# 國立交通大學

## 應用數學系

## 碩 士 論 文

由局限距離正則圖構造強正則圖的一種方法

Construct Strongly Regular Graphs from

D-bounded Distance-Regular Graphs

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## 摘要

在數學的領域中，透過不同數學方法而得到相同的理論是常見的。本論文的目的是探討，透過不同的兩個方法來証明強正則圖的參數, 一個利用變數的計算(如定理 7.6 所示)，另一個是利用線性代數的方法(如定理 8.3). 結果顯示此強正則圖的參數呈現是一樣的。

# Construct Strongly Regular Graphs from D-bounded Distance-Regular Graphs

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Abstract

In the field of mathematics, it is common to achieve the same conclusion of the theory via various approaches. The purpose of this thesis is to probe the parameters of a strongly regular graph via two different methods, one (Theorem 7.6) with the use of counting argument and the other (Theorem 8.3) with a linear algebric method. The result shows that these parameters are determined explicitly and are the same.

# Contents

# 1 Introduction

In mathematics, it is common to get the same theory through different approaches. In this thesis, we adopt two different methods to construct a strongly regular graph, one (Theorem 7.6) with the use of counting argument and the other (Theorem 8.3) with a linear algebric method.

We consider a distance-regular graph $\Gamma$, called a *D-bounded* distance-regular graph. This graph $\Gamma$ contains many *weak-geodetically closed subgraphs*. We fix a weak-geodetically closed subgraph $\Delta$ of diameter $s$ and let $P$ (resp. $\mathcal{B}$) be the set of all weak-geodetically closed subgraphs containing $\Delta$ of diameter $s+1$ (resp. $s+2$). We show that $(P, \mathcal{B})$ is a 2-design. We show that the block graph of $(P, \mathcal{B})$ is a strongly regular graph and determine its parameters explicitly by two methods.

To do this, we need some concepts about graphs and designs. Hence, in Chapter 2, we review some definitions and basic concepts of graphs, such as regular graphs, subgraphs, eigenvalues, etc. In Example 2.2, a special graph, Petersen graph, is introduced since it is an essential example in graph theory.

Next, Chapter 3 introduces the distance-regular graphs along with a special class of distance-regular graphs, the class of strongly regular graphs, which will be used in Chapter 5. Meanwhile, a classic theorem and its proof, Theorem 3.5, are mentioned to characterize strongly regular graph by its eigenvalues.

In Chapter 4, we give some definitions of designs and its basic concepts. Firstly, a $t$-design is presented accompanied with a special case, 2-design later utilized in the following chapter.

Chapter 5 is about Quasisymmetric Designs and its relation to strongly regular graphs. Quasisymmetric Design is known as a 2-design $(P, \mathcal{B})$, which constructs a graph with vertex set $\mathcal{B}$. In Theorem 5.4, we know that this graph is a strongly regular graph. Next, in Lemma 5.5, its parameters are determined explicitly. Within Lemma 5.5, a special case, Corollay 5.6, will be generated as well and used in Theorem 8.3.

In Chapter 6, we give some definitions and properties about D-bounded distance-regular, which will be used in the next Chapter.

In Chapter 7, a strongly regular graph is constructed from a $D$-bounded distance-regular graph. Firstly, we define a graph $G(\Delta, 2)$ which is known to be either a clique or a strongly regular graph in Theorem 7.1. The parameters of a strongly regular graph will be obtained. It is worth noticing that Theorem 7.6 is the main theorem in this thesis.

Finally, in Chaptet 8, we use a linear algebraic method to prove Theorem 7.6 again as Theorem 8.3 shown.

## 2 Preliminaries of graphs

In this section we review some definitions and basic concepts of graphs. The reader can refer to [13] for more details.

**Definition 2.1.** A *graph* $G$ is a pair consisting of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where $E(G)$ is a set containing some 2-subsets of $V(G)$. For a vertex $u \in V(G)$ and an edge $uv \in E(G)$, we say $u$ is *incident* to $uv$ and $u$ is *adjacent* to $v$.

**Example 2.2.** Let $V(G) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $E(G) = \{01, 12, 23, 34, 40, 05, 16, 27, 38, 49, 57, 58, 69, 68, 79\}$. Then $G = (V(G), E(G))$ is a graph. See Figure 1 for the drawing of this graph. $G$ is called the *Petersen graph*.

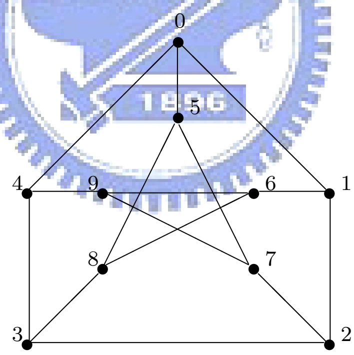


Figure 1: Petersen graph

**Definition 2.3.** The *degree* of vertex $v$ in a graph $G$ is the number of edges incident to $v$. $G$ is *regular* if every vertex has the same degree. It is $b_0$-*regular* if the common degree is $b_0$.

**Example 2.4.** The Petersen graph is 3-regular.

**Definition 2.5.** A *path* in a graph $G$ is a sequence of vertices so that any two consecutive vertices are adjacent. For two vertices $u, v$ in $G$, a *$u, v$-path* is a path with the first vertex $u$ and the last vertex $v$.

**Definition 2.6.** A graph $G$ is *connected* if each pair of vertices in $G$ belongs to a path.

**Definition 2.7.** If $G$ has a $u, v$-path, then the *distance* from $u$ to $v$, written $\partial_G(u, v)$ or simply $\partial(u, v)$, is the least length of a $u, v$-path. If $G$ has no such path, then $\partial(u, v) = \infty$. The *diameter* (diam $G$) is

$$\max\{\partial(u, v) | u, v \in V(G)\}.$$

**Example 2.8.** The Petersen graph has diameter 2, since nonadjacent vertices have a common neighbor.

**Definition 2.9.** A *subgraph* of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* is a subgraph $H$ of a graph $G$ such that if $u, v \in V(H)$ and $uv \in E(G)$ then $uv \in E(H)$.

**Definition 2.10.** Let $G$ be a graph with vertex set $V(G)$ of size $n$ and edge set $E(G)$ of size $m$. The *adjacency matrix* of $G$, written $A(G)$, is the $n$-by-$n$ matrix in which entry $a_{ij}$, where $i, j \in V(G)$, is 1 if $ij \in E(G)$ and otherwise is 0. The *incidence matrix* $M(G)$ is the $n$-by-$m$ matrix in which entry $m_{ie}$, where $i \in V(G)$ and $e \in E(G)$, is 1 if $i$ is incident to $e$ and otherwise is 0.

**Definition 2.11.** The *eigenvalues* of a graph $G$ are the eigenvalues of its adjacency matrix $A(G)$.

**Definition 2.12.** The *complement* $\overline{G}$ of a graph $G$ is the graph with vertex set $V(\overline{G})$=$V(G)$ and the edge set $E(\overline{G}) = \{uv \mid u, v \in V(G), uv \notin E(G)\}$.

