国立交通大學

應 用 數 學 系

碩 士 論 文

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碩士論文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
in Partial Fulfillment of the Requirements
for the Degree of
Master
in

Applied Mathematics

July, 2008

Hsinchu, Taiwan, Republic of China

中華民國九十七年七月

S. D. G.

# 具多重解的自律微分方程鏈回歸集

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## 摘　　要

1967 年, Strauss 和 Yorke 引介漸進自律微分方程當中的廣義正向極限集。在這篇論文中, 我們研究可能不具有惟一解的自律微分方程鏈回歸集。在某些條件底下, 我們可以進一步得到鏈回歸集的半不變性。而鏈回歸集和廣義正向極限集之間的關係將會一併探討。

# Chain Recurrent Sets for Autonomous Differential Equations with Multiple Solutions

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## ABSTRACT

In 1967, Strauss and Yorke introduced the concept of generalized positive limit sets for asymptotic autonomous differential equations. In this thesis, we study chain recurrent sets for autonomous differential equations possibly with multiple solutions. Under certain conditions, we obtain semi-invariance of the chain recurrent set. Relations between chain recurrent sets and generalized positive limit sets are also concerned.

# 誌　　謝

愚頑如筆者, 竟完成了碩士班的學業。只能說, 這是上主無比的厚恩。所以, 先感謝上帝, 不念筆者的愚頑和固執, 一路導引, 直到如今。

除了感謝上帝直接的眷佑, 也感謝上帝在這段時間當中, 讓筆者和一些很棒的師長們, 同學們, 學弟妹們, 以及學生們相遇。除了感謝上帝, 更感謝你們願意在這些日子的陪伴及支援。

感謝指導教授李明佳老師。在這一年半的日子當中, 老師給了我很大的自由, 給了我需要的資源, 也給了我許多的包容。因著老師, 在研究生涯的起步, 接觸了許多不一樣的觀念, 嘗試了許多不一樣的想法; 因著老師, 在學習和思考上有許多的挑戰及突破。謝謝老師, 在這段時間當中, 給了筆者許多探索的時空, 點出了筆者在學習和思考上的諸多盲點; 謝謝老師, 願意如此包容, 如此付出。

感謝 Zhivko S. Athanassov 教授。若非 Athanassov 教授願意主動關心及回饋, 這篇論文恐怕無法及時完成。雖然到最後, 還是沒有按著教授的建議完成極限方程的探討, 但教授對數學及學生的主動及熱誠一再的成爲筆者學習和研究的鼓舞。到現在印象依然深刻的是, Athanassov 教授親自到期刊室示範查找論文的方法, 並介紹相關的期刊資源。千言萬語, 實在是無法描述 Athanassov 教授的提點之恩及筆者至深的謝忱。

感謝莊重老師, 陳怡全老師, 鄭文巧老師願意撥冗審查筆者的論文。特別感謝老師們對於筆者校稿一再出錯的包容。

感謝交大信望愛社 (冠賢哥、姵樺姐、以及所有的學弟妹) 這兩年來對筆者如此接納, 完全沒有距離。交大信望愛眞的是很棒的一群! 要是高中認眞些, 早一點來到這裡不知道有多好! 這兩年, 和你們一起笑, 一起哭, 一起探索, 一起期待... 迎新, 查經, 宵夜, 大山背螢火蟲, 康希特會, 第十屆青宣... 這兩年, 在這裡, 交大信望愛讓筆者知道團契是何等的美好。甚願, 在天家, 我們能永遠在一起。

感謝協同會新竹教會, 以及台東教會。首先感謝兄姐們對小弟兄的包容, 願意傾聽小弟兄許許多多亂七八糟的心聲, 並付上禱告的代價。其次感謝弟妹們, 願意接納並信任筆者, 讓筆者見證你們在這裡的改變及成長。感謝和筆者一起同工的黛琳姐, 因著黛琳姊如此諒解及接納, 在新竹充滿波折的第二年才能平順許多。感謝在台東的小組, 負笈北上的這段期間, 見面及通訊的機會很少, 但依然得蒙弟兄姊妹多次多方的紀念, 甚願學成之後, 不管在哪裡都能按著上帝及弟兄姊妹們的託付, 貢獻所學, 盡力拉拔提攜後進。

最後, 若有, 願一切榮耀歸與上帝。

# 目　　錄

# 一、Introduction

To generalize the result of invariance of positive limit sets for autonomous differential equations whose solutions are uniquely determined by initial conditions in [4], in 1967, Strauss and Yorke introduced the concepts of semi-invariance sets and generalized positive limit sets for asymptotically autonomous differential equations possibly with multiple solutions and obtained the semi-invariance of generalized positive limit sets in [1]. A set is called semi-invariant for an autonomous differential equation if for each point of this set, there is a trajectory of given point lies in given set.

