Chapter 5

Distributional and Inferential Properties of Loss Indices

In this chapter, we consider three loss function indices L_{pe}, L_{ot} and L_e, and investigate the statistical properties of their natural estimators. For L_{pe}, we show that the natural estimator is the UMVUE (uniformly minimum variance unbiased estimator), which is consistent and asymptotically efficient. We also obtain the MLE (maximum likelihood estimator), which has smaller mean square error than the UMVUE, hence it is more reliable, particularly, for short production run applications. For L_{ot}, we show that the natural estimator is the MLE. We also obtain the UMVUE, which is shown to be more reliable (has smaller mean squared error) than the MLE for applications with $n \geq 4$. We show that the UMVUE is consistent and asymptotically efficient. For L_e, we show that the natural estimator is the MLE and also the UMVUE, which is consistent and asymptotically efficient. In addition, we construct tables of 90%, 95%, and 99% upper confidence limits for L_e based on the UMVUE. We also construct tables of the maximum values of $\hat{L}_e$ under $\mu = T$ for which the process is capable 90%, 95%, and 99% of the time. An efficient UMP test based on the UMVUE of L_e is derived. Using the UMP test, a testing procedure is proposed. The estimators we recommend have all the desired statistical properties, and are considered reliable in determining whether a process meets the capability requirement.

5.1 Estimating Process Relative Inconsistency Loss

To estimate the process relative inconsistency loss, we consider the natural estimator $\hat{L}_{pe}$ defined as follows, where $S_{n-1} = [\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)]^{1/2}$ is the conventional estimator of the process standard deviation σ,

$$\hat{L}_{pe} = \frac{1}{n-1}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{d^2} = \frac{S_{n-1}^2}{d^2}. \tag{5.1}$$

The natural estimator $\hat{L}_{pe}$ can be rewritten as

$$\hat{L}_{pe} = \frac{L_{pe}}{n-1}\frac{(n-1)\hat{L}_{pe}}{L_{pe}} = \frac{L_{pe}}{n-1}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{\sigma^2}. \tag{5.2}$$

If the process follows the normal distribution, then $\hat{L}_{pe}$ is distributed as $[L_{pe}/(n-1)]\chi^2_{n-1}$, where χ^2_{n-1} is a chi-squared distribution with $(n-1)$ degrees of freedom. The probability density function of $\hat{L}_{pe}$ can be easily derived as

$$f(x) = \{[(n-1)x/(2L_{pe})]^{(n-1)/2}\exp[-(n-1)x/(2L_{pe})]\}\{x\Gamma[(n-1)/2]\}^{-1}, \text{for } x > 0. \tag{5.3}$$

The rth moment, the expected value, the variance, and the mean squared error of $\hat{L}_{pe}$ can be obtained as follows:

$$\mathrm{E}(\hat{L}_{pe})^r = \frac{\Gamma\{[(n-1)/2]+r\}}{\Gamma[(n-1)/2]}\left(\frac{2L_{pe}}{n-1}\right)^r, \tag{5.4}$$

$$\mathrm{E}(\hat{L}_{pe}) = \left(\frac{L_{pe}}{n-1}\right)\mathrm{E}(\chi^2_{n-1}) = L_{pe}, \tag{5.5}$$

$$\mathrm{Var}(\hat{L}_{pe}) = \left(\frac{L_{pe}}{n-1}\right)^2 \mathrm{Var}(\chi^2_{n-1}) = 2(n-1)\left(\frac{L_{pe}}{n-1}\right)^2 = \frac{2L_{pe}^2}{n-1}, \tag{5.6}$$

$$\mathrm{MSE}(\hat{L}_{pe}) = \mathrm{E}(\hat{L}_{pe}-L_{pe})^2 = \mathrm{Var}(\hat{L}_{pe}) + [\mathrm{E}(\hat{L}_{pe})-L_{pe}]^2 = \frac{2L_{pe}^2}{n-1}. \tag{5.7}$$

If the process characteristic is normally distributed, an $100(1-\alpha)\%$ upper confidence limit on L_{pe} can be established in terms of the estimator $\hat{L}_{pe}$ as $[(n-1)\hat{L}_{pe}/\chi^2_{n-1}(\alpha)]$, where $\chi^2_{n-1}(\alpha)$ is the (lower) αth percentile of the χ^2_{n-1} distribution. A capability testing can then be conducted. In addition, we can show that the natural estimator $\hat{L}_{pe}$ is the UMVUE of L_{pe}, which is consistent. We can also show that $\sqrt{n}(\hat{L}_{pe}-L_{pe})$ converges to $N(0,2L_{pe}^2)$ in distribution, and that $\hat{L}_{pe}$ is asymptotically efficient (see Theorem 5.1 for proofs). Thus, in real-world applications using $\hat{L}_{pe}$, which has all the desired statistical properties, as an estimate of L_{pe} would be reasonable.

