Chapter 6

Measuring Process Loss for Asymmetric Tolerances

Most research in quality assurance literature has focus on cases in which the manufacturing tolerance is symmetric. A process is said to have a symmetric tolerance if the target value T is set to be the midpoint of the specification interval [LSL, USL], i.e. $T = M = (USL + LSL)/2$. Investigations on symmetric case can be found in Kane (1986), Chan *et al.* (1988), Choi and Owen (1990), Boyles (1991), Pearn *et al.* (1992), Vännman (1995), Vännman and Kotz (1995), and Spiring (1997). Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances also may occur in the manufacturing industry.

From the customer's point of view, asymmetric tolerances reflect that deviations from the target are less tolerable in one direction than in the other (see Boyles (1994)), Vännman (1997), and Wu and Tang (1998)). Usually they are not related to the shape of the supplier's process distribution. However, asymmetric tolerances can also arise in situations where the tolerances are symmetric to begin with, but the process distribution is skewed or follows a non-normal distribution. Dealing with this, the data have been transformed to achieve approximate normality, as shown by Chou *et al.* (1998) who have used Johnson curves to transform non-normal process data. Excluding Boyles (1994), Vännman (1997), Pearn *et al.* (1998, 1999), and Chen *et al.* (1999), unfortunately, there has been comparatively little research published on cases with asymmetric tolerances.

Under asymmetric tolerances situation, using L_e would be risky and probably the results obtained are misleading. Consider the following example with asymmetric tolerance (LSL, T, USL), where $T = (3USL + LSL)/4$ and $\sigma = d/3$. For processes A and B with $\mu_A = T - d/2 = M$ (the midpoint of the specification interval) and $\mu_B = T + d/2 = USL$. Both processes have the index value L_e = $13/36$ and equal degree of clustering around the target, that is, $|\mu - T| = d/2$ for both processes A and B. However, the expected proportions non-conforming are approximately 0.27% for process A and 50% for process B. Obviously, L_e inconsistently measures process capability in this case, and is inappropriate for those with asymmetric tolerances. This problem calls for a need to generalize the index L_e to cover situations with asymmetric tolerances so that appropriate use of the process loss index can be continued.

6.1 A New Generalization L_e''

In this section, we consider a new generalization of L_e to handle processes with asymmetric tolerances. We refer to this generalization as L_e'', which may be defined as follows:

$$L_e'' = \left(\frac{A}{d^*}\right)^2 + \left(\frac{\sigma}{d^*}\right)^2 \tag{6.1}$$

where $A = \max\{(\mu - T)\cdot d / D_u, (T - \mu)\cdot d / D_l\}$, $D_u = USL - T$, $D_l = T - LSL$, $d^* = \min\{D_u, D_l\}$. We denoted $(A/d^*)^2$ by L''_{ot}, $(\sigma/d^*)^2$ by L''_{pe} and hence $L''_e = L''_{ot} + L''_{pe}$. Obviously, if the tolerances are symmetric, then $A = |\mu - T|$, $D_u = D_l = d$, and $d^* = d = (USL - LSL)/2$. Accordingly, the new generalization defined in Equation (6.1) reduces to the original index L_e as in Equation (1.7).

In developing the new generalization, we have replaced the term $|\mu - T|$ in L_e by A. This ensures that the new index obtains the minimal value at $\mu = T$ regardless of whether the tolerances are symmetric or asymmetric. By substituting the half specification width d by d^*, L''_e is sensitive to target value T and obtains larger value when T is away from M. For processes with asymmetric tolerances, the corresponding loss function is also asymmetric in T. We take into account the asymmetry of the loss function by adding the factors (d/D_u) and $(-d/D_l)$ to $(\mu - T)$ according to whether μ is greater or less than T. The factors (d/D_u) and $(-d/D_l)$ ensures that if processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfy $(\mu_A - T)/D_u = (T - \mu_B)/D_l$, then the index values given to A and B are the same. Also, it is easy to verify that if the process is on target, then $L''_e = L''_{pe} = (\sigma/d^*)^2$ is the minimum value.

6.2 Comparisons of L''_e and L_e

To examine some basic difference between L''_e and L_e, in the following, the generalization L''_e is compared with the original index L_e. We consider the following example with manufacturing specifications $LSL = T - 1.50d$, $USL = T + 0.50d$. Table 25 displays the values of L_e, L_{ot}, L_{pe}, L''_e, L''_{ot}, and L''_{pe} for various values of μ, with fixed $\sigma = d/4$. And these index values of L''_e, L''_{ot}, L_e, L_{ot}, L''_{pe} and L_{pe} versus μ are plotted in Figure 54 (from bottom to top in plot). We note that L_e and L_{ot} have the minimum value at the target. But their values at the upper specification limit (say, when the expected proportion nonconforming is 50%) are equal to those at the midpoint M. See Table 25, the values of L_e and L_{ot} are 0.313 and 0.250, respectively, either for $\mu = USL = T + 0.5d$ or $\mu = M = T - 0.5d$. These indices, being symmetric abut the target value, do not take into account the location of the process mean.