In this thesis, we study the chain recurrent sets for autonomous differential equations possibly whih multiple solutions. Under certain conditions, we obtain semi-invariance of the chain recurrent set. Because the relations between chain recurrent sets and generalized positive limit sets are also concerned, we introduce the work of Strauss and Yorke in [1] before introducing our study.

# 二、Basic Settings

In this thesis, $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers, and $|\cdot|$ is a norm of $\mathbb{R}^n$.

Let $Q \subseteq \mathbb{R}^n$ be an open set, $f \in C(Q, \mathbb{R}^n)$, $t_0 \in \mathbb{R}$, and $a, b \in Q$.

Consider the equation

$$x' = f(x), \tag{A}$$

Let $x(\cdot)$ (sometimes $x(\cdot)$ is abbreviated to $x$) be a solution of (A), and let $x(\cdot\,; t_0, a)$ be a solution of equation (A) with initial condition $x(t_0) = a$ and maximal interval $(\alpha^-, \alpha^+)$, where $-\infty \le \alpha^- < t_0 < \alpha^+ \le +\infty$, provided the given initial value problem has a solution.

In this thesis, we assume that (A) satisfies the following three hypotheses of:

**(H1)** (A) with the initial condition $x(0) = a$ has a solution.

**(H2)** If (A) with initial condition $x(t_0) = a$ has a solution, and $x(\cdot\,; t_0, a)$ is a solution of given initial value problem, then $\alpha^+ = +\infty$, and $0 \in (\alpha^-, \alpha^+)$.

**(H3)** For $\varepsilon > 0$, there is a $\delta > 0$ such that if $|a - b| < \delta$, then

$$|x(1; 0, a) - x(1; 0, b)| < \varepsilon.$$

Note that we do not assume the uniqueness of (A) with given initial condition. Different solutions of (A) may have different maximal intervals, even if they have the same initial time and position.

**Example 1** (pp. 79-84 in [3])**.** Consider the logistic equation

$$x' = x(1 - x) \tag{1}$$

in $(0, 1)$. The solutions are $x(t) = \frac{a}{a + (1-a)e^{-t}}$ for $t \in \mathbb{R}$, where $a \in (0, 1)$. See Figure 1. Hence (1) is an automonous differential equation satisfies (H1) to (H3).

**Example 2** (pp. 177 in [1])**.** Consider the equation

$$x' = \begin{cases} (2-x)^{\frac{1}{2}} & \text{if } 1 \le x \le 2 \\ x^{\frac{1}{2}} & \text{if } 0 \le x < 1 \\ 0 & \text{if } x < 0 \text{ or } x > 2 \end{cases} \tag{2}$$

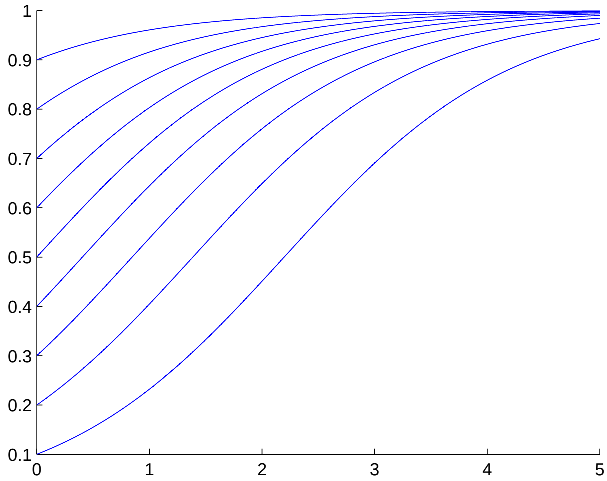


Figure 1: Solutions of (1)