Theorem 5.1. If the process characteristic is normally distributed, then:

(a) $\hat{L}_{pe}$ is the UMVUE of L_{pe};

(b) $\hat{L}_{pe}$ is consistent;

(c) $\sqrt{n}(\hat{L}_{pe}-L_{pe})$ converges to $N(0,2L_{pe}^2)$ in distribution;

(d) $\hat{L}_{pe}$ is asymptotically efficient .

Proof. (a) Since S^2_{n-1} is a sufficient and complete statistic for σ^2, and the unbiased estimator $\hat{L}_{pe}$ is a function of S^2_{n-1} only, then by Lehmann-Scheffé Theorem, $\hat{L}_{pe}$ is the UMVUE.

(b) For every $\varepsilon > 0$,

$$\mathrm{P}(|\hat{L}_{pe}-L_{pe}| > \varepsilon) < \mathrm{E}(\hat{L}_{pe}-L_{pe})^2/\varepsilon^2. \tag{5.8}$$

Since

$$\mathrm{E}(\hat{L}_{pe}-L_{pe})^2 = \mathrm{Var}(\hat{L}_{pe}) = 2L_{pe}^2/(n-1) \tag{5.9}$$

converges to zero, then $\hat{L}_{pe}$ must be consistent.

(c) Under general conditions, $\sqrt{n}(\hat{L}_{pe}-L_{pe})$ converges to $N(0,\sigma^2_{pe})$ in distribution, where $\sigma^2_{pe} = (\mu_4-\sigma^4)/d^4$. The result follows directly since for a normal distribution, $\mu_4 = 3\sigma^4$ and $L_{pe} = (\sigma/d)^2$.

(d) Under normality assumption, the information matrix can be calculated as shown below. Since the information lower bound is achieved, then $\hat{L}_{pe}$ must be asymptotically efficient:

$$I(\theta) = I(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix},$$

$$\left[\frac{\partial L_{pe}}{\partial \mu} \frac{\partial L_{pe}}{\partial \sigma^2}\right] \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \dfrac{\partial L_{pe}}{\partial \mu} \\ \dfrac{\partial L_{pe}}{\partial \sigma^2} \end{bmatrix} = \frac{2L_{pe}^2}{n}. \tag{5.10}$$

We note that by multiplying the UMVUE $\hat{L}_{pe}$ by the constant $c_n = (n-1)/n$, we obtain the MLE of L_{pe}. We can show that the MLE $\tilde{L}_{pe}$ is consistent, and is asymptotically unbiased. We can show that $\sqrt{n}(\tilde{L}_{pe} - L_{pe})$ converges to $N(0, 2L_{pe}^2)$ in distribution, and that $\tilde{L}_{pe}$ is asymptotically efficient. Since $c_n < 1$, then $\tilde{L}_{pe} = c_n \hat{L}_{pe}$ underestimates the index L_{pe}, but it has smaller variance. In fact, we may calculate

$$\mathrm{MSE}(\tilde{L}_{pe}) = [(2n-1)/n^2](L_{pe})^2, \tag{5.11}$$

$$\mathrm{MSE}(\hat{L}_{pe}) - \mathrm{MSE}(\tilde{L}_{pe}) = \{(3n-1)/[n^2(n-1)]\}(L_{pe})^2 > 0, \text{ for all } n. \tag{5.12}$$

Therefore, the MLE $\tilde{L}_{pe}$ has smaller mean squared error than the UMVUE $\hat{L}_{pe}$, hence it is more reliable, particular for short production run applications. We consider some commonly used values of L_{pe} = 0.11, 0.06, 0.05, 0.04, and 0.03, equivalent to C_p = 1.00, 1.33, 1.50, 1.67, and 2.00, covering the widespread range of the precision requirements for most applications (see Table 17). Tables 18(a) and 18(b) display the relative error of the UMVUE $\hat{L}_{pe}$, defined as $[\mathrm{MSE_R}(\hat{L}_{pe})]^{½} = \{\mathrm{E}[(\hat{L}_{pe} - L_{pe})/L_{pe}]^2\}^{½}$, for sample size n = 2(1)50, and 60(10)500, 600(100)1000 with those commonly used values of L_{pe} = 0.11, 0.06, 0.05, 0.04, 0.03.

Table 17. Recommended range of L_{pe} for various precision requirements.

Range	Precision Requirement
$0.06 \le L_{pe} \le 0.11$	Capable
$0.05 \le L_{pe} \le 0.06$	Satisfactory
$0.04 \le L_{pe} \le 0.05$	Good
$0.03 \le L_{pe} \le 0.04$	Excellent
$L_{pe} \le 0.03$	Super

The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true L_{pe}. We note that for UMVUE $\hat{L}_{pe}$, $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2} = [2/(n-1)]^{1/2}$, which is a function of the sample size n only. Therefore, $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2}$ values are the same for all L_{pe} values. For example, with $n = 300$ we has $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2} = 0.0818$. Thus, for $n = 300$, we expect that the average error of $\hat{L}_{pe}$ would be no greater than 8.18% of the true L_{pe}. Tables 19(a) and 19(b) display the relative error, $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2}$, of the MLE $\tilde{L}_{pe}$. We note that $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2} = [(2n-1)/n^2]^{1/2}$, which is also a function of the sample size n only. Thus, $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2}$ values are the same for all L_{pe} values.