**Definition 2.13.** Assume $G$ is a connected graph with diameter $D$. For all vertices $x$ in $G$ and for $0 \le i \le D$, set

$$G_i(x) := \{y \in G \mid \partial(x, y) = i\}.$$

# 3 Distance-regular graphs

Assume $\Gamma$ is a connected graph with diameter $D$. $\Gamma$ is said to be *distance-regular* whenever for $0 \le h, i, j \le D$ and for vertices $x, y$ in $\Gamma$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := \left|\Gamma_i(x) \cap \Gamma_j(y)\right|$$

is independent of $x, y$. The constants $p^h_{ij}$ are known as the *intersection numbers* of $\Gamma$. For convenience, set $c_i := p^i_{1i-1}, a_i := p^i_{1i}, b_i := p^i_{1i+1}$ and $k_i := p^0_{ii}$. Note that $c_1 = 1$, $a_0 = 0$, $b_D = 0$ and

$$k_1 = c_i + a_i + b_i \quad \text{for} \quad 0 \le i \le D.$$

The Petersen graph described in Figure 1 is a distance-regular graph with diameter $D = 2$ and intersection numbers $c_1 = 1, c_2 = 1, a_1 = 0, a_2 = 2, b_0 = 3, b_1 = 2$.

Next, we give a special class of distance-regular graphs. Its diameter is two.

**Definition 3.1.** A *strongly regular graph* $SRG(v, b_0, a_1, c_2)$ is a $b_0$-regular graph which has $v$ vertices and the following properties hold:

(i) For any two adjacent vertices $x, y$, there are exactly $a_1$ vertices adjacent to $x$ and to $y$.

(ii) For any two nonadjacent vertices $x, y$, there are exactly $c_2$ vertices adjacent to $x$ and $y$.

Next, we give two examples.

**Example 3.2.** A pentagon is an $SRG(5, 2, 0, 1)$.

**Example 3.3.** The Petersen graph is an $SRG(10, 3, 0, 1)$.

Next, we introduce the lemma, which will be used later.

**Lemma 3.4.** *A connected graph with diameter $d$ has at least $d+1$ distinct eigenvalues.*

*Proof.* Let $A = A(G)$ be the adjacency matrix. Suppose $A$ has distinct eigenvalues $\theta_0, \theta_1, ..., \theta_m$ where $m < d$. Then $m(x) = \prod_{i=0}^{m}(x - \theta_i)$ is the minimal polynomial of $A$. Hence $A^{d-m-1}m(A) = 0$. Expanding to find

$$A^d = C_{d-1}A^{d-1} + C_{d-2}A^{d-2} + ... + C_0 \text{ for some } C_i \in R.$$

Pick two vertices $x, y \in G$ with $\partial(x, y) = d$. We check the $xy$ position in the above equation and find

$$0 \neq (A^d)_{xy} = (C_{d-1}A_{xy}^{d-1} + C_{d-2}A_{xy}^{d-2} + ... + C_0) = 0,$$

a contradiction. $\square$

The following theorem characterize a strongly regular graph by its eigenvalues.

**Theorem 3.5.** ([1, Problem 31H]). *Let $G$ denote a connected $b_0$-regular graph of diameter 2. Then $G$ is a strongly regular graph $SRG(v, b_0, a_1, c_2)$ for some scalars $v$, $b_0$, $a_1$, $c_2$ if and only if $G$ has three distinct eigenvalues.*

*Proof.* ($\Rightarrow$)Let $A = A(G)$ be the adjacency matrix.

Observe

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_0 \\ \vdots \\ b_0 \end{bmatrix} = b_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Claim(i): $A^2 + (c_2 - a_1)A + (c_2 - b_0)I = c_2 J$, where $J$ denotes the all 1's matrix.

Pick $x, y \in G$. Compare $x, y$ entries on both sides, according to the following three cases.

Case 1. $\partial(x, y) = 0:$

$$b_0 + 0 + (c_2 - b_0) \cdot 1 = c_2;$$

Case 2. $\partial(x, y) = 1:$

$$a_1 + (c_2 - a_1) \cdot 1 + 0 = c_2;$$

Case 3. $\partial(x, y) = 2:$

$$c_2 + 0 + 0 = c_2.$$

Claim(ii): $(A - b_0 I)(A^2 + (c_2 - a_1)A + (c_2 - b_0)I) = 0.$

The left hand side acts on $(1,1,\ldots,1)^t$ is 0. Let $u=(1,1,\ldots,1)^t$. Other eigenvectors are orthogonal to $u$. Hence by claim(i)

$$\begin{aligned}(A-b_0I)(A^2+(c_2-a_1)A+(c_2-b_0)I)u &= (A-b_0I)(c_2J)u \\ &= 0.\end{aligned}$$

By claim(ii), $A$ has eigenvalues among $b_0,(a_1-c_2\pm\sqrt{(c_2-a_1)^2-4(c_2-b_0)})/2$. By Lemma 3.4, $G$ has at least three eigenvalues. Hence $G$ has exactly three eigenvalues.

($\Leftarrow$) Clearly, $G$ has a eigenvalue $b_0$. Suppose $g<s$ are the other two. Then $(A-gI)(A-sI)=(b_0-g)(b_0-s)J/|G|$. (As above, apply both sides to all eigenvectors of $A$.) Pick $x,y\in G$.

Case 1. $\partial(x,y)=0$ :

$$\begin{aligned}A^2_{xy} &= (g+s)A_{xy}-gsI_{xy}+\frac{(b_0-g)(b_0-s)}{|G|}J_{xy} \\ &= -gs+\frac{(b_0-g)(b_0-s)}{|G|}\end{aligned}$$

is independent of $x=y$, and $b_0=-gs+(b_0-g)(b_0-s)/|G|$.

Case 2. $\partial(x,y)=1$ :

$$\begin{aligned}A^2_{xy} &= (g+s)A_{xy}-gsI_{xy}+\frac{(b_0-g)(b_0-s)}{|G|}J_{xy} \\ &= g+s+\frac{(b_0-g)(b_0-s)}{|G|}\end{aligned}$$

is independent of $x,y$ with $\partial(x,y)=1$ and $a_1=g+s+(b_0-g)(b_0-s)/|G|$.

Case 3. $\partial(x,y)=2$ :

$$\begin{aligned}A^2_{xy} &= (g+s)A_{xy}-gsI_{xy}+\frac{(b_0-g)(b_0-s)}{|G|}J_{xy} \\ &= \frac{(b_0-g)(b_0-s)}{|G|}\end{aligned}$$

is independent of $x, y$ with $\partial(x, y) = 2$ and $c_2 = (b_0 - g)(b_0 - s)/|G|$.

$G$ has diameter 2. Since $G$ has three eigenvalues. Hence $b_1 := b_0 - a_1 - 1$ is $|G_1(x) \cap G_2(y)|$ for any $x, y \in G$ with $\partial(x, y) = 1$. Then $G$ is a strongly regular graph.