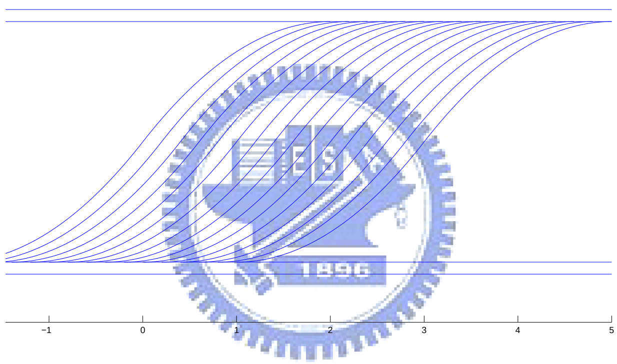


Figure 2: Solutions of (2)

in $\mathbb{R}$. The solutions are $x(t) = a$ for $t \in \mathbb{R}$, where $a \geq 2$ or $a \leq 0$, and

$$x(t) = \begin{cases} 0 & \text{if } t < a \\ \frac{1}{4}(t-a)^2 & \text{if } a \leq t < a+2 \\ -\frac{1}{4}[t-(a+4)]^2 + 2 & \text{if } a+2 \leq t \leq a+4 \\ 2 & \text{if } t > a+4 \end{cases} \quad \text{for } t \in \mathbb{R}, \text{ where } a \in \mathbb{R}.$$

See Figure 2.

Also (2) is an automonous differential equation satisfies (H1) to (H3).

# 三、Semi-invariance and Generalized Positive Limit Sets

Before introducing our study, we introduce the concept of semi-invariance and generalized positive limit sets for autonomous differential equations. This is special case of work of Strauss and Yorke in [1].

First we introduce the concept of semi-invariance for autonomous differential equations.

## 3.1 Semi-invariance and Invariance

**Definition 3** (**Semi-invariance and Invariance**). Let $\Gamma \subseteq Q$.

$\Gamma$ is *semi-invariant* for (A) if for $a \in \Gamma$, there is a solution $x(t;\ 0,\ x_0)$ such that

$$x((\alpha^-,\ \alpha^+);\ 0,\ a) \subseteq \Gamma.$$

If the solutions of (A) are uniquely determined by initial time and initial position, then $\Gamma$ is *invariant* for (A).

**Remark 4.** It's easily seen that the union of a collection of (semi-)invariant sets is also (semi-)invariant.

Second we introduce the concept of generalized positive limit sets for autonomous differential equations.

## 3.2 Generalized Positive Limit Sets

**Definition 5** (**Generalized Positive Limit Sets**). Let $F$ belongs to the family of all solutions of (A).

The *generalized positive limit set* of $F$, denoted by $\Omega_{\mathrm{A}}(F)$, is the set of those points $b \in Q$ which there are sequences $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ with $\lim_{n\to\infty} t_n = +\infty$ and $\{x_n(\cdot)\}_{n\in\mathbb{N}}$ in $F$ such that

$$\lim_{n\to\infty} x_n(t_n) = b.$$

For convenience, denote $\Omega_{\mathrm{A}}(\{x(\cdot)\})$ by $\Omega_{\mathrm{A}}(x)$.

**Remark 6.** If the solutions of (A) are uniquely determined by initial conditions, then $\Omega_{\mathrm{A}}(x) = \Omega_{\mathrm{A}}(x(0))$, where $\Omega_{\mathrm{A}}(x(0))$ is the positive limit set of $x(0)$ for (A). That is why we use the adjective "generalized".

**Remark 7.** By definition 5, it's seen that for each family $F$ of solutions of (A), $\bigcup_{x\in F}\Omega_A(x)\subseteq\Omega_A(F)$. However $\bigcup_{x\in F}\Omega_A(x)\neq\Omega_A(F)$ in general. See example 8 and example 9.

**Example 8.** Consider the logistic equation (1) in $(0,\ 1)$.

By example 1, for each solution $x$ of (1) in $(0,\ 1)$, $\Omega_1(x)=\emptyset$. However if $F$ is the family of all solutions of (1) in $(0,\ 1)$, then $\Omega_1(F)=(0,\ 1)$.