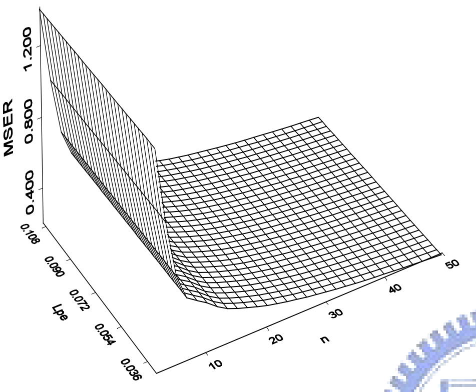

Figure 48. Plot of $[\mathrm{MSE_R}(\hat{L}_{pe})]^{0.5}$ with $L_{pe} =$ 0.11, 0.06, 0.05, 0.04, 0.03, versus sample size $n = 2(1)50$.

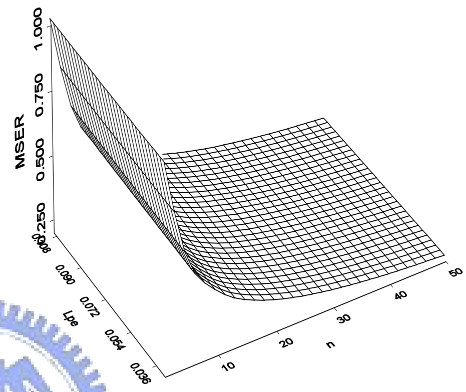

Figure 49. Plot of $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{0.5}$ with $L_{pe} =$ 0.11, 0.06, 0.05, 0.04, 0.03, versus sample size $n = 1(1)50$.

Figure 48 plots $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2}$ with $L_{pe} = 0.11, 0.06, 0.05, 0.04, 0.03$, versus sample size $n = 2(1)50$ and Figure 49 plots $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2}$ with $L_{pe} = 0.11, 0.06, 0.05, 0.04, 0.03$, versus sample size $n = 1(1)50$. The sensitivity of square root of the relative mean squared error for both estimators due to the process relative inconsistency loss L_{pe}, as well as sample size n can then be easily understood. For short run applications (such as accepting a supplier providing short production runs in QS-9000 certification), the difference between the two relative errors is considered significant for sample size $n \le 35$, and we strongly recommend using the MLE $\tilde{L}_{pe}$ rather than the UMVUE $\hat{L}_{pe}$. For other applications with sample size $n > 35$, the difference between the two estimators is negligible (less than 0.52%).

5.2 Estimating Process Relative Off-Target Loss

To estimate the relative off-target loss, we consider the natural estimator $\hat{L}_{ot}$ defined as the following, where $\bar{X} = \sum_{i=1}^{n} X_i / n$ is the conventional estimator of the process mean μ. We note that the estimator $\hat{L}_{ot}$ can also be written as a function of L_{pe}:

$$\hat{L}_{ot} = \frac{(\bar{X}-T)^2}{d^2} = \frac{L_{pe}}{n}\frac{n\hat{L}_{ot}}{L_{pe}} = \frac{L_{pe}}{n}\frac{n(\bar{X}-T)^2}{\sigma^2}. \tag{5.13}$$

If the process characteristic is normally distributed, then the estimator $\hat{L}_{ot}$ is distributed as $[L_{pe}/n]\chi_1^2(\delta)$, where $\chi_1^2(\delta)$ is a non-central chi-squared distribution with one degree of freedom and non-centrality parameter $\delta = n(\mu-T)^2/\sigma^2$. Therefore, the probability density function of $\hat{L}_{ot}$ can be expressed as

$$g(x) = \sum_{k=0}^{\infty}\left\{\frac{[(nx)/(2L_{pe})]^{k+\frac{1}{2}}\exp[-(nx)/(2L_{pe})]}{x\Gamma(k+\frac{1}{2})}\cdot\frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}, \text{ for } x>0. \tag{5.14}$$

The rth moment, the expected value, the variance, and the mean squared error of $\hat{L}_{ot}$, therefore, can be calculated as

$$\mathrm{E}(\hat{L}_{ot})^r = \left(\frac{L_{ot}}{n}\right)^r \mathrm{E}[\chi_1^2(\delta)]^r = \sum_{k=0}^{\infty}\left\{\left(\frac{2L_{ot}}{n}\right)^r\frac{\Gamma(k+\frac{1}{2}+r)}{\Gamma(k+\frac{1}{2})}\cdot\frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}, \tag{5.15}$$

$$\mathrm{E}(\hat{L}_{ot}) = \left(\frac{L_{pe}}{n}\right)\mathrm{E}[\chi_1^2(\delta)] = \left(\frac{L_{pe}}{n}\right)(1+\delta) = \frac{L_{pe}}{n} + L_{ot}, \tag{5.16}$$

