R(\phi) = 1 - \tau$$

Thus $h : In6 \mapsto In4 \cup In5$.

Corollary 5.9. *For each $\phi \in In6$, $h : In6 \mapsto In4 \cup In5$.*

Configuration I_3

Dynamics in $In0 : \phi \in I_3$

$$h(\phi) = 1 - \phi$$

Thus, $h : In0 \mapsto I_1$.

Corollary 5.10. *$In0$ is the same as $Ex0$ and we have $h : In0 \mapsto I_1$.*

5.3. **Construction of Phase Diagrams.**

Since for all $\phi \in In6$, $h(\phi) = 1-\tau$, and the line $1-\tau$ belongs to $In4$, $In5$, $In6$. Let's find out how the oscillators run when $\phi = 1-\tau$. We find that different τ leads to three different cases.

Theorem 5.1.
Case 1: $\tau \leq g(\frac{1}{2}) - \frac{1}{2}$ *(Figure 5.1: $\tau = 0.25$)*
$\Rightarrow \exists G_1, G_2$ *s.t.* $f(2\tau)-1 < G_1 \leq G_2 < -f(2\tau)$, *and* $G_1 = G_2$ *when* $\tau = g(\frac{1}{2}) - \frac{1}{2}$.
(i) If $\epsilon \in [G_1, G_2]$, *then* $h(1-\tau) \in In6$.
(ii)If $\epsilon \in [f(2\tau)-1, G_1) \cup (G_2, 0)$, *then* $h(1-\tau) \in In5$.
(iii)If $\epsilon \in (-1, f(2\tau)-1)$, *then* $h(1-\tau) \in In0$.
Case 2: $g(\frac{1}{2}) - \frac{1}{2} < \tau \leq \frac{1}{2}g(\frac{1}{2})$ *(Figure 5.2: $\tau = 0.3$)*
$\Rightarrow$ *(i) If* $\epsilon \in [f(2\tau)-1, 0)$, *then* $h(1-\tau) \in In5$.
(ii)If $\epsilon \in (-1, f(2\tau)-1)$, *then* $h(1-\tau) \in In0$.
Case 3: $\tau > \frac{1}{2}g(\frac{1}{2})$ *(Figure 5.3: $\tau = 0.4$)*
$\Rightarrow$ *(i) If* $\epsilon \in [f(2\tau)-1, 0)$, *then* $h(1-\tau) \in In5$.
(ii)If $\epsilon \in (-1, f(2\tau)-1)$, *then* $h(1-\tau) \in In0$.

Proof. We clearly know that the line $1-\tau \in In4, In5, In6$. If $\phi \in In4$, then $\epsilon > f(2\tau) - f(\tau+\phi)$ and $\phi \in I_2$, i.e., if $\phi \in In4$, then $\epsilon > f(2\tau) - 1$. On the other hand, If $\phi \in In6$, then $\epsilon \leq -f(\tau+\phi)$ and $\phi \in I_2$, i.e., if $\phi \in In6$, then $\epsilon < -f(2\tau)$.
Case 1:
$\forall \phi = 1-\tau \in In4$, by Eq. (5.20),

$$h(1-\tau) < l_4 \Rightarrow h(1-\tau) \in In5 \vee In6. \tag{5.24}$$

(i)

$$h(1-\tau) \in In6 \Leftrightarrow \begin{cases} f(2\tau)-1 < \epsilon < -f(2\tau) \\ |\epsilon| \geq f(\tau + h(1-\tau)). \end{cases}$$

On the one hand, since $f(2\tau)-1 < \epsilon < -f(2\tau)$, $f(2\tau)-1 < -f(2\tau)$, or equivalently, $\tau < \frac{1}{2}g(\frac{1}{2})$.
On the other hand, since $h(1-\tau) = 1 - g(1-\epsilon) + \tau$,

$$\begin{aligned} & |\epsilon| \geq f(\tau + h(1-\tau)) \\ \Leftrightarrow\ & -\epsilon \geq f(2\tau + 1 - g(1+\epsilon)) \\ \Leftrightarrow\ & g(-\epsilon) \geq 2\tau + 1 - g(1+\epsilon) \\ \Leftrightarrow\ & g(-\epsilon) + g(1+\epsilon) \geq 2\tau + 1. \end{aligned} \tag{5.25}$$