Indeed, for each solution $x$ of (1) in $(0,\ 1)$, there is a $a\in(0,\ 1)$ such that $x(t)=\frac{a}{a+(1-a)e^{-t}}$ for all $t\in\mathbb{R}$. Since $x(t)=\frac{a}{a+(1-a)e^{-t}}$ increases to 1 as $t$ approaches to $+\infty$, and $1\notin(0,\ 1)$, $\Omega_1(x)=\emptyset$.

For $0<b<1$ and $T>0$, $0<\frac{b}{(1-b)e^T+b}<1$ and $x(T;\ 0,\ \frac{b}{(1-b)e^T+b})=b$. Hence $\lim_{n\to\infty}x(n;\ 0,\ \frac{b}{(1-b)e^n+b})=b$. Therefore $\Omega_1(F)=(0,\ 1)$.

**Example 9.** Consider equation (2) in $\mathbb{R}$.

By example 2, for each nonconstant solution $x$ of (2), $\Omega_2(x)=\{2\}$. However if $F$ is the family of all nonconstant solution of (2), then $\Omega_2(F)=[0,\ 2]$.

Indeed, for each nonconstant solution $x$ of (2), there is a $a\in\mathbb{R}$ such that

$$x(t)=\begin{cases}0 & \text{if } t<a\\ \frac{1}{4}(t-a)^2 & \text{if } a\le t<a+2\\ -\frac{1}{4}[t-(a+4)]^2+2 & \text{if } a+2\le t\le a+4\\ 2 & \text{if } t>a+4.\end{cases}$$

Since $x(t)$ increases to 2 as $t$ approaches to $+\infty$, $\Omega_2(x)=\{2\}$.

For $0\le b\le 2$, since all nonconstant solution of (2) are horizental shift of

$$x_0(t)=\begin{cases}0 & \text{if } t<0\\ \frac{1}{4}t^2 & \text{if } 0\le t<2\\ -\frac{1}{4}(t-4)^2+2 & \text{if } 2\le t\le 4\\ 2 & \text{if } t>4,\end{cases}$$

and $x_0(\mathbb{R})=[0,\ 2]$, $\Omega_2(F)=[0,\ 2]$.

Here is a special cace that $\Omega_A(F)=\bigcup_{x\in F}\Omega_A(x)$. It's seen by definition 5.

**Property 10.** *If F is a finite family of solutions of* (A), *then* $\Omega_A(F)=\bigcup_{x\in F}\Omega_A(x)$.

Third we introduce the semi-invariance of generalized positive limit sets for autonomous differential equations.

### 3.3 Semi-invariance of Generalized Positive Limit Sets

In [1], Strauss and Yorke obtained the remarkable property of generalized limit sets for asymptotically autonomous differential equations. Of course, autonomous differential equations are one kind of asymptotically autonomous differential equations. Here we just list the version for autonomous differential equations.

**Theorem 11** (**Special Case of Theorem 2.4 in** [**1**])**.** *For every family $F$ of solutions of* (A)*, the generalized positive limit set* $\Omega_{\mathrm{A}}(F)$ *is semi-invariant for* (A).

# 四、$\varepsilon$-chains and Chain Recurrent Sets

After introducing the work of Strauss and Yorke, we introduce our study.

First we introduce the $\varepsilon$-chains for autonomous differential equations. This is necessary for defining the chain recurrent sets for autonomous differential equations.

**Definition 12** (**$\varepsilon$-chains**). Let $a,\ b \in Q$, $\varepsilon > 0$, $T \geq 1$, and $k \in \mathbb{N}$.

A finite sequence $\{x_0 = a,\ x_1,\ \ldots,\ x_k = b\}$ in $Q$ is called an *$\varepsilon$-chain of length $T$ from $a$ to $b$* for (A) if there is a finite sequence $\{t_0 = 0,\ t_1,\ \ldots,\ t_k\}$ in $\mathbb{R}_+$ such that for all $j = 1,\ \ldots,\ k$, $t_j \geq 1$, there is an $x(\cdot;\ t_0 + \ldots + t_{j-1},\ x_{j-1})$, such that

$$|x(t_0 + \ldots + t_j;\ t_0 + \ldots + t_{j-1},\ x_{j-1}) - x_j| < \varepsilon, \text{ and } t_0 + \ldots + t_k = T.$$

Second we define the chain recurrent sets for autonomous differential equations by the term of $\varepsilon$-chains.