$$\mathrm{Var}(\hat{L}_{ot}) = \left(\frac{L_{pe}}{n}\right)^2\mathrm{Var}[\chi_1^2(\delta)] = \left(\frac{L_{pe}}{n}\right)^2(2+4\delta) = \frac{4L_{pe}L_{ot}}{n} + \frac{2L_{pe}^2}{n}, \tag{5.17}$$

$$\mathrm{MSE}(\hat{L}_{ot}) = \mathrm{Var}(\hat{L}_{ot}) + [\mathrm{E}(\hat{L}_{ot}) - L_{ot}]^2 = \frac{4L_{pe}L_{ot}}{n} + \frac{3L_{pe}^2}{n^2}. \tag{5.18}$$

If the process characteristic is normally distributed, an $100(1-\alpha)\%$ upper confidence limit on L_{ot} can be expressed in terms of the estimator $\hat{L}_{ot}$ as $(\delta\hat{L}_{ot})/\chi_1^2(\alpha,\delta)$, where $\chi_1^2(\alpha,\delta)$ is the (lower) αth percentile of the $\chi_1^2(\delta)$ distribution. A capability testing can then be conducted. In practice, we note that parameter δ is unknown and should be estimated by the sample data.

Since $\bar{X}$ is the MLE of μ, then by the invariance property of MLE, the natural estimator $\hat{L}_{ot}$ is the MLE of L_{ot}. Noting that $\mathrm{E}(\hat{L}_{ot}) = L_{ot} + (L_{pe}/n)$, and $\mathrm{E}(\hat{L}_{pe}) = L_{pe}$, the corrected estimator $\tilde{L}_{ot} = \hat{L}_{ot} - (\hat{L}_{pe}/n)$ must be unbiased for L_{ot}. We can show that $\tilde{L}_{ot}$ is the UMVUE of L_{ot}, which is consistent. We can also show that $\sqrt{n}(\tilde{L}_{ot} - L_{ot})$ converges to $N(0, 4L_{pe}L_{ot})$ in distribution, and $\tilde{L}_{ot}$ is asymptotically efficient (see Theorem 5.2 for proofs). Thus, in real-world applications using the UMVUE $\tilde{L}_{ot}$, which has all the desired statistical properties, as an estimate of L_{ot} would be reasonable.

Theorem 5.2 If the process characteristic is normally distributed, then:

(a) $\tilde{L}_{ot}$ is the UMVUE of L_{ot};

(b) $\tilde{L}_{ot}$ is consistent;

(c) $\sqrt{n}(\tilde{L}_{ot} - L_{ot})$ converges to $N(0, 4L_{pe}L_{ot})$ in distribution;

(d) $\tilde{L}_{ot}$ is asymptotically efficient .

Proof. (a) Noting that $\bar{X}$ is a sufficient and complete statistic for μ, and that the unbiased estimator $\tilde{L}_{ot}$ is a function of $\bar{X}$ and S_{n-1}^2 only. By Lehmann-Scheffé Theorem, $\tilde{L}_{ot}$ is the UMVUE of L_{ot}.

(b) For every $\varepsilon > 0$,

$$\mathrm{P}(|\tilde{L}_{ot} - L_{ot}| > \varepsilon) < \mathrm{E}(\tilde{L}_{ot} - L_{ot})^2 / \varepsilon^2 . \tag{5.19}$$

Since

$$\mathrm{E}(\tilde{L}_{ot} - L_{ot})^2 = [4L_{pe}L_{ot} / n] + [2L_{pe}^2 / (n^2 - n)] \tag{5.20}$$

converges to zero, then $\tilde{L}_{ot}$ must be consistent.

(c) Under general conditions, $\sqrt{n}(\hat{L}_{ot} - L_{ot})$ converges to $N(0, \sigma_{ot}^2)$ in distribution, where $\sigma_{ot}^2 = 4(\mu - T)^2\sigma^2 / d^4$. Under normality assumption, $\sqrt{n}(\hat{L}_{ot} - L_{ot})$ converges to $N(0, 4L_{pe}L_{ot})$ in distribution. Since $\sqrt{n}(\tilde{L}_{ot} - \hat{L}_{ot})$ converges to zero in probability, then by Slutsky's Theorem,

$$\sqrt{n}(\tilde{L}_{ot} - L_{ot}) = \sqrt{n}(\tilde{L}_{ot} - \hat{L}_{ot}) + \sqrt{n}(\hat{L}_{ot} - L_{ot}) \tag{5.21}$$

converges to $N(0, 4L_{pe}L_{ot})$ in distribution.