Define $K(\epsilon) := g(-\epsilon) + g(1+\epsilon)$. Then we have

$$K'(\epsilon) = -g'(-\epsilon) + g'(1+\epsilon)$$

and

$$K''(\epsilon) = g''(-\epsilon) + g''(1+\epsilon) < 0.$$

Here $K'(\epsilon) = 0$ as $\epsilon = -\frac{1}{2}$, and the maximum value of K is $K(-\frac{1}{2}) = 2g(\frac{1}{2})$. Since $K(-\frac{1}{2}+x) = K(-\frac{1}{2}-x)$ i.e., K is symmetric at $\epsilon = -\frac{1}{2}$ and $[f(2\tau)-1]+[-f(2\tau)] = 2\cdot(-\frac{1}{2})$, for a fixed τ, $\exists\epsilon \in (f(2\tau)-1, -f(2\tau))$ s.t. $g(-\epsilon)+g(1+\epsilon) \geq 2\tau+1$, if and only if $2g(\frac{1}{2}) \geq 2\tau+1$, i.e., $\tau \leq g(\frac{1}{2}) - \frac{1}{2} \leq \frac{1}{2}g(\frac{1}{2})$. Moreover, the collection of such ϵ is an interval.

Thus, there is a ϵ such that $h(1-\tau) \in In6$ with $1-\tau \in In4$ if and only if $\tau \leq g(\frac{1}{2}) - \frac{1}{2}$, and as $\tau \leq g(\frac{1}{2}) - \frac{1}{2}$, set $S = \{1-\tau \in In4, h(1-\tau) \in In6\} = [G_1, G_2] \neq \emptyset$ for some $G_1, G_2 \in (f(2\tau)-1, -(f\tau))$.

(ii)If $\epsilon \in [f(2\tau)-1, G_1) \cup (G_2, 0)$, then $1-\tau \in In4$. By Eq. (5.24) and (i), we obtain $h(1-\tau) \in In5$.

(iii)If $\epsilon \in (-1, f(2\tau)-1)$, then $1-\tau \in In5$ by the definition of $In5$. Thus, $h(1-\tau) \in In0$.

Case 2:

If $\tau \leq \frac{1}{2}g(\frac{1}{2})$, then $f(2\tau)-1 \leq -f(2\tau)$. Hence, $\nexists[G_1, G_2] \in (f(2\tau)-1, -f(2\tau))$.

(i) The proof is the same as Case 1-(ii).

(ii)The proof is the same as Case 1-(iii).

Case 3:

If $\tau > \frac{1}{2}g(\frac{1}{2})$, then$f(2\tau)-1 > -f(2\tau)$.

(i) The proof is the same as Case 1-(ii).

(ii)The proof is the same as Case 1-(iii). □

In the following Figure 5.1 $\sim$ 5.3, we use the particular energy function f and inverse function g which are used in Section 4.3 (The inverse of Eq. (4.21)). Also, we fix $s := 1$, $r := 0.9$, and three different τ in Figure 5.1 $\sim$ 5.3.

Moreover, we add Figure 5.4 which is the phase diagram we make from Ernst [10] as a contrast. And it's energy function f and inverse function g is exactly the Mirollo-Strogatz-type (Eq. 4.21) oscillators we mentioned before. Also, we fixed $s := 1$, $r := 0.4$, and $\tau = 0.2$ in Figure 5.4.

And there are some notations for the following figures:

$$\begin{aligned}
l_1 &: \epsilon = -f(\tau-\phi) \\
l_2 &: \epsilon = -f(\tau+\phi) \\
l_3 &: \epsilon = f(2\tau) - f(\tau+\phi) \\
D_1 &: \forall\phi \in D_1,\ \phi \text{ is an attractor.} \\
D_2 &: \forall\phi \in D_2,\ \text{the complete synchronization occurs.} \\
D_3 &: \forall\phi \in D_3,\ \phi \text{ is a repellor.} \\
D_4 &: \forall\phi \in D_4,\ \text{the complete synchronization occurs.}
\end{aligned}$$

In the following tables, we fixed an arbitrary $\epsilon < 0$, and shift ϕ in [0,1]. Therefore, we clearly see which area (InN) does the oscillator begin and figure out where it ends.