**Definition 13** (**Chain Recurrent Sets**). The *chain recurrent set* of (A), denoted by $\mathscr{R}_{\mathrm{A}}$, is the set of those points $c \in Q$ which for $\varepsilon > 0$ , there are $\varepsilon$-chains from $c$ to itself.

In fact, all points in the chain recurrent set of (A) have additional property as follows.

**Property 14.** *The chain recurrent set of* (A) *is the set of those points $c \in Q$ which for $\varepsilon > 0$ and $T \geq 1$, there are $\varepsilon$-chains of length greater than $T$ from $c$ to itself.*

*Proof.* Let $\mathscr{R}$ be the set of those points $c \in Q$ which for $\varepsilon > 0$ and $T \geq 1$, there are $\varepsilon$-chains of length greater than $T$ from $c$ to itself.

It's obvious that $\mathscr{R} \subseteq \mathscr{R}_{\mathrm{A}}$.

For $c \in \mathscr{R}_{\mathrm{A}}$ and $\varepsilon > 0$, there exist an $\varepsilon$-chain $\{x_0 = c,\ x_1,\ \cdots,\ x_k = c\}$ from $c$ to itself. If this $\varepsilon$-chain is of length $T_0$, then

$$\{x_0 = c,\ x_1,\ \ldots,\ x_k = c,\ x_{k+1} = x_1,\ x_{2k-1} = x_{k-1},\ x_{2k} = x_k\}$$

is an $\varepsilon$-chain of length $2 \cdot T_0$ from $c$ to itself. By similar manner and induction,

$$\{x_0 = c,\ x_1,\ \ldots,\ x_k = c,\ \ldots,\ x_{(l-1)k+1} = x_1,\ x_{lk-1} = x_{k-1},\ x_{lk} = x_k\}$$

is an $\varepsilon$-chain of length $l \cdot T_0$ from $c$ to itself for $l \in \mathbb{N}$. Thus by Archimedean property, $c \in \mathscr{R}$. Hence $\mathscr{R}_{\mathrm{A}} \subseteq \mathscr{R}$.

Therefore $\mathscr{R} = \mathscr{R}_{\mathrm{A}}$ □

**Remark 15.** $\mathscr{R}_A$ may be an empty set. See example 15.

**Example 16.** Consider the logistic equation (1) in $(0, 1)$. Then $\mathscr{R}_1 = \emptyset$.

Indeed, since for $0 < a < 1$, $x(t;\ 0,\ a) = \frac{a}{a+(1-a)e^{-t}}$ for $t \in \mathbb{R}$ is strictly incresing on $\mathbb{R}$, by definition 12 and definition 13, $a \notin \mathscr{R}_1$. Hence $\mathscr{R}_1 = \emptyset$.

**Example 17.** Consider equation (2) in $\mathbb{R}$, then $\mathscr{R}_2 = (-\infty, 0] \cup [2, +\infty)$.

Indeed, for $c \le 0$ or $c \ge 2$, take $x(t;\ 0,\ c) = c$ for $t \in \mathbb{R}$. Then for $\varepsilon > 0$ and $T \ge 1$, $\{c,\ c\}$ with corresponding time sequence $\{0,\ T+1\}$ is an $\varepsilon$-chain of length $T+1$, since $x(T+1;\ 0,\ c) = c$. Hence $(-\infty, 0] \cup [2, +\infty) \subseteq \mathscr{R}_2$.

For $0 < a \le 1$,

$$x(t;\ 0,\ a) = \begin{cases} 0 & \text{if } t < 2a^{\frac{1}{2}} \\ \frac{1}{4}(t - 2a^{\frac{1}{2}})^2 & \text{if } 2a^{\frac{1}{2}} \le t < 2a^{\frac{1}{2}} + 2 \\ -\frac{1}{4}[t - (2a^{\frac{1}{2}} + 4)]^2 + 2 & \text{if } 2a^{\frac{1}{2}} + 2 \le t \le 2a^{\frac{1}{2}} + 4 \\ 2 & \text{if } t > 2a^{\frac{1}{2}} + 4. \end{cases}$$