(d) Under normality assumption, the information matrix can be calculated as follows. Since the information lower bound is achieved, then $\tilde{L}_{ot}$ must be asymptotically efficient:

$$I(\theta) = I(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix}, \tag{5.22}$$

$$\begin{bmatrix} \dfrac{\partial L_{ot}}{\partial \mu} & \dfrac{\partial L_{ot}}{\partial \sigma^2} \end{bmatrix} \dfrac{I^{-1}(\theta)}{n} \begin{bmatrix} \dfrac{\partial L_{ot}}{\partial \mu} \\ \dfrac{\partial L_{ot}}{\partial \sigma^2} \end{bmatrix} = \dfrac{4L_{pe}L_{ot}}{n} . \tag{5.23}$$

We note that the MLE $\hat{L}_{ot}$ has smaller variance than the UMVUE $\tilde{L}_{ot}$. However, we can show that $\mathrm{MSE}(\tilde{L}_{ot}) = 4L_{pe}L_{ot} / n + \{2/[n(n-1)]\}(L_{pe})^2$, and so $\mathrm{MSE}(\tilde{L}_{ot}) - \mathrm{MSE}(\hat{L}_{ot}) = \{(3-n)/[n^2(n-1)]\}(L_{pe})^2$, which is greater than 0 for n = 2, equal to 0 for n = 3, and less than 0 for $n \geq 4$. Therefore, the UMVUE $\tilde{L}_{ot}$ has smaller mean squared error than the MLE $\hat{L}_{ot}$, and is more reliable for applications with $n \geq 4$. Tables 20(a) and 20(b) display the relative error,

$[\mathrm{MSE_R}(\tilde{L}_{ot})]^{1/2}$, of the UMVUE $\tilde{L}_{ot}$ for L_{pe} = 0.11, 0.06, 0.05, 0.04, 0.03, and fixed L_{ot} = 0.25. This value of L_{ot} is equivalent to $C_a = 0.50$. The relative errors, $[\mathrm{MSE_R}(\tilde{L}_{ot})]^{1/2}$, for other values of L_{ot} are available from the authors. We note that if the process is perfectly centered, then L_{ot} = 0.00 (equivalently, $C_a = 1.00$). For example, for L_{pe} =0.11, L_{ot} =0.25, and n = 300 we have $[\mathrm{MSE_R}(\tilde{L}_{ot})]^{1/2}$ = 0.0770. Thus, the average error (average relative deviation) of $\tilde{L}_{ot}$ would be no greater than 7.70% of the true L_{ot}.

Tables 21(a) and 21(b) display the relative error, $[\mathrm{MSE_R}(\hat{L}_{ot})]^{1/2}$, of the MLE $\hat{L}_{ot}$ for L_{pe} = 0.11, 0.06, 0.05, 0.04, 0.03, and L_{ot} = 0.25 is fixed (tables of $[\mathrm{MSE_R}(\hat{L}_{ot})]^{1/2}$ for other values of L_{ot} are available from the authors). We note that for $n \le 30$, the difference between the two relative errors (percentage of deviations) is significant, and we recommend using the UMVUE $\tilde{L}_{ot}$ rather than the MLE $\hat{L}_{ot}$. However, for $n > 30$, the difference between the two is negligible (less than 0.04%), and using either of the two estimators is equally reliable.

5.3 Estimating Process Expected Relative Loss

To estimate the process expected relative loss (a combined measure of process relative inconsistency loss and process relative off-target loss), we consider the nature estimator $\hat{L}_e$ defined as the following, where $\bar{X} = \sum_{i=1}^{n} X_i / n$, which can also be written as a function of L_{pe}:

$$\hat{L}_e = \frac{1}{n}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{d^2} + \frac{(\bar{X} - T)^2}{d^2} = \frac{L_{pe}}{n}\frac{n\hat{L}_e}{L_{pe}} = \frac{L_{pe}}{n}\sum_{i=1}^{n}\frac{(X_i - T)^2}{\sigma^2}. \tag{5.24}$$

If the process characteristic is normally distributed, then the estimator $\hat{L}_e$ is distributed as $[L_{pe}/n]\chi_n^2(\delta)$, where $\chi_n^2(\delta)$ is a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nL_{ot}/L_{pe}$. Therefore, the probability density function of $\hat{L}_e$ can be expressed as

$$h(x) = \sum_{k=0}^{\infty}\left\{\frac{[(nx)/(2L_{pe})^{k+(n/2)}]\exp[(-nx)/(2L_{pe})]}{x\Gamma(k+(n/2))}\cdot\frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}, \text{ for } x > 0. \tag{5.25}$$

The rth moment (hence the expected value, the variance, and the mean squared error) of $\hat{L}_e$, therefore, can be calculated as follows:

$$\mathrm{E}(\hat{L}_e)^r = \left(\frac{L_{pe}}{n}\right)^r \mathrm{E}[\chi_n^2(\delta)]^r = \sum_{k=0}^{\infty}\left\{\left(\frac{2L_{pe}}{n}\right)^r\frac{\Gamma(k+\frac{n}{2}+r)}{\Gamma(k+\frac{n}{2})}\cdot\frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}, \tag{5.26}$$

$$\mathrm{E}(\hat{L}_e) = \left(\frac{L_{pe}}{n}\right)\mathrm{E}[\chi_n^2(\delta)] = \left(\frac{L_{pe}}{n}\right)(n+\delta) = L_{pe} + L_{ot} = L_e, \tag{5.27}$$

$$\mathrm{Var}(\hat{L}_e) = \left(\frac{L_{pe}}{n}\right)^2 \mathrm{Var}[\chi_n^2(\delta)] = \left(\frac{L_{pe}}{n}\right)^2 (2n+4\delta) = \frac{2L_{pe}}{n}(L_{ot}+L_e), \tag{5.28}$$

$$\mathrm{MSE}(\hat{L}_e) = \mathrm{Var}(\hat{L}_e) + [\mathrm{E}(\hat{L}_e) - L_e]^2 = \frac{2L_{pe}}{n}(L_{ot}+L_e). \tag{5.29}$$