$\square$

**Example 3.6.** The Petersen graph is a 3-regular graph of diameter 2. And from Example 3.3, we know it is a strongly regular graph . By Theorem 3.5, the Petersen graph has three distinct eigenvalues.

# 4 Preliminaries of designs

We first give the definition of a design and its basic concepts.

**Definition 4.1.** $(P, \mathcal{B})$ is a $t$-$(v, k, \lambda)$ *design* whenever the following (i)-(iv) hold.

(i) $P$ is a finite set of $v$ elements.

(ii) $\mathcal{B}$ is a class of subsets, called *blocks*, of $P$.

(iii) $|B| = k$ for all $B \in \mathcal{B}$.

(iv) For any $t$ distinct elements $p_1, p_2, ..., p_t \in P$ there are exactly $\lambda$ blocks $B_1, B_2, ..., B_\lambda \in \mathcal{B}$ such that $p_i \in B_j$ for all $i, j$.

A $t$-design is a $t$-$(v, k, \lambda)$ design for some positive integers $v$,$k$,$\lambda$.

Next, we see two examples.

**Example 4.2.** $P = \{1, 2, 3, \ldots, n\}$, $\mathcal{B} = \{P\}$. Then $(P, \mathcal{B})$ is a $t$-$(n, n, 1)$ design for any $t = 1, 2, \ldots, n$.

**Example 4.3.** $P = \{0,1\}^2 = \{0,1\} \times \{0,1\}$. $\mathcal{B} = \{\{(0,0),(1,0)\},\{(0,1),(1,1)\},\{(0,0),(0,1)\},\{(1,0),(1,1)\},\{(0,0),(1,1)\},\{(1,0),(0,1)\}\}$. Then $(P, \mathcal{B})$ is a 2-$(4, 2, 1)$ design.

We give a few properties of 2-design which will be used later.

**Lemma 4.4.** ([1, Theorem 19.2]). *In a* 2-$(v, k, \lambda)$ *design, there are* $b := \lambda \binom{v}{2} / \binom{k}{2}$ *blocks.*

*Proof.* We count the pairs $(S, B)$, where $S \subseteq B \in \mathcal{B}$ and $|S| = 2$, by two ways:

$$\binom{v}{2}\lambda = b\binom{k}{2}.$$

Hence $b = \lambda \binom{v}{2} / \binom{k}{2}$. □

**Lemma 4.5.** ([1, Theorem 19.3]). *For* $i = 0, 1, 2$, *any* $i$ *points in a* 2-$(v, k, \lambda)$ *design* $(P, \mathcal{B})$ *are contained in* $\gamma_i := \lambda \binom{v-i}{2-i} / \binom{k-i}{2-i}$ *blocks.*

*Proof.* Let $I \subseteq P$ with $|I| = i$. Count the pair $(S, B)$, where $S \cup I \subseteq B \in \mathcal{B}$, $S \cap I = \phi$ and $|S| = 2 - i$, by two ways:

$$\binom{v-i}{2-i}\lambda = \gamma_i \binom{k-i}{2-i}.$$

Hence $\gamma_i = \lambda \binom{v-i}{2-i} / \binom{k-i}{2-i}$. □

**Definition 4.6.** In a 2-$(v, k, \lambda)$ design, $\gamma_1 = \lambda(v-1)/(k-1)$ is usually denoted by $r$, referred as the *replication number.*

We give the following example of 2-design. To calculate $b$ and $r$.

**Example 4.7.** Let $P = \{1,2,3,4,5,6,7\}$, $\mathcal{B} = \{\{1,2,3\},\{1,6,7\},\{1,4,5\},\{3,4,7\},\{2,5,7\},\{2,4,6\},\{3,5,6\}\}$. Then $(P, \mathcal{B})$ is 2-$(7,3,1)$ design. And $b = 1\begin{pmatrix} 7 \\ 2 \end{pmatrix} / \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 7$, $r = 1(7-1)/(3-1) = 3$.

# 5 Quasisymmetric designs and strongly regular graphs

In this section, we introduce the concept of Quasisymmetric Designs and the graph which is constructed from a quasisymmetric design. This graph is known to be a strongly regular graph. We determine its parameters explicitly.

We give a special class of 2-*design.*

**Definition 5.1.** A *quasisymmetric design* $(QSD)$ *with parameters* $\rho < \alpha$ is a 2-design $(P, \mathcal{B})$ such that $|B \cap B'| = \rho$ or $\alpha$ for all distinct blocks $B$ and $B'$ in $\mathcal{B}$.

We construct a graph from a quasisymmetric design.

**Definition 5.2.** Suppose that $(P, \mathcal{B})$ is a $QSD$ with parameters $\rho < \alpha$. Let $G$ be a graph with vertex set $\mathcal{B}$ and two vertices $B$ and $B'$ of $\mathcal{B}$ are adjacent if and only if $|B \cap B'| = \rho$. Then $G$ is called the *block graph* of $(P$,$\mathcal{B})$.

Next, we introduce the lemma, which will be used later.

**Lemma 5.3.** *Let $N$ denote the $v \times b$ incidence matrix of $(P, \mathcal{B})$ where $v = |P|$ and $b = |\mathcal{B}|$ and $J$ denote the all 1's matrix, $r$ is the replication number. Then $NN^t = (r - \lambda)I + \lambda J$.*

*Proof.*

$$
\begin{aligned}
(NN^t)_{xy} &= \sum_{\ell \in \mathcal{B}} N_{x\ell} N^t_{\ell y} \\
&= \sum_{\ell \in \mathcal{B}} N_{x\ell} N_{y\ell} \\
&= \begin{cases} r, & \text{if } x = y \text{ ;} \\ \lambda, & \text{if } x \neq y \text{ ;} \end{cases} \\
&= ((r - \lambda)I + \lambda J)_{xy}.
\end{aligned}
$$

□

The following theorem is the main theorem in this section.

**Theorem 5.4.** ([1, Theorem 21.2]). *The block graph $G$ of a QSD $(P, \mathcal{B})$ with parameters $\rho < \alpha$ is a strongly regular graph.*

*Proof.* Let $A(G)$ denote the adjacency matrix of $G$, and $N$ denote the $v \times b$ incidence matrix of $(P, \mathcal{B})$ where $v = |P|$ and $b = |\mathcal{B}|$. Let $k$ be the cardinality of a block in $\mathcal{B}$ and $J$ denotes the all 1's matrix.

Claim: $N^t N = kI + \rho A + \alpha(J - I - A)$.

Pick two block $B, B' \in \mathcal{B}$. Compare the $BB'$-entry on both sides.

Case 1. $B = B'$ :

$$k = |B \cap B'| = k + 0 + \alpha(1 - 1 - 0);$$

Case 2. $|B \cap B'| = \rho$ :

$$\rho = 0 + \rho + \alpha(1 - 0 - 1);$$

Case 3. $|B \cap B'| = \alpha$ :

$$\alpha = 0 + 0 + \alpha(1 - 0 - 0).$$

The claim follows from this.