For $1 < a < 2$,

$$x(t;\ 0,\ a) = \begin{cases} 0 & \text{if } t < 2(2-a)^{\frac{1}{2}} - 4 \\ \frac{1}{4}\{t - [2(2-a)^{\frac{1}{2}} - 4]\}^2 & \text{if } 2(2-a)^{\frac{1}{2}} - 4 \le t < 2(2-a)^{\frac{1}{2}} - 2 \\ -\frac{1}{4}[t - 2(2-a)^{\frac{1}{2}}]^2 + 2 & \text{if } 2(2-a)^{\frac{1}{2}} - 2 \le t \le 2(2-a)^{\frac{1}{2}} \\ 2 & \text{if } t > 2(2-a)^{\frac{1}{2}}. \end{cases}$$

Thus for $a \in (0, 2)$, $x(\,\cdot\,;\ 0,\ a)$ is increasing on $\mathbb{R}$ and strictly increasing on a neighborhood of 0. Hence $(0, 2) \notin \mathscr{R}_2$.

Therefore, $\mathscr{R}_2 = (-\infty, 0] \cup [2, +\infty)$.

# 五、Semi-invariance of Chain Recurrent Sets

The semi-invariance of chain recurrent set is the main result of this thesis. To obtain this result, we declare relations between chain recurrent sets and generalized positive limit sets.

## 5.1 Chain Recurrent Sets and Generalized Positive Limit Sets

First we declare the chain recurrent set of (A) contain all generalized positive limit sets of single solution.

**Theorem 18.** $\bigcup_{x'=f(x)} \Omega_{\mathrm{A}}(x) \subseteq \mathscr{R}_{\mathrm{A}}$.

*Proof.* If $\bigcup_{x'=f(x)} \Omega_{\mathrm{A}}(x) = \emptyset$, it's readily. Hence $\bigcup_{x'=f(x)} \Omega_{\mathrm{A}}(x) \neq \emptyset$ is assumed.

For $b \in \bigcup_{x'=f(x)} \Omega_{\mathrm{A}}(x)$, $b \in \Omega_{\mathrm{A}}(x)$ for some solution $x$ of (A). Hence by definition 5, there exist a sequence $\{t_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}_+$ with $\lim_{n\to\infty} t_n = +\infty$ such that $\lim_{n\to\infty} x(t_n) = b$.

Thus for $\varepsilon > 0$ and $T \geq 1$, by (H1) to (H3), there is a $\delta > 0$ such that if $a \in Q$ and $|a - b| < \delta$, then for each $x(\cdot\,; 0, a)$, $x(\cdot\,; 0, b)$,

$$|x(1;\, 0,\, a) - x(1;\, 0,\, b)| < \varepsilon.$$

Since $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} x(t_n) = b$, there are $K,\ N \in \mathbb{N}$ so that

$$|x(t_K) - b| < \delta,\ |x(t_N) - b| < \varepsilon, \text{ and } t_N - t_K > T.$$

Note that $x(\cdot + t_K)$ is a solution of (A) with initial condition $x(0) = x(t_K)$ and $x(1) = x(1 + t_K)$, and $x(t_N) = x([t_N - (t_K + 1) + (t_K + 1)])$. Hence denote $x(\cdot + t_K)$ by $x(\cdot\,; 0,\, x(t_K))$ and $x(\cdot\,; 1,\, x(t_K + 1))$.

Conclude above arguments, it's seen that

$$|x(1;\, 0, b) - x(t_K + 1)| = |x(1;\, 0, b) - x(1;\, 0,\, x(t_K))| < \varepsilon \quad (|b - x(t_K)| < \delta),$$

and

$$|x(t_N - t_K;\, 1, x(t_K)) - b| = |x(t_N) - b| < \varepsilon.$$

Hence by definition 12, $\{b,\ x(t_K + 1),\ b\}$ is an $\varepsilon$-chain of length greater than $T$ with corresponding time sequence $\{0,\ 1,\ t_N - (t_K + 1)\}$.