(I) $f(\phi) = \dfrac{\ln(\frac{s-r\phi}{s})}{\ln(\frac{s-r}{s})}$, $f' > 0$, $f'' > 0$

Case 1: $\tau \le g(\dfrac{1}{2}) - \dfrac{1}{2}$

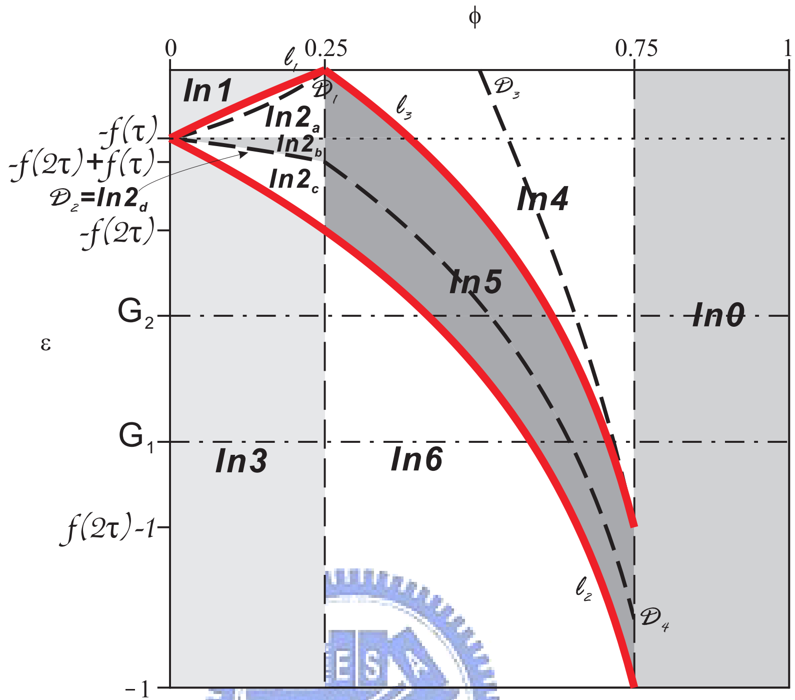


FIGURE 5.1. $s := 1$, $r := 0.9$, $\tau = 0.25$, where $s > r > 0$

ϵ	Dynamics	Multistability
$-f(\tau) \le \epsilon < 0$	$\phi \to \phi^*_{In2_a}$, if $\phi \in I_1, I_2 \backslash D_3, I_3$ $\phi \to \phi$, if $\phi \in D_3$	Lag Syn. with Lag $\phi^*_{In2_a}$ Lag Syn. with Lag ϕ
$G_2 < \epsilon < -f(\tau)$	$\phi \to$Period$(\phi^*_{In3}, 1-\phi^*_{In3})$, if $\phi \in I_1 \backslash D_2, I_2 \backslash \{D_3 \cup D_4\}, I_3$ $\phi \to 0$, if $\phi \in D_2, D_4$ $\phi \to \phi$, if $\phi \in D_3$	Lag Syn. with Lag ϕ^*_{In3} Complete Syn. Lag Syn. with Lag ϕ
$G_1 \le \epsilon \le G_2$	$\phi \to$Period$(\phi^*_{In3}, 1-\phi^*_{In3})$, if $\phi \in I_1, I_3$ $\phi \to$Period$(\phi^*_{In3}, 1-\phi^*_{In3})$, if $\phi \in In4 \backslash \{l_{1-\tau} \cup D_3\}, In5 \backslash D_4$ $\phi \to$Period$(1-\tau, h(1-\tau))$, if $\phi \in l_{1-\tau}, In6$ $\phi \to \phi$, if $\phi \in D_3$ $\phi \to 0$, if $\phi \in D_4$	Lag Syn. with Lag ϕ^*_{In3} Lag Syn. with Lag ϕ^*_{In3} Lag Syn. with Lag $1-\tau$ Lag Syn. with Lag ϕ Complete Syn.
$f(2\tau) - 1 < \epsilon < G_1$, $-1 \le \epsilon < f(2\tau) - 1$	$\phi \to$Period$(\phi^*_{In3}, 1-\phi^*_{In3})$, if $\phi \in I_1, I_2 \backslash \{D_3 \cup D_4\}, I_3$ $\phi \to \phi$, if $\phi \in D_3$ $\phi \to 0$, if $\phi \in D_4$	Lag Syn. with Lag ϕ^*_{In3} Lag Syn. with Lag ϕ Complete Syn.
$\epsilon = f(2\tau) - 1$	$\phi \to 1-\tau$, if $\phi \in l_{1-\tau}, In6$ $\phi \to$Period$(\phi^*_{In3}, 1-\phi^*_{In3})$, if $\phi \in I_1, I_2 \backslash \{D_4 \cup l_{1-\tau}\}, I_3$ $\phi \to 0$, if $\phi \in D_4$	Lag Syn. with Lag $1-\tau$ Lag Syn. with Lag ϕ^*_{In3} Complete Syn.