If the process characteristic follows the normal distribution, then we can show that $\hat{L}_e$ is the MLE, which is also the UMVUE of L_e. We can also show that $\hat{L}_e$ is consistent, $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0, 2L_{pe}L_{ot} + 2L_{pe}L_e)$ in distribution, and $\hat{L}_e$ is asymptotically efficient (see Theorem 5.3 for proofs). Since the estimator has all the desired statistical properties, in practice using $\hat{L}_e$ to estimate process expected relative loss would be reasonable.

Theorem 5.3. If the process characteristic is normally distributed, then:

(a) $\hat{L}_e$ is the MLE of L_e;

(b) $\hat{L}_e$ is the UMVUE of L_e;

(c) $\hat{L}_e$ is consistent;

(d) $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0, 2L_{pe}L_{ot} + 2L_{pe}L_e)$ in distribution;

(e) $\hat{L}_e$ is asymptotically efficient .

Proof. (a) Since $(\bar{X}, S_n^2)$ is the MLE of (μ, σ^2), where $S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$, and $\hat{L}_e = (S_n^2/d^2) + [(\bar{X}-T)^2/d^2]$, then by the invariance property of MLE, $\hat{L}_e$ is the MLE of L_e.

(b) We note that $(\bar{X}, S_n^2)$ is sufficient and complete for (μ, σ^2). Since the unbiased estimator $\hat{L}_e$ is a function of $(\bar{X}, S_n^2)$ only, then by the Lehmann -Scheffé Theorem, $\hat{L}_e$ is the UMVUE.

(c) For every $\varepsilon > 0$,

$$\mathrm{P}(|\hat{L}_e - L_e| > \varepsilon) < \mathrm{E}(\hat{L}_e - L_e)^2/\varepsilon^2. \tag{5.30}$$

Since

$$\mathrm{E}(\hat{L}_e - L_e)^2 = \mathrm{Var}(\hat{L}_e) = 2L_{pe}(L_{ot}+L_e)/n \tag{5.31}$$

converges to zero, then $\hat{L}_e$ must be consistent.

(d) Under general conditions, $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0, \sigma_e^2)$ in distribution, where $\sigma_e^2 = [4(\mu-T)^2\sigma^2/d^4] + [4\mu_3(\mu-T)/d^4] + [(\mu_4 - \sigma^4)/d^4]$. Therefore, $\sqrt{n}$ $(\hat{L}_e - L_e)$ converges to $N(0, 2L_{pe}L_{ot} + 2L_{pe}L_e)$ in distribution if the process is normal.

(e) Under normality assumption, the information matrix can be calculated as shown below. Since the information lower bound is achieved, then the estimator $\hat{L}_e$ must be asymptotically efficient:

$$I(\theta) = I(\mu,\sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix}, \tag{5.32}$$

$$\begin{bmatrix} \frac{\partial L_e}{\partial \mu} & \frac{\partial L_e}{\partial \sigma^2} \end{bmatrix} \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \frac{\partial L_e}{\partial \mu} \\ \frac{\partial L_e}{\partial \sigma^2} \end{bmatrix} = \frac{2L_{pe}}{n}(L_{ot} + L_e). \tag{5.33}$$

5.4 Testing Process Capability Based on Process Loss

Under normality assumption, $n\hat{L}_e/(L_e - L_{ot})$ is distributed as $\chi_n^2(\delta)$, a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nL_{ot}/L_{pe}$. Let $U = U(X_1, X_2, \ldots, X_n)$ be a statistic calculated from the sample data satisfying $\mathrm{P}(L_e \le U) = 1 - \alpha$, where the confidence level $1-\alpha$ does not depend on L_e. Then, U is an 100 $(1-\alpha)\%$ upper confidence limit for L_e. We note that

$$\begin{aligned} \mathrm{P}(L_e \le U) &= \mathrm{P}(L_e - L_{ot} \le U - L_{ot}) \\ &= \mathrm{P}(n\hat{L}_e/(L_e - L_{ot}) \ge n\hat{L}_e/(U - L_{ot})) \\ &= \mathrm{P}(\chi_n^2(\delta) \ge n\hat{L}_e/(U - L_{ot})) = 1 - \alpha. \end{aligned} \tag{5.34}$$