With the fact that $a \in \bigcup_{x'=f(x)} \Omega_{\mathrm{A}}(x)$, $\varepsilon > 0$, and $T \geq 1$ are arbitrary given, by property 14, $\bigcup_{x'=f(x)} \Omega_{\mathrm{A}}(x) \subseteq \mathscr{R}_{\mathrm{A}}$. □

Second we declare the chain recurrent set of (A) is contained by some union of generalized positive limit sets of some families of solutions of (A). Under additional assumptions, the chain recurrent set is equal to some union of generalized positive limit sets of some families of solutions of (A).

**Theorem 19.** *If* $\mathscr{R}_{\mathrm{A}} \neq \emptyset$*, then for* $c \in \mathscr{R}_{\mathrm{A}}$*, there is a family* $F_c$ *of solutions of* (A) *such that*

$$\mathscr{R}_{\mathrm{A}} \subseteq \bigcup_{c \in \mathscr{R}_{\mathrm{A}}} \Omega_{\mathrm{A}}(F_c).$$

*Moreover, if for each* $c \in \mathscr{R}_{\mathrm{A}}$*,* $F_c$ *can be chosen that* $\Omega_{\mathrm{A}}(F_c) = \bigcup_{x \in F_c} \Omega_{\mathrm{A}}(x)$*, then*

$$\mathscr{R}_{\mathrm{A}} = \bigcup_{c \in \mathscr{R}_{\mathrm{A}}} \Omega_{\mathrm{A}}(F_c).$$

*Proof.* Fix $c \in \mathscr{R}_{\mathrm{A}}$. By property 14, for $n \in \mathbb{N}$, there is an $\frac{1}{n}$-chain

$$\{x_{n,0} = c,\ x_{n,1},\ \ldots,\ x_{n,k_n} = c\}$$

with corresponding time sequence

$$\{t_{n,0} = 0,\ t_{n,1},\ t_{n,k_n}\}$$

such that the length of the $\frac{1}{n}$-chain is greater than $n$ if $n > 1$.
For $n \in \mathbb{N}$, let

$$x_{c,n}(\cdot) = x(\cdot;\ t_{n,0} + t_{n,1} + \ldots + t_{n,k_n-1},\ x_{n,k_n-1}),$$

where $x(\cdot;\ t_{n,0} + t_{n,1} + \ldots + t_{n,k_n-1},\ x_{n,k_n-1})$ is given by definition 12, and

$$t_{c,n} = t_{n,0} + t_{n,1} + \ldots + t_{n,k_n}.$$

Then by setting above and definition 12, for $n \in \mathbb{N}$ $\{t_{c,n}\}_{n \in \mathbb{N}}$ is in $\mathbb{R}_+$ with $\lim_{n\to\infty} t_{c,n} = +\infty$, and $|x_{c,n}(t_{c,n}) - c| = |x(t_{c,n};\ t_{n,0} + t_{n,1} + \ldots + t_{n,k_n-1},\ x_{n,k_n-1}) - x_{n,k_n}| < \frac{1}{n}$. Hence $\{x_{c,n}(\cdot)\}_{n \in \mathbb{N}}$ is a sequence of solutions of (A) with

$$\lim_{n\to\infty} x_{c,n}(t_{c,n}) = c.$$

Let $F_c = \{x_{c,n}(\cdot)\}_{n \in \mathbb{N}}$. Then by above argument, $\mathscr{R}_{\mathrm{A}} \subseteq \bigcup_{c \in \mathscr{R}_{\mathrm{A}}} \Omega_{\mathrm{A}}(F_c)$.

The second statement is obviously by theorem 17. □

## 5.2 Semi-invariance of Chain Recurrent Sets

After declaring the relations between chain recurrent sets and generalized positive limit sets, the semi-invariance of chain recurrent sets is corollary of theorem 19, theorem 11, and remark 4.

**Theorem 20.** *If all hypotheses in theorem 19 holds, then $\mathscr{R}_{\mathrm{A}}$ is semi-invariant.*

**Example 21.** Consider (2) in $\mathbb{R}$. By example 17, $\mathscr{R}_2 = (-\infty, 0] \cup [2, +\infty)$. Since for $c \in \mathscr{R}_2$, $\{c\} = \Omega_2(x_c)$, where $x_c(t) = c$ for $t \in \mathbb{R}$. Hence by theorem 18, theorem 19, and theorem 20, $\mathscr{R}_2$ is semi-invariant.

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