Case 2: $g(\frac{1}{2}) - \frac{1}{2} < \tau \leq \frac{1}{2}g(\frac{1}{2})$

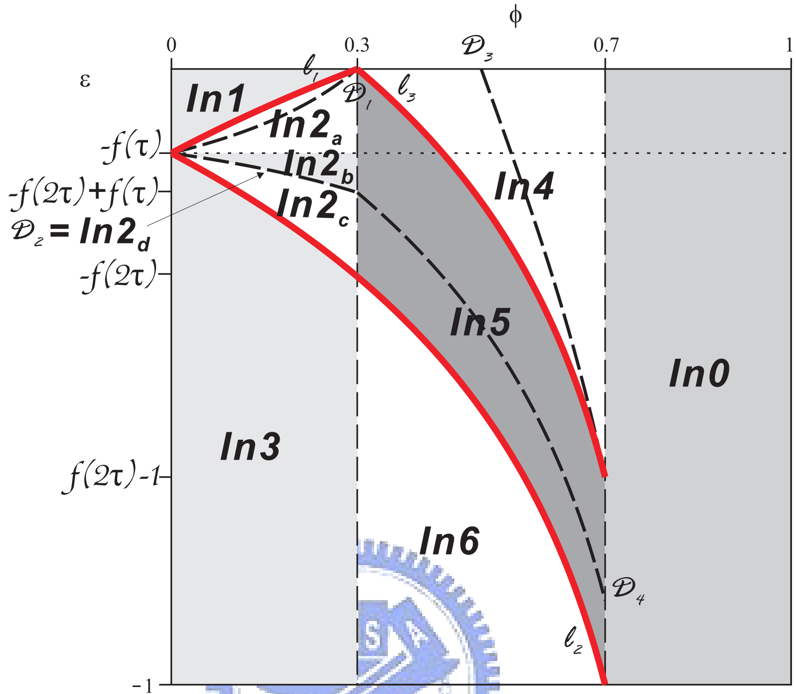


FIGURE 5.2. $s := 1$, $r := 0.9$, $\tau = 0.3$, where $s > r > 0$

ϵ	Dynamics	Multistability
$-f(\tau) \leq \epsilon < 0$	$\phi \to \phi^*_{In2_a}$, if $\phi \in I_1, I_2\backslash D_3, I_3$ $\phi \to \phi$, if $\phi \in D_3$	Lag Syn. with Lag $\phi^*_{In2_a}$ Lag Syn. with Lag ϕ
$f(2\tau) - 1 < \epsilon < -f(\tau)$, $-1 \leq \epsilon < f(2\tau) - 1$	$\phi \to$Period$(\phi^*_{In3}, 1 - \phi^*_{In3})$, if $\phi \in I_1\backslash D_2, I_2\backslash\{D_3 \cup D_4\}, I_3$ $\phi \to 0$, if $\phi \in D_2, D_4$ $\phi \to \phi$, if $\phi \in D_3$	Lag Syn. with Lag ϕ^*_{In3} Complete Syn. Lag Syn. with Lag ϕ
$\epsilon = f(2\tau) - 1$	$\phi \to 1 - \tau$, if $\phi \in l_{1-\tau}, In6$ $\phi \to$Period$(\phi^*_{In3}, 1 - \phi^*_{In3})$, if $\phi \in I_1, I_2\backslash\{D_4 \cup l_{1-\tau}\}, I_3$ $\phi \to 0$, if $\phi \in D_4$	Lag Syn. with Lag $1 - \tau$ Lag Syn. with Lag ϕ^*_{In3} Complete Syn.