Note that $A = (N^t N + (\alpha - k)I - \alpha J)/(\rho - \alpha)$.

Recall that $NN^t = (r - \lambda)I + \lambda J$. We know that both $NN^t$ and $N^t N$ have all-one eigenvectors $j$(of different lengths!) with eigenvalue $r - \lambda + \lambda v$ $(= kr)$. We know that $NN^t$ has only the eigenvalue $r - \lambda$ with multiplicity $v - 1$. Therefore $N^t N$ has this same eigenvalue, with the same multiplicity, and the eigenvalue 0 with multiplicity $b - v$. Observe

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{kr + \alpha - k - \alpha b}{\rho - \alpha} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Hence $G$ is regular. Observe the eigenvectors of $N^t N$ are eigenvectors of I and J. Hence the other 2 eigenvalues of $A$ are

$$\frac{r - \lambda + \alpha - k + 0}{\rho - \alpha}, \quad \frac{0 + \alpha - k + 0}{\rho - \alpha}.$$

Hence G has exactly three eigenvalues. By Theorem 3.5, we obtain $G$ is a strongly regular graph. $\square$

In Theorem 5.4, we know "$G$ is a strongly regular regular." Next, in the following lemma, we will determine its parameters explicitly.

**Lemma 5.5.** *Let $G$ denote the strongly regular graph obtained in Theorem 5.4 with parameters $(v, b_0, a_1, c_2)$. Then the following (i)-(iv) hold.*

*(i)* $v = |G|$,

*(ii)* $b_0 = \dfrac{kr + (\alpha - k) - \alpha b}{\rho - \alpha}$,

*(iii)* $a_1 = \dfrac{(\rho-\alpha)(r-\lambda+2(\alpha-k)+kr+\alpha-k-\alpha b)+(\alpha-k)(r-\lambda+\alpha-k)}{(\rho-\alpha)^2}$,

*(iv)* $c_2 = \dfrac{(\alpha-k)(r-\lambda+\alpha-k)+(\rho-\alpha)(kr+\alpha-k-\alpha b)}{(\rho-\alpha)^2}$.

*Where $b = |\mathcal{B}|$, $r$ is the replication number, and $k$ is the cardinality of a block in $\mathcal{B}$.*

*Proof.* Clearly, $G$ has a eigenvalue $b_0$. Suppose $g < s$ are the other two eigenvalues of $G$. Where $g = (r-\lambda+\alpha-k)/(\rho-\alpha)$, $s = (\alpha-k)/(\rho-\alpha)$.

(ii) Observe that

$$\begin{aligned} b_0 &= \frac{kr+\alpha-k-\alpha b}{\rho-\alpha} \\ &= -gs + \frac{(k-g)(k-s)}{\mid G \mid}. \end{aligned}$$

(iii) Observe that

$$\begin{aligned}
a_1 &= g + s + \frac{(k-g)(k-s)}{|G|} \\
&= g + s + b_0 + gs \\
&= \frac{r-\lambda+\alpha-k}{\rho-\alpha} + \frac{\alpha-k}{\rho-\alpha} + \frac{kr+\alpha-k-\alpha b}{\rho-\alpha} + \frac{(r-\lambda+\alpha-k)(\alpha-k)}{(\rho-\alpha)^2} \\
&= \frac{(\rho-\alpha)(r-\lambda+2(\alpha-k)+kr+\alpha-k-\alpha b)+(\alpha-k)(r-\lambda+\alpha-k)}{(\rho-\alpha)^2}.
\end{aligned}$$

(iv) Observe that

$$\begin{aligned}
c_2 &= \frac{(k-g)(k-s)}{|\, G \,|} \\
&= b_0 + gs \\
&= \frac{kr+\alpha-k-\alpha b}{\rho-\alpha} + \frac{(r-\lambda+\alpha-k)(\alpha-k)}{(\rho-\alpha)^2} \\
&= \frac{(\rho-\alpha)(kr+\alpha-k-\alpha b)+(\alpha-k)(r-\lambda+\alpha-k)}{(\rho-\alpha)^2}.
\end{aligned}$$

□

We give a special case of lemma 5.5, which will be used later.

**Corollary 5.6.** *From Lemma 5.5, and let* $\rho = 0, \alpha = 1, \lambda = 1$, *we have*

*(ii)* $b_0 = -kr + k - 1 + b$;

*(iii)* $a_1 = 2k - 2kr - 2 + b + k^2$;

*(iv)* $c_2 = r - 2kr + k^2 + b - 1$.

*Proof.* From Lemma 5.5, and let $\rho = 0, \alpha = 1, \lambda = 1$.

(ii) We have

$$\begin{aligned} b_0 &= \frac{kr + \alpha - k - \alpha b}{\rho - \alpha} \\ &= \frac{kr + 1 - k - b}{0 - 1} \\ &= -kr - 1 + k + b. \end{aligned}$$

(iii) We have

$$\begin{aligned} a_1 &= \frac{(\rho - \alpha)(r - \lambda + 2(\alpha - k) + kr + \alpha - k - \alpha b) + (\alpha - k)(r - \lambda + \alpha - k)}{(\rho - \alpha)^2} \\ &= \frac{(0 - 1)(r - 1 + 2(1 - k) + kr + 1 - k - b) + (1 - k)(r - 1 + 1 - k)}{(0 - 1)^2} \\ &= (-1)(r - 1 + 2 - 2k + kr + 1 - k - b) + r - k - kr + k^2 \\ &= (-1)(r + 2 - 3k + kr - b) + r - k - kr + k^2 \\ &= -r - 2 + 3k - kr + b + r - k - kr + k^2 \\ &= 2k - 2kr - 2 + b + k^2. \end{aligned}$$

(iv) We have

$$\begin{aligned} c_2 &= \frac{(\alpha - k)(r - \lambda + \alpha - k) + (\rho - \alpha)(kr + \alpha - k - \alpha b)}{(\rho - \alpha)^2} \\ &= \frac{(1 - k)(r - 1 + 1 - k) + (0 - 1)(kr + 1 - k - b)}{(0 - 1)^2} \\ &= (1 - k)(r - k) + (-1)(kr + 1 - k - b) \\ &= r - k - kr + k^2 - kr - 1 + k + b \\ &= r - 2kr + k^2 + b - 1. \end{aligned}$$

□

# 6 D-bounded distance-regular graphs

Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. A sequence of vertices $x, y, z$ of $\Gamma$ is *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1,$$

where $\partial$ is the distance function of $\Gamma$. A subgraph $\Delta$ of $\Gamma$ is *weak-geodetically closed* whenever for all weak-geodetic sequences of vertices $x, y, z$ of $\Gamma$ we have

$$x, z \in \Delta \Longrightarrow y \in \Delta.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [7]. We refer the reader to [9], [2], [5], [8], [10], [6] for the constructions of weak-geodetically closed subgraphs of $\Gamma$. It is immediate from the definition that a weak-geodetically closed subgraph $\Delta$ is an induced subgraph of $\Gamma$ and the distance function on $\Delta$ is induced from that on $\Gamma$. $\Gamma$ is *D-bounded* if (i) all of the weak-geodetically closed subgraphs of $\Gamma$ are regular; and (ii) for all vertices $x, y$ of $\Gamma$, $x, y$ are contained in a common weak-geodetically closed subgraph $\Delta(x, y)$ of diameter $\partial(x, y)$. In fact $\Delta(x, y)$ is uniquely determined by the vertices $x$ and $y$ [10, Corollary 5.4], and is distance-regular [10, Corollary 5.3]. Regular near polygons [2], [6], [9] and Hermitian forms graphs [5] are examples of $D$-bounded distance-regular graphs. The classification of $D$-bounded distance-regular graphs with some additional assumptions can be found in [11], [12].