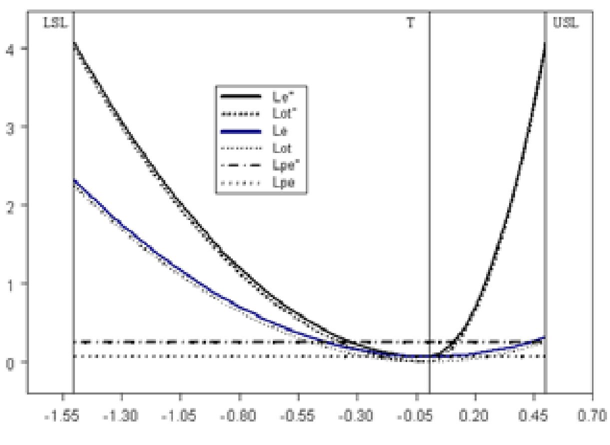

Figure 54. Plots of L''_e, L''_{ot}, L_e, L_{ot}, L''_{pe} and L_{pe} versus μ (top to bottom in plot).

On the other hand, the new index L_e'' we propose takes into account the location of the process mean for asymmetric tolerances. Thus, given two processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfying $(\mu_A - T) = (T - \mu_B)$ and $D_l > D_u$, B has significantly higher yield that A, so the index value of the new generalization L_e'' of A is greater than the index value of B. We note that L_e'' is of the smaller-the-better type as one may expect, since process loss is smaller the better. An illustrative example is L_e'' = 4.063 for $\mu_A = T + 0.5d$ and L_e'' = 0.507 for $\mu_B = T - 0.5d$ in Table 25. These two process means have equal departure from the target, but B has significantly higher yield than A, so intuitively A should score higher than B. Therefore, we conclude that the proposed new generalization L_e'' is superior to the original index L_e.

6.3 Estimation of the Process Loss Indices for Asymmetric Tolerances

We consider the case when the characteristic of the underlying process is normally distributed. Let $X_1, X_2, \ldots, X_n$ be a random sample drawing from a normal distribution with mean μ and variance σ^2 measuring the characteristic under investigation.

6.3.1 Estimation of L_e''

To estimate the new generalization of loss index L_e'', we consider the natural estimator which can be defined as follows:

$$\hat{L}_e'' = \left(\frac{\hat{A}}{d^*}\right)^2 + \left(\frac{S_n}{d^*}\right)^2, \tag{6.2}$$

where $\hat{A} = \max\{(\bar{X} - T)\cdot d / D_u, (T - \bar{X})\cdot d / D_l\}$, the mean μ is estimated by the sample mean, $\bar{X} = \sum_{i=1}^{n} X_i / n$, and the variance σ^2 by $S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 / n$, the maximum likelihood estimator. For the case where the production tolerance is symmetric, $\hat{A}$ may be simplified as $|\bar{X} - T|$. Therefore, the estimator $\hat{L}_e''$ reduces to $\hat{L}_e = (n^{-1}d^{-2})\sum_{i=1}^{n}(X_i - T)^2$, the natural estimator of L_e discuss by Johnson (1992). Consequently, we may view the estimator $\hat{L}_e''$ as a direct extension of $\hat{L}_e$. Now we focus on some statistical properties of this natural estimator $\hat{L}_e''$.

Theorem 6.1. Let $X_1, X_2, \ldots, X_n$ be a random sample form $N(\mu, \sigma^2)$, $Y = \max^2\{d_u Z, -d_l Z\}$, where $Z = \sqrt{n}(\bar{X} - T)/\sigma$ is distributed as $N(\delta, 1)$ and $\delta = \sqrt{n}\ (\mu - T)/\sigma$. Then Y has a weighted non-central chi-square distribution with one degree of freedom (*d.f.*) and non-centrality parameter δ. The probability density function of Y is:

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j)\Gamma\left(\frac{1+j}{2}\right)\left[(d_u^{-2}) f_{Y_j}(y_u) + (-1)^j (d_l^{-2}) f_{Y_j}(y_l)\right], \tag{6.3}$$

where $P_j = (\sqrt{2}\delta)^j / (j!)$, $d_u = d / D_u, d_l = d / D_l$, $y_u = (y / d_u^2)$, $y_l = (y / d_l^2)$, $\lambda =$

δ^2, $\delta = \sqrt{n}(\mu - T)/\sigma$, and Y_j is distributed as χ^2_{1+j}. For the case when $d_u = d_l = 1$, this formula reduces to the probability density function of a non-central chi-square distribution with one *d.f.* and non-centrality parameter δ.