Case 3: $\tau > \frac{1}{2}g(\frac{1}{2})$

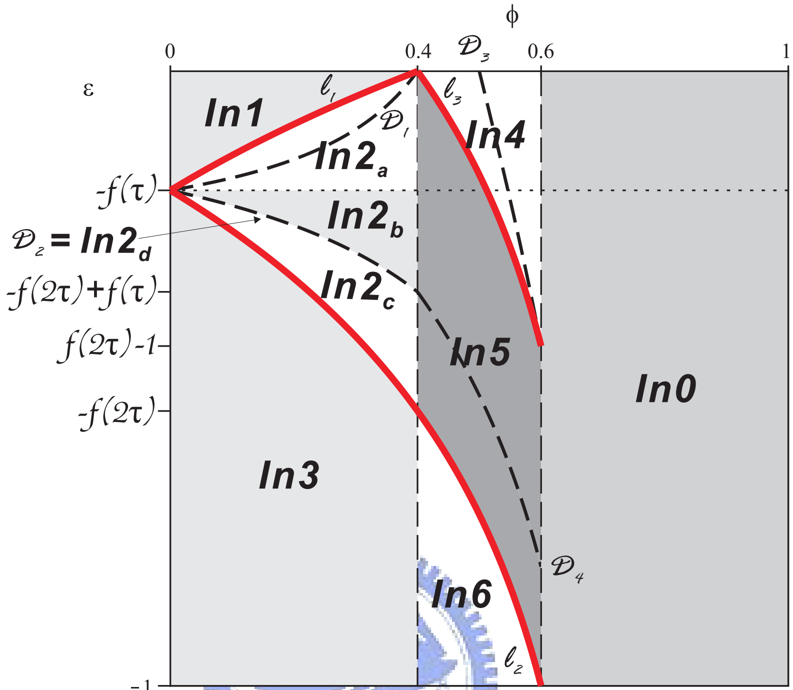


FIGURE 5.3. $s := 1$, $r := 0.9$, $\tau = 0.4$, where $s > r > 0$

ϵ	Dynamics	Multistability
$-f(\tau) \le \epsilon < 0$	$\phi \to \phi^*_{In2_a}$, if $\phi \in I_1, I_2 \backslash D_3, I_3$ $\phi \to \phi$, if $\phi \in D_3$	Lag Syn. with Lag $\phi^*_{In2_a}$ Lag Syn. with Lag ϕ
$f(2\tau) - 1 < \epsilon < -f(\tau)$, $-1 \le \epsilon < f(2\tau) - 1$	$\phi \to$Period$(\phi^*_{In3}, 1 - \phi^*_{In3})$, if $\phi \in I_1 \backslash D_2, I_2 \backslash \{D_3 \cup D_4\}, I_3$ $\phi \to 0$, if $\phi \in D_2, D_4$ $\phi \to \phi$, if $\phi \in D_3$	Lag Syn. with Lag ϕ^*_{In3} Complete Syn. Lag Syn. with Lag ϕ
$\epsilon = f(2\tau) - 1$	$\phi \to 1 - \tau$, if $\phi \in l_{1-\tau}, In6$ $\phi \to$Period$(\phi^*_{In3}, 1 - \phi^*_{In3})$, if $\phi \in I_1, I_2 \backslash \{D_4 \cup l_{1-\tau}\}, I_3$ $\phi \to 0$, if $\phi \in D_4$	Lag Syn. with Lag $1 - \tau$ Lag Syn. with Lag ϕ^*_{In3} Complete Syn.