Below we recall a few properties in a $D$-bounded distance-regular graph, which will be used in the next section. Let $\Gamma$ denote a $D$-bounded distance-regular graph where $D \geq 3$ is the diameter of $\Gamma$. Let $a_i, b_i, c_i$ denote the intersection numbers of $\Gamma$ for $0 \leq i \leq D$. Let $\Delta$ denote a weak-geodetically closed subgraph of diameter $s$ for $0 \leq s \leq D$. Note that $\Delta$ is regular by the assumption (i) of $D$-bounded definition. In fact $\Delta$ is distance-regular with intersection numbers

$$\begin{aligned} a_i(\Delta) &= a_i(\Gamma) \\ c_i(\Delta) &= c_i(\Gamma) \\ b_i(\Delta) &= b_i(\Gamma) - b_s(\Gamma) \end{aligned}$$

for $0 \leq i \leq s$ [10, Corollary 5.3]. In particular a weak-geodetically closed subgraph of diameter 1 is a clique of size $b_0 - b_1 + 1$, and we refer such a clique to a *line*. The intersection of weak-geodetically closed subgraphs is either an empty set or a weak-geodetically closed subgraph. Hence $|\Delta \cap \ell| \in \{0, 1, b_0 - b_1 + 1\}$ for any line $\ell$ in $\Gamma$. Let $x$ denote a vertex in $\Delta$. Then $\Delta_1(x)$ is a disjoint union of $(b_0 - b_s)/(b_0 - b_1)$ cliques of the form $\ell - \{x\}$, where $\ell \subseteq \Delta$ is a line containing $x$. There are

$$\frac{b_0}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}$$

lines $\ell'$ containing $x$ such that $\ell' \not\subseteq \Delta$. For such a line $\ell'$, there exists a unique weak-geodetically closed subgraph $\Delta'$ of diameter $s+1$ containing $\Delta$ and $\ell'$. There are

$$\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}$$

lines $\ell''$ (including $\ell'$) containing $x$ such that $\ell'' - \{x\} \subseteq \Delta' - \Delta$.

# 7 Construct strongly regular graphs from $D$-bounded distance-regular graphs

Throughout the section, let $\Gamma$ denote a $D$-bounded distance-regular graph with intersection numbers $b_i, c_i$ for $0 \leq i \leq D$. Note that $b_i > b_{i+1}$ for $0 \leq i \leq D-1$ [11, Lemma 2.6]. Fix an integer $0 \leq s \leq D-3$ and a weak-geodetically closed subgraph $\Delta$ of $\Gamma$ with diameter $s$. Let $\mathcal{P} = \mathcal{P}(\Delta)$ denote the collection of weak-geodetically closed subgraphs containing $\Delta$. If $\Delta = \{x\}$ for some vertex $x$ of $\Gamma$ then we write $\mathcal{P}(x)$ for $\mathcal{P}(\Delta)$. It was shown that $\mathcal{P}$ is a ranked atomic lattice [3], where $\mathrm{rank}(\Omega)$ is $\mathrm{diameter}(\Omega) - s$ for $\Omega \in \mathcal{P}$. Let $\mathcal{P}_j = \mathcal{P}_j(\Delta)$ denote the set of rank $j$ elements in $\mathcal{P}$ for $0 \leq j \leq D-s$. For each $1 \leq i \leq D-s$ we define a graph $G(\Delta, i)$ whose vertex set is $\mathcal{P}_i$, and vertex $\Omega$ is adjacent to vertex $\Omega'$ in $G(\Delta, i)$ if and only if $\Omega \cap \Omega' \in \mathcal{P}_{i-1}$, where $\Omega, \Omega' \in \mathcal{P}_i$.

**Theorem 7.1.** $G(\Delta, 2)$ *is either a clique or a strongly regular graph with parameters*

$$b_0(G) = \frac{b_{s+2}(b_s - b_{s+2})}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})}, \tag{7.1}$$

$$a_1(G) = (\frac{b_{s+1} - b_{s+2}}{b_s - b_{s+1}})^2 + \frac{b_{s+2}}{b_{s+1} - b_{s+2}} - 1, \tag{7.2}$$

$$c_2(G) = (\frac{b_s - b_{s+2}}{b_s - b_{s+1}})^2. \tag{7.3}$$

*Proof.* Fix $x \in \Delta$ and $\Omega \in \mathcal{P}_2$. Then $x \in \Delta \subseteq \Omega$ by the construction. First we prove the number $b_0(G) = b_0(G)(\Omega)$ as expressed in (7.1). We do this by counting the triples $(\Omega', \ell, \ell')$ in the order and its reversed order where $\Omega' \in \mathcal{P}_2$ such that $\Omega \cap \Omega' \in \mathcal{P}_1$, and $\ell, \ell' \subseteq \Omega'$ are lines containing $x$ such that $\ell - \{x\} \subseteq \Omega \cap \Omega' - \Delta$ and $\ell' - \{x\} \subseteq \Omega' - \Omega$. We find

$$\begin{aligned}
& b_0(G) \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \\
= \; & (\frac{b_0}{b_0 - b_1} - \frac{b_0 - b_{s+2}}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times 1
\end{aligned}$$

to obtain (7.1).

Second we fix another $\Omega' \in \mathcal{P}_2$ such that $\Omega \cap \Omega' \in \mathcal{P}_1$. We prove the number $a_1(G) = a_1(G)(\Omega, \Omega')$ as expressed in (7.2). Let $\lambda_1$ (resp. $\lambda_2$) denote the number of $\Omega'' \in \mathcal{P}_2$ such that

$$\Omega'' \cap \Omega = \Omega'' \cap \Omega' = \Omega' \cap \Omega \tag{7.4}$$

(resp.

$$\Omega'' \cap \Omega \in \mathcal{P}_1, \Omega'' \cap \Omega' \in \mathcal{P}_1, \Omega \cap \Omega' \cap \Omega'' = \Delta). \tag{7.5}$$

Note that

$$a_1(G) = \lambda_1 + \lambda_2. \tag{7.6}$$

To determine $\lambda_1$ we count the pairs $(\Omega'', \ell'')$ in the order and its reversed order, where $\Omega'' \in \mathcal{P}_2$ satisfies (7.4) and $\ell'' \subseteq \Omega''$ is a line such that $\ell'' \not\subseteq \Omega \cup \Omega'$. We find

$$\begin{aligned}
& \lambda_1 \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \\
= \; & (\frac{b_0}{b_0 - b_1} - 2\frac{b_0 - b_{s+2}}{b_0 - b_1} + \frac{b_0 - b_{s+1}}{b_0 - b_1}) \times 1.
\end{aligned} \tag{7.7}$$

To determine $\lambda_2$ we count the triples $(\Omega'', \ell, \ell')$ in the order and its reversed order, where $\Omega'' \in \mathcal{P}_2$ satisfies (7.5), and $\ell, \ell' \subseteq \Omega''$ are lines containing $x$ such that $\ell - \{x\} \subseteq \Omega - \Omega'$ and $\ell' - \{x\} \subseteq \Omega' - \Omega$. We find

$$\begin{aligned} & \lambda_2 \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times ((\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \\ = & (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \times 1. \end{aligned} \tag{7.8}$$

(7.2) is immediate by solving (7.6)-(7.8) for $a_1(G)$.