Proof. Based on the notation of Theorem 6.1, the cumulative distribution function of Y is:

$$F_Y(y) = \int_{-\sqrt{y}/d_l}^{\sqrt{y}/d_u} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(z-\delta)^2}{2}\right] dz \,. \tag{6.4}$$

Then

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \left[\frac{(d_u^{-2})}{2\sqrt{y_u}} \cdot e^{-y_u/2} \cdot e^{\delta\sqrt{y_u}} + \frac{(d_l^{-2})}{2\sqrt{y_l}} \cdot e^{-y_l/2} \cdot e^{\delta\sqrt{y_l}} \right]. \tag{6.5}$$

Expanding e^y in power series, we obtain

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \Gamma\left(\frac{1+j}{2}\right) \left[(d_u^{-2}) f_{Y_j}(y_u) + (-1)^j (d_l^{-2}) f_{Y_j}(y_l) \right]. \tag{6.6}$$

Theorem 6.2. The rth moment about zero of $\hat{L}_e''$ is:

$$E(\hat{L}_e'')^r = \left(\frac{\sigma^2}{nd^{*2}}\right)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \cdot 2^r \cdot \Gamma\left(\frac{n+j}{2} + r\right) \cdot$$

$$\left\{ \sum_{i=0}^{r} \binom{r}{i} \frac{\Gamma([(1+j)/2]+i)}{\Gamma([(n+j)/2]+i)} \cdot \left[(d_u^2 - 1)^i + (-1)^j (d_l^2 - 1)^i \right] \right\}, \tag{6.7}$$

Proof. For the sake of deriving the rth moment of $\hat{L}_e''$, the following notation is introduced:

1. $B = \sigma^2/(nd^{*2})$,
2. $K = (nS_n^2)/(\sigma^2)$,
3. $Y = \max^2\{d_u Z, -d_l Z\}$.

Assume that the process is normally distributed with mean μ and variance σ^2, then K is distributed as χ^2_{n-1}, Y is distributed as a weighted non-central chi-square distribution with one *d.f.* and non-centrality parameter δ (see Theorem 6.1). In the notation the estimator $\hat{L}_e''$ can be represented as $\hat{L}_e'' = B(Y+K)$. Thus, the rth moment of $\hat{L}_e''$ is $E(\hat{L}_e'')^r = (B^r)E(Y+K)^r$. Since Y is distributed as a weighted non-central shi-square distribution with one *d.f.* and non-centrality parameter δ, we have

$$E(\hat{L}_e'')^r = (B^r) \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \Gamma\left(\frac{1+j}{2}\right) \{ E[K + (d_u^2) Y_j]^r + (-1)^j E[K + (d_l^2) Y_j]^r \}, \tag{6.8}$$

where Y_j is distributed as χ^2_{1+j}. Let $H_j = Y_j/(K+Y_j)$ and $W_j = K+Y_j$. Under the assumption of normality, H_j and W_j are independent random variables (see, for instance, Johnson and Kotz (1970)), and H_j is distributed according to $\beta((1+j)/2, (n-1)/2)$. Furthermore, W_j has a chi-square distribution with $(n + j)$ degrees of freedom. Therefore

$$E(K+vY_j)^r = E(W_j)^r E(1+(v-1)H_j)^r , \tag{6.9}$$

$$E(W_j)^r = \frac{2^r\Gamma((n+j)/2+r)}{\Gamma((n+j)/2)}, \tag{6.10}$$

and

$$E(1+(v-1)H_j)^r = \sum_{i=0}^{r}\binom{r}{i}(v-1)^i\,\frac{\Gamma([(1+j)/2]+i)\Gamma((n+1)/2)}{\Gamma([(n+j)/2]+i)\Gamma((1+j)/2)}. \tag{6.11}$$

Combining the results, we can obtain the rth moment of $\hat{L}_e''$ as stated in Theorem 6.2.

From the derivation given in Theorem 6.1 and Theorem 6.2, we have the rth moment of $\hat{L}_e''$ as:

$$E(\hat{L}_e'')^r = \left(\frac{\sigma^2}{nd^{*2}}\right)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\cdot 2^r\cdot\Gamma\left(\frac{n+j}{2}+r\right)\cdot$$
$$\left\{\sum_{i=0}^{r}\binom{r}{i}\frac{\Gamma([(1+j)/2]+i)}{\Gamma([(n+j)/2]+i)}\cdot\left[(d_u^2-1)^i+(-1)^j(d_l^2-1)^i\right]\right\}, \tag{6.12}$$

where $P_j = (\sqrt{2}\delta)^j/(j!)$, $d_u = d/D_u, d_l = d/D_l$, $y_u = (y/d_u^2)$, $y_l = (y/d_l^2)$, $\lambda = \delta^2$, $\delta = \sqrt{n}(\mu-T)/\sigma$. In particular, the expected value and the variance of $\hat{L}_e''$ can be obtained as follows:

$$E(\hat{L}_e'') = \left(\frac{(n-1)\sigma^2}{nd^{*2}}\right)+\left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\cdot\Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot\left[d_u^2+(-1)^j d_l^2\right], \tag{6.13}$$

$$Var(\hat{L}_e'') = \left(\frac{\sigma^4}{n^2d^{*4}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\cdot\Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot(3+j)\cdot\left[d_u^4+(-1)^j d_l^4\right]$$
$$-\left\{\left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\cdot\Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot\left[d_u^2+(-1)^j d_l^2\right]\right\}^2+\left(\frac{2(n-1)\sigma^4}{n^2d^{*4}}\right). \tag{6.14}$$

We note that the estimator $\hat{L}_e''$ is biased. The bias of $\hat{L}_e''$ may be computed as $Bias(\hat{L}_e'') = E(\