(II) $f(\phi) = \frac{s}{r} - \frac{s}{r}(\frac{s-r}{s})^{\phi}$, $f' > 0$, $f'' < 0$

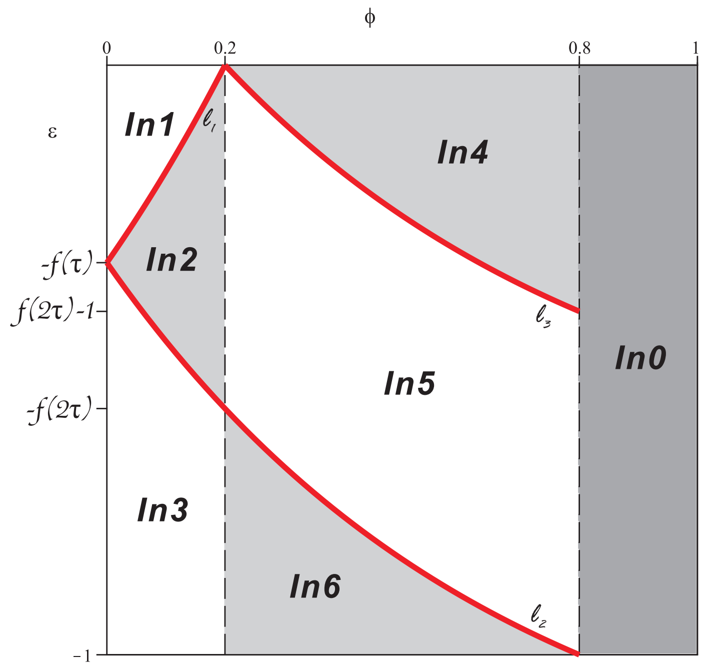


FIGURE 5.4. $s := 1$, $r := 0.4$, $\tau = 0.2$, where $s > r > 0$

ϵ	Dynamics	Multistability
$-f(\tau) \le \epsilon < 0$	$\phi \to 0$, if $\phi \in I_1, In5, I_3$ $\phi \to \phi^*_{In4}$, if $\phi \in In4$	Complete Syn. Lag Syn. with Lag ϕ^*_{In4}
$f(2\tau) - 1 < \epsilon < -f(\tau)$	$\phi \to \phi^*_{In3}$, if $\phi \in I_1, In5, I_3$ $\phi \to \phi^*_{In4}$, if $\phi \in In4$	Lag Syn. with Lag ϕ^*_{In3} Lag Syn. with Lag ϕ^*_{In4}
$-1 \le \epsilon < f(2\tau) - 1$	$\phi \to \phi^*_{In3}$, if $\phi \in I_1, I_2, I_3$	Lag Syn. with Lag ϕ^*_{In3}

6. **Conclusion**

It was numerically demonstrated in [10] that with $N \gg 2$ convex oscillators, the system reveal multistable phase clustering for inhibitory couplings. For $N \gg 2$ concave oscillators the corresponding system also has multistable phase clustering for excitatory couplings. Since for $N = 2$, the corresponding system has stable in-phase synchronization. It will be interesting to treat N as a parameter so as to see how the system evolves from synchronization to clustering synchronization as N increases.

References

[1] S. Bottani, *Pulse-Coupled Relaxation Oscillators: From Biological Synchronization to Self-Organized Criticality*, Phys. Rev. Lett., **74**, 4189 (1995).

[2] J. Buck, *Synchronous Rhythmic Flashing of Fireflies. II.*, Q. Rev. Biol., **63**, 265 (1988).

[3] Y. C. Chang, and J. Juang, *Stable Synchrony In Globally-Coupled Integrate-And-Fire Oscillators*, preprint.

[4] H. Daido, *Lower Critical Dimension for Populations of Oscillators with Randomly Distributed Frequencies: A Renormalization-Group Analysis*, Phys. Rev. Lett., **61**, 231 (1988).