Third we fix $\Omega'' \in \mathcal{P}_2$ such that $\Omega \cap \Omega'' = \Delta$. We prove the number $c_2(G) = c_2(G)(\Omega, \Omega'')$ as expressed in (7.3). We do this by counting the triples $(\Omega''', \ell, \ell'')$ in the order and its reversed order, where $\Omega''' \in \mathcal{P}_2$ such that $\Omega''' \cap \Omega, \Omega''' \cap \Omega'' \in \mathcal{P}_1$, and $\ell, \ell'' \subseteq \Omega'''$ are lines containing $x$ such that $\ell - \{x\} \subseteq \Omega - \Omega''$ and $\ell'' - \{x\} \subseteq \Omega'' - \Omega$. We find

$$\begin{aligned} & c_2(G) \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \\ = & (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times 1. \end{aligned} \tag{7.9}$$

(7.3) follows from (7.9). □

Theorem 7.1 is a generalization of [4], which proves in the case $\Delta = \{x\}$ for some vertex $x$ of $\Gamma$ and some additional assumptions.

**Theorem 7.2.** ([13, Theorem 8.6.33]). *The complement $\overline{G}$ of an $SRG(v, b_0, a_1, c_2)$ $G$ is an $SRG(v, v - b_0 - 1, v - 2b_0 + c_2 - 2, v - 2b_0 + a_1)$*

*Proof.* For each adjacent pair $u, w$ in $G$, there are $2(b_0 - 1) - a_1$ other vertices in $G_1(u) \cup G_1(w)$, so $u$ and $w$ have $v - 2 - 2(b_0 - 1) + a_1 = v - 2b_0 + a_1$ common nonneighbors. When $u, w$ are not adjacent, there are $2b_0 - c_2$ vertices in $G_1(u) \cup G_1(w)$ and thus $v - 2b_0 + c_2$ common nonneighbors. □

**Example 7.3.** The Petersen graph $G$, an SRG $(10, 3, 0, 1)$. Its complement $\overline{G}$ is an SRG $(10, 6, 3, 4)$.

Next, we introduce the following lemma, which will be used later.

**Lemma 7.4.** *Set*

$P := \{\Delta' | \Delta' \supseteq \Delta$ is a weak − geodetically closed subgraph of diameter $s+1$ *in* $\Gamma\}$.

*Then* $|P| = \dfrac{b_s}{b_s - b_{s+1}}$.

*Proof.* Observe

$$\begin{aligned} |P| &= \left(\frac{b_s}{b_0 - b_1}\right)/\left(\frac{b_s - b_{s+1}}{b_0 - b_1}\right) \\ &= \frac{b_s}{b_s - b_{s+1}} \end{aligned}$$

by counting arguments.

□

**Lemma 7.5.** *Set*

$\mathcal{B} = \{\Delta'' | \Delta'' \supseteq \Delta$ *is a weak-geodetically closed subgraph of diameter* $s+2$ *in* $\Gamma\}$.

*Then* $|\mathcal{B}| = \dfrac{b_s b_{s+1}}{(b_{s+1} - b_{s+2})(b_s - b_{s+2})}$.

*Proof.* We count the pair $(\Delta', \Delta'')$ such that $\Delta \in P$, $\Delta' \in \mathcal{B}$ and $\Delta' \subseteq \Delta''$ to find

$$|P| \times \left(\frac{b_{s+1}}{b_{s+1} - b_{s+2}}\right) = |\mathcal{B}| \times \frac{b_s - b_{s+2}}{(b_s - b_{s+2}) - (b_{s+1} - b_{s+2})}.$$

By Lemma 7.4

$$\frac{b_s}{b_s - b_{s+1}} \times (\frac{b_{s+1}}{b_{s+1} - b_{s+2}}) \quad = \quad |\mathcal{B}| \times \frac{b_s - b_{s+2}}{b_s - b_{s+1}}.$$

Hence

$$\begin{aligned} |\mathcal{B}| &= \frac{b_s}{b_s - b_{s+1}} \times \frac{b_{s+1}}{b_{s+1} - b_{s+2}} \times \frac{b_s - b_{s+1}}{b_s - b_{s+2}} \\ &= \frac{b_s b_{s+1}}{(b_{s+1} - b_{s+2})(b_s - b_{s+2})}. \end{aligned}$$

□

**Theorem 7.6.** $\overline{G(\Delta, 2)}$ *is a strongly regular graph with parameters*

$$\begin{aligned} b_0(\bar{G}) &= \frac{b_s b_{s+2}^2 - b_{s+2}^3 - b_{s+1}^2 b_{s+2} + b_{s+1} b_{s+2}^2}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})}. \\ a_1(\bar{G}) &= \frac{2b_s^2 b_{s+1}^2 - b_s b_{s+1}^3 - 3b_s^2 b_{s+1} b_{s+2} - 2b_s b_{s+1}^2 b_{s+2} + 2b_{s+1}^3 b_{s+2} - b_s^3 b_{s+2}}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\ &\quad + \frac{5b_s^2 b_{s+2}^2 + 3b_s b_{s+1} b_{s+2}^2 - 2b_{s+1}^2 b_{s+2}^2 - 5b_s b_{s+2}^3 + b_{s+1} b_{s+2}^3 + b_{s+2}^4}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})}. \\ c_2(\bar{G}) &= \frac{b_s b_{s+1}^3 + 2b_s^2 b_{s+2}^2 - 3b_s b_{s+2}^3 - b_s^2 b_{s+1} b_{s+2} + 3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\ &\quad + \frac{-3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3 - 3b_s b_{s+1}^2 b_{s+2} + b_{s+1}^2 + b_{s+2}^4}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})}. \end{aligned}$$