Thus, $n\hat{L}_e/(U - L_{ot})) = \chi_n^2(\alpha,\delta)$, where $\chi_n^2(\alpha,\delta)$ is the (lower) αth percentile of the $\chi_n^2(\delta)$ distribution. An $100(1-\alpha)\%$ upper confidence limit on L_e can be expressed, in terms of $\hat{L}_e$, as $U = L_{ot} + [n\hat{L}_e/\chi_n^2(\alpha,\delta)]$. On the other hand, to test $H_0: L_e \ge C\,(incapable)$ versus $H_1: L_e < C\,(capable)$, we claim that the process is capable for at least $100(1-\alpha)\%$ of the time if $\hat{L}_e \le c_0$. We can show that the critical value $c_0 = [\chi_n^2(\alpha,\delta)\cdot C]/(n+\delta)$, where C is the capability requirement preset. Then, $c_0 = [\chi_n^2(\alpha,\delta)\cdot C]/(n+\delta)$ is the maximum value of the estimated expected relative loss $\hat{L}_e$ in order that the process is considered capable at least $100(1-\alpha)\%$ of the time.

By letting $\xi = (\mu - T)/\sigma$, we have $\delta = n(\mu-T)^2/\sigma^2 = n\xi^2$. The formula for calculating critical value c_0 can be written as $c_0 = [\chi_n^2(\alpha, n\xi^2)\cdot C]$ $/[n(1+\xi^2)]$, which is easy to understand and straightforward to apply. But, since the process measurement μ and σ must be estimated from the sampled data to obtain the characteristic parameter ξ, a great degree of uncertainty may be introduced to capability assessments due to sampling errors. Johnson (1992) suggested to estimate μ and σ by $\bar{X}$ and S_n, respectively, to obtain upper confidence limit $[(n+n\xi^2)/\chi_n^2(\alpha,n\xi^2)]\hat{L}_e$ (which is equivalent to our expression $U = L_{ot} +$ $[n\hat{L}_e/\chi_n^2(\alpha,\delta)]$) for L_e. Such approach introduces additional sampling errors from estimating ξ, and would be less reliable. Consequently, any decisions made would provide less quality assurance to the customers.

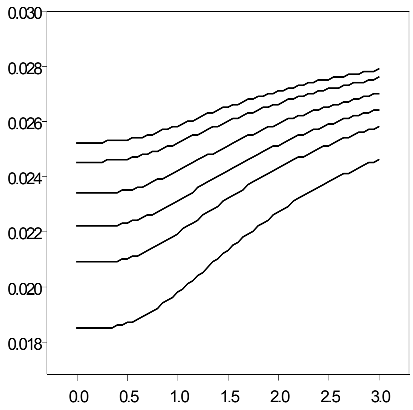

Figure 50. Plots of c_0 vs $|\xi|$ for $L_e = 0.03$, n = 30, 50, 70, 100, 150, 200 (bottom to top).

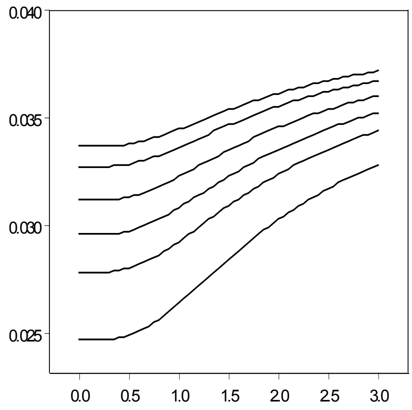

Figure 51. Plots of c_0 vs $|\xi|$ for $L_e = 0.04$, n = 30, 50, 70, 100, 150, 200 (bottom to top).

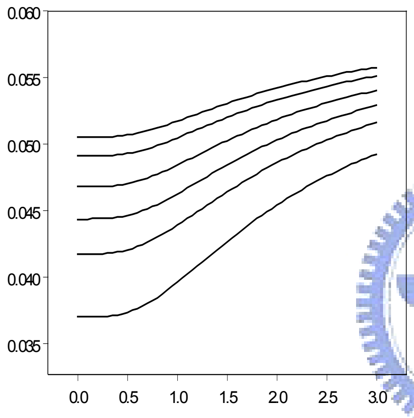

Figure 52. Plots of c_0 vs $|\xi|$ for $L_e = 0.06$, n = 30, 50, 70, 100, 150, 200 (bottom to top).

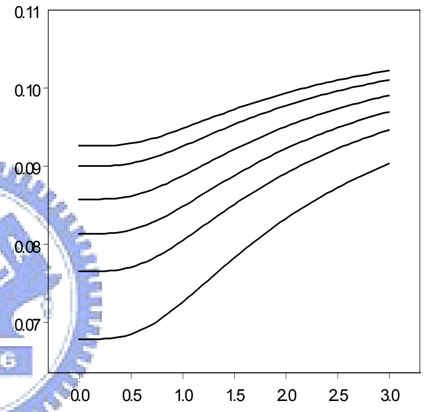

Figure 53. Plots of c_0 vs $|\xi|$ for $L_e = 0.11$, n = 30, 50, 70, 100, 150, 200 (bottom to top).