[5] J. Deppisch, H. U. Bauer, T. Schillen, P. König, K. Pawelzik, and T. Geisel, Network **4**, 243 (1993).

[6] R. Eckhorn, R. Bauer, W. Jordan, M. Brosch, W. Kruse, M. Munk, and H. J. Reitboeck, Biol. Cybern. **60**, 121 (1988).

[7] A. K. Engel, P. König, A. K. Kreiter, and W. Singer, Science **252**, 1177 (1991).

[8] G. Ermentrout, *An adaptive model for synchrony in the firefly Pteroptyx malaccae*, J. Math. Biol., **29**, 571 (1991).

[9] U. Ernst, K. Pawelzik, and T. Geisel, Phys. Rev. Lett. **74**, 1570 (1995).

[10] U. Ernst, K. Pawelzik, and T. Geisel, *Delay-induced multistable synchronization of biological oscillators*, Phys. Rev. E, **57**, 2150 (1998).

[11] C. M. Gray, P. Krönig, A. K. Engel, and W. Singer, Nature (London) **338**, 334 (1989).

[12] J. Jalife, *Mutual entrainment and electrical coupling as mechanisms for synchronous firing of rabbit sinoatrial pacemaker cells*, J. Physiol., **356**, 221 (1984).

[13] Y. Kuramoto and I. Nishikawa, *Statistical macrodynamics of large dynamical systems. Case of a phase transition in oscillator communities*, J. Stat. Phys., **49**, 596 (1987).

[14] Y. Kuramoto, *Collective synchronization of pulse-coupled oscillators and excitable units*, Physica D, **50**, 15 (1991).

[15] W. W. Lytton and T. J. Sejnowski, J. Neurophys. **66**,1059 (1991); C. van Vreeswijk, L. F. Abbott, and G. B. Ermentrout, J. Comp. Neurosci. (to be published); A. Nischwitz, H. Glünder, A. von Oertzen, and P. Klausner, *ANN Proceedings* (Springer-Verlag, Berlin, 1993), p. 622.

[16] C. v. d. Malsburg, Technical Report No. 81-2, MPI for Biophysical Chemistry, 1981.

[17] C. v. d. Malsburg and W. Schneider, Biol. Cybern. **54**, 29 (1986).

[18] D. C. Michaels, E. P. Matyas, J. Jalife, *Mechanisms of sinoatrial pacemaker synchronization: a new hypothesis*, Circulation Res., **61**, 704 (1987).

[19] R. E. Mirollo and S. H. Strogatz, *Synchronization of pulse-coupled biological oscillators*, SIAM J. Appl. Math., **50**, 1645 (1990).

[20] C. S. Peskin, *Mathematical Aspects of Heart Physiology*, Courant Inst. Math. Sci., New York Univ., 268 (1975).

[21] J. Rubin and D. Terman, *Geometric analysis of population rhythms in synaptically coupled neuronal networks*, Neural Computation, **12**, 597 (2000).

[22] A. Sherman, J. Rinzel, J. Keizer, *Emergence of organized bursting in clusters of pancreatic beta-cells by channel sharing*, Biophys. J., **54**, 411 (1988).

[23] A. Sherman and J. Rinzel., *Collective properties of insulin secreting cells*, in “Cell to Cell Signalling: From Experiments to Theoretical Models”, A. Goldbeter, ed., Academic Press, London, 61 (1989).

[24] S. H. Strogatz, C. M. Marcus, R. M. Westervelt, and R. E. Mirollo, *Simple Model of Collective Transport with Phase Slippage*, Phys. Rev. Lett., **61**, 2380 (1988).

[25] S. Strogatz, *Lecture Notes in Biomathematics*, Springer, Berlin, **100**, (1993).

[26] Vincent Torre, *A theory of synchronization of heart pace-maker cells*, J. Theoret. Biol., **61**, 55 (1976).

[27] A. T. Winfree, *The Geometry of Biological Time* (Springer-Verlag, New York, 1980).

[28] C. W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems*, World Scientific series on nonlinear science, Series A, **41**, World Scientific, Singapore, (2002).