*Proof.* Observe

$$v = |\mathcal{P}_2| = |\mathcal{B}| = \frac{b_s b_{s+1}}{(b_{s+1} - b_{s+2})(b_s - b_{s+2})}$$

and by Theorem 7.2, we have

$$
\begin{aligned}
b_0(\bar{G}) &= v - b_0(G) - 1 \\
&= \frac{b_s b_{s+1}}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})} - \frac{b_{s+2}(b_s-b_{s+2})^2}{(b_s-b_{s+1})(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&- \frac{(b_s-b_{s+1})(b_{s+1}-b_{s+2})(b_s-b_{s+2})}{(b_s-b_{s+1})(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&= \frac{b_s b_{s+2}^2 - b_{s+2}^3 - b_{s+1}^2 b_{s+2} + b_{s+1} b_{s+2}^2}{(b_s-b_{s+1})(b_{s+1}-b_{s+2})(b_s-b_{s+2})}.
\end{aligned}
$$

$$
\begin{aligned}
a_1(\bar{G}) &= v - 2b_0(G) + c_2 - 2 \\
&= \frac{b_s b_{s+1}}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})} - 2\frac{b_{s+2}(b_s-b_{s+2})^2}{(b_s-b_{s+1})(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&+ \frac{(b_s-b_{s+2})^2}{(b_s-b_{s+1})^2} - 2 \\
&= \frac{b_s b_{s+1}(b_s-b_{s+1})^2}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&-2\frac{b_{s+2}(b_s-b_{s+2})^2(b_s-b_{s+1})}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&+\frac{(b_s-b_{s+2})^3(b_{s+1}-b_{s+2})}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&-2\frac{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&= \frac{2b_s^2 b_{s+1}^2 - b_s b_{s+1}^3 - 3b_s^2 b_{s+1} b_{s+2} - 2b_s b_{s+1}^2 b_{s+2} + 2b_{s+1}^3 b_{s+2} - b_s^3 b_{s+2}}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&+\frac{5b_s^2 b_{s+2}^2 + 3b_s b_{s+1} b_{s+2}^2 - 2b_{s+1}^2 b_{s+2}^2 - 5b_s b_{s+2}^3 + b_{s+1} b_{s+2}^3 + b_{s+2}^4}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})}.
\end{aligned}
$$

$$
\begin{aligned}
c_2(\bar{G}) &= v - 2b_0(G) + a_1 \\
&= \frac{b_s b_{s+1}}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})} - 2\frac{b_{s+2}(b_s-b_{s+2})^2}{(b_s-b_{s+1})(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&\quad +(\frac{b_{s+1}-b_{s+2}}{b_s-b_{s+1}})^2 + \frac{b_{s+2}}{b_{s+1}-b_{s+2}} - 1 \\
&= \frac{b_s b_{s+1}(b_s-b_{s+1})^2}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&\quad -\frac{2b_{s+2}(b_s-b_{s+2})^2(b_s-b_{s+1})}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&\quad +\frac{(b_{s+1}-b_{s+2})^3(b_s-b_{s+2})}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&\quad +\frac{b_{s+2}(b_s-b_{s+1})^2(b_s-b_{s+2})}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&\quad -\frac{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2}{(b_{s+1}-b_{s+2})(b_s-b_{s+2})(b_s-b_{s+1})^2} \\
&= \frac{b_s b_{s+1}^3 + 2b_s^2 b_{s+2}^2 - 3b_s b_{s+2}^3 - b_s^2 b_{s+1} b_{s+2} + 3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&\quad +\frac{-3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3 - 3b_s b_{s+1}^2 b_{s+2} + b_{s+1}^2 + b_{s+2}^4}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})}.
\end{aligned}
$$

$\square$

# 8 Another proof of Theorem 7.6

We prove Theorem 7.6 by a linear algebraic method in this section. Let $\Gamma, \Delta, \mathcal{P} = \mathcal{P}(\Delta), G(\Delta, i)$ be as in Section 7, and $P, \mathcal{B}$ be as in Lemma 7.4 and Lemma 7.5.

A quasisymmetric design ($QSD$) with parameters $\rho < \alpha$ is a $2-(v,k,\lambda)$ design (such that $|B \cap B'| = \rho$ or $\alpha$ for all distinct blocks $B$ and $B'$.)

From Lemma 7.4, we immediately have the following Lemma.

**Lemma 8.1.** *$(P, \mathcal{B})$ is a 2-$(v, k, 1)$ quasisymmetric design with parameters $0, 1$, where*

$$
\begin{aligned}
v &= b_s/(b_s - b_{s+1}), \\
k &= (b_s - b_{s+2})/(b_s - b_{s+1}).
\end{aligned}
$$

□

**Corollary 8.2.** *$(P, \mathcal{B})$ is a 2-$(v, k, 1)$ quasisymmetric design with parameters $0, 1$. Then*

*(i)*

$$b = \frac{b_s b_{s+1}}{(b_s - b_{s+2})(b_{s+1} - b_{s+2})}.$$

*(ii)*

$$r = \frac{b_{s+1}}{b_{s+1} - b_{s+2}}.$$



*Proof.*

(i) By Lemma 4.4. Hence

$$
\begin{aligned}
b &= \frac{\lambda v(v-1)}{k(k-1)} \\
&= \frac{v(v-1)}{k(k-1)} \\
&= \frac{v}{k} \times \frac{v-1}{k-1} \\
&= \frac{b_s b_{s+1}}{(b_s - b_{s+2})(b_{s+1} - b_{s+2})}.
\end{aligned}
$$

(ii) By Lemma 4.5. Hence

$$\begin{aligned} r &= \frac{\lambda(v-1)}{k-1} \\ &= \frac{v-1}{k-1} \\ &= \frac{b_{s+1}}{b_{s+1}-b_{s+2}}. \end{aligned}$$

□

Below we will proof the Theorem 7.6 by another method.

**Theorem 8.3.** *$\overline{G(\Delta,2)}$ is a quasisymmetric design (QSD) with parameters $\rho = 0, \alpha = 1, \lambda = 1$. By Corollary 5.6, $\overline{G(\Delta,2)}$ is a strongly regular graph with parameters*

*(ii)*

$$b_0(\overline{G}) = \frac{b_s b_{s+2}^2 - b_{s+2}^3 - b_{s+1}^2 b_{s+2} + b_{s+1} b_{s+2}^2}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})}.$$

*(iii)*

$$\begin{aligned} a_1(\overline{G}) = {} & \frac{2b_s^2 b_{s+1}^2 - b_s b_{s+1}^3 - 3b_s^2 b_{s+1} b_{s+2} - 2b_s b_{s+1}^2 b_{s+2} + 2b_{s+1}^3 b_{s+2} - b_s^3 b_{s+2}}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\ & + \frac{5b_s^2 b_{s+2}^2 + 3b_s b_{s+1} b_{s+2}^2 - 2b_{s+1}^2 b_{s+2}^2 - 5b_s b_{s+2}^3 + b_{s+1} b_{s+2}^3 + b_{s+2}^4}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})}. \end{aligned}$$

*(iv)*

$$\begin{aligned} c_2(\overline{G}) = {} & \frac{b_s b_{s+1}^3 + 2b_s^2 b_{s+2}^2 - 3b_s b_{s+2}^3 - b_s^2 b_{s+1} b_{s+2} + 3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\ & + \frac{-3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3 - 3b_s b_{s+1}^2 b_{s+2} + b_{s+1}^2 + b_{s+2}^4}{(b_s - b_{s+1})^2 (b_{s+1} - b_{s+2})(b_s - b_{s+2})}. \end{aligned}$$

*Proof.* Where $k = (b_s - b_{s+2})/(b_s - b_{s+1})$, $b = b_s b_{s+1}/(b_s - b_{s+2})(b_{s+1} - b_{s+2})$, $r = b_{s+1}/(b_{s+1} - b_{s+2})$.