To eliminate the need for further estimating the characteristic parameter $\xi = (\mu - T)/\sigma$, we examine the sensitivity of the critical value c_0 against the parameter ξ. The results indicate that the critical value c_0 is increasing in ξ and reaches its minimum at $\xi = 0$ (hence $\mu = T$) in all cases. Figures 50 - 53 plot the curves of the critical value c_0 versus the parameter $\xi = 0(0.05)3.00$, n = 30, 50, 70, 100, 150, 200 with confidence level $\gamma = 0.95$, for $L_e = 0.03$, 0.04, 0.06 and 0.11, respectively. Hence, for practical purpose we may calculate the critical value c_0 by setting $\hat{\xi} = \xi = 0$ for given L_e, n, and γ, without having to further estimate the parameter ξ. Thus, based on such approach, the γ confidence level can be ensured and the decisions made are indeed more reliable.

Uniformly Most Powerful Test

For testing hypothesis about L_e, $H_0: L_e \geq C\,(incapable)$ versus $H_1: L_e < C\,(capable)$, we define a test as $\phi^*(x) = 1$ (reject H_0) if $\hat{L}_e < c_0$, and $\phi^*(x) = 0$ otherwise, is the uniformly most powerful (UMP) test of level α under $\xi = 0$ (hence $\mu = T$), where c_0 is determined by $\mathrm{E}_C[\phi^*(X)] = \alpha$. The proof is shown as follows. For the test, the power function is

$$\beta(L_e,\phi^*) = \mathrm{E}_{L_e}[\phi^*(X)] = \mathrm{P}_{L_e}[\chi_n^2 < (nc_0)/L_e]. \tag{5.35}$$

For $\alpha(c_0) = \alpha$, $c_0 = [\chi_n^2(\alpha)\cdot C]/n$, where $\chi_n^2(\alpha)$ is the (lower) αth percentile of the χ_n^2 distribution. From the probability density function of $\hat{L}_e$, we define $\lambda(x)$ as:

$$\lambda(x) = f_{\hat{L}_e}(x, L_e')/f_{\hat{L}_e}(x, L_e) = (L_e/L_e')^{n/2}\exp\left[\frac{n}{2}\left(\frac{1}{L_e} - \frac{1}{L_e'}\right)\cdot x\right]. \tag{5.36}$$

Since for $L_e' > L_e > 0$, the ratio $\lambda(x)$ is an increasing function of x, then $\{f_{\hat{L}_e}(x, L_e) \mid L_e > 0\}$ has monotone likelihood ratio (MLR) property in L_e. Therefore, the test ϕ^* must be the UMP test.

Making Decisions

Tables 22(a), 23(a), and 24(a) give 90%, 95%, and 99% upper confidence limits for L_e under $\mu = T$ with n given, and $\hat{L}_e$ calculated from the sample data. On the other hand, we note that $\hat{L}_e = \chi_n^2(\alpha,\delta)(U - L_{ot})/n$ depends on U and L_{ot}. In the special case where $\mu = T$ and U equals the recommended maximum value for L_e, the probability that $L_e \le U$ would be either 1 or 0 if L_e were known. In practice, since L_e is unknown, we take a random sample of size n and calculate $\hat{L}_e$. Tables 22(b), 23(b), and 24(b) give critical values of $\hat{L}_e$ in the case $\mu = T$, for the process to be considered capable (i.e., $L_e \le C$) 90%, 95%, and 99% of the time. The following example illustrates the use of these tables. To determine whether the process meets the capability requirement, we first determine C, and the α-risk. Then, we calculate the estimator $\hat{L}_e$ from the sample. From the appropriate table, we find the critical value c_0 based on the α-risk, capability requirement C, and sample size n. If the estimated value $\hat{L}_e$ is less than the critical value c_0, then we conclude that the process meets the preset capability requirement.

An Example of Testing L_e

A practice that is becoming increasingly common in industry is to require a supplier to demonstrate process capability as part of the contractual agreement. Suppose a customer has told his supplier that, in order to quality for business with his company, the supplier must demonstrate that his process capability L_e is less than 0.06. This problem may be formulated as a hypothesis-testing problem:

$$H_0:\ L_e \ge 0.06\ (incapable),$$

$$H_1:\ L_e < 0.06\ (capable).$$

In statistical hypothesis testing rejection of H_0 is always a strong conclusion. The supplier would like to reject H_0, thereby demonstrating that his process is capable. Moreover, he wants to be sure that if the process capability is below 0.06 there will be a high probability of judging the process capable (say, 0.95). One takes a random sample of size n, and calculates the value of $\hat{L}_e$. Using Table 23(b) based on the random sample of size $n = 50$, for example, we obtain $c_0 = 0.0435$. Thus, if the calculated $\hat{L}_e \leq 0.0435$, then we claim that the process is capable at least 95% of the time, or equivalently, at the significant level $\alpha = 0.05$.