(ii) From Corollary 5.6, we have

$$b_0(\overline{G}) = -kr + k - 1 + b.$$

Hence

$$\begin{aligned}
b_0(\overline{G}) &= -kr + k - 1 + b \\
&= \frac{-(b_s - b_{s+2})b_{s+1}}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})} + \frac{b_s - b_{s+2}}{(b_s - b_{s+1})} - 1 + \frac{b_s b_{s+1}}{(b_s - b_{s+2})(b_{s+1} - b_{s+2})} \\
&= \frac{-b_{s+1}(b_s - b_{s+2})^2}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})} + \frac{(b_s - b_{s+2})^2(b_{s+1} - b_{s+2})}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad - \frac{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})} + \frac{b_s b_{s+1}(b_s - b_{s+1})}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&= \frac{b_s b_{s+2}^2 - b_{s+2}^3 - b_{s+1}^2 b_{s+2} + b_{s+1} b_{s+2}^2}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})(b_s - b_{s+2})}.
\end{aligned}$$

(ii) From Corollary 5.6, we have

$$a_1(\overline{G}) = 2k - 2kr - 2 + b + k^2.$$

Hence

$$\begin{aligned}
a_1(\overline{G}) &= 2k-2kr-2+b+k^2 \\
&= \frac{2(b_s-b_{s+2})}{(b_s-b_{s+1})} - \frac{2(b_s-b_{s+2})b_{s+1}}{(b_s-b_{s+1})(b_{s+1}-b_{s+2})} - 2 + \frac{b_s b_{s+1}}{(b_s-b_{s+2})(b_{s+1}-b_{s+2})} \\
&\quad + \frac{(b_s-b_{s+2})^2}{(b_s-b_{s+1})^2} \\
&= \frac{(2b_{s+2}^2-2b_sb_{s+1}+2b_{s+1}^2-2b_{s+1}b_{s+2})(b_s-b_{s+1})(b_s-b_{s+2})}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&\quad + \frac{b_sb_{s+1}(b_s-b_{s+1})^2}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})} + \frac{(b_s-b_{s+2})^3(b_{s+1}-b_{s+2})}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&= \frac{2b_s^2b_{s+1}^2-b_sb_{s+1}^3-3b_s^2b_{s+1}b_{s+2}-2b_sb_{s+1}^2b_{s+2}+2b_{s+1}^3b_{s+2}-b_s^3b_{s+2}}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})} \\
&\quad + \frac{5b_s^2b_{s+2}^2+3b_sb_{s+1}b_{s+2}^2-2b_{s+1}^2b_{s+2}^2-5b_sb_{s+2}^3+b_{s+1}b_{s+2}^3+b_{s+2}^4}{(b_s-b_{s+1})^2(b_{s+1}-b_{s+2})(b_s-b_{s+2})}.
\end{aligned}$$

(iv) From Corollary 5.6, we have

$$c_2(\overline{G}) = r - 2kr + k^2 + b - 1.$$

Hence

$$
\begin{aligned}
c_2(\overline{G}) &= r - 2kr + k^2 + b - 1 \\
&= \frac{b_{s+1}}{(b_{s+1} - b_{s+2})} - 2\frac{(b_s - b_{s+2}b_{s+1})}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})} + \frac{(b_s - b_{s+2})^2}{(b_s - b_{s+1})^2} \\
&\quad + \frac{b_s b_{s+1}}{(b_s - b_{s+2})(b_{s+1} - b_{s+2})} - 1 \\
&= \frac{b_{s+1}(b_s - b_{s+1})^2(b_s - b_{s+2})}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad - \frac{2(b_s - b_{s+1})(b_s - b_{s+2})^2 b_{s+1}}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad + \frac{(b_s - b_{s+2})^3(b_{s+1} - b_{s+2})}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad + \frac{b_s b_{s+1}(b_s - b_{s+1})^2}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad - \frac{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&= \frac{b_s b_{s+1}^3 + 2b_s^2 b_{s+2}^2 - 3b_s b_{s+2}^3 - b_s^2 b_{s+1} b_{s+2} + 3b_s b_{s+1} b_{s+2}^2}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad + \frac{-b_{s+1}b_{s+2}^3 - b_{s+2})(b_s - b_{s+2})}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})} \\
&\quad \frac{-3b_s b_{s+1} b_{s+2}^2 - b_{s+1} b_{s+2}^3 - 3b_s b_{s+1}^2 b_{s+2} + b_{s+1}^2 + b_{s+2}^4}{(b_s - b_{s+1})^2(b_{s+1} - b_{s+2})(b_s - b_{s+2})}.
\end{aligned}
$$

$\square$

From the proof of Theorem 8.3, we obtain result similar to theorem 7.6.

# References


[1] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press (Second Edition 2001).

[2] A. E. Brouwer and H. A. Wilbrink, The structure of near polygons with quads., *Geom. Dedicata*, 14(1883)145-176.

[3] J. Guo and S. Gao, Lattices Generated by Join of Strongly Closed Subgraphs in d-bounded Distance-regular Graphs, preprint.

[4] J. Guo and S. Gao, A Construction of a Distance-regular Graphs with Diameter 2, preprint.

[5] A. A. Ivanov and S. V. Shpectorov, Characterization of the association schemes of Hermitian forms over $GF(2^2)$, *Geom. Dedicata*, 30(1989)23-33.

[6] A. Hiraki, Strongly closed subgraphs in a regular thick near polygon, *European J. Combin.*, 20(8)(1999) 789-796.

[7] H. Suzuki, On strongly closed subgraphs of highly regular graphs, *European J. Combin.*, 16(1995)197-220.

[8] H. Suzuki, Strongly closed subgraphs of a distance-regular graph with geometric girth five, *Kyushu Journal of Mathematics*, 50(2)(1996)371-384.

[9] E. E. Shult and A. Yanushka, Near $n$-gons and line systems, *Geom. Dedicata*, 9(1980)1-72.

[10] C. Weng, Weak-Geodetically Closed Subgraphs in Distance-Regular Graphs, *Graphs and Combinatorics*, 14(1998)275-304.

[11] C. Weng, D-bounded Distance-Regular Graphs, *Europ. J. Comb.*, 18(1997)211-229.

[12] C. Weng, Classical Distance-Regular Graphs of Negative Type, *J. Combin. Theory Ser. B*, 76(1996)93-116.

[13] Douglas B. West. *Introduction to Graph Theory*, Prentice Hall , Second Edition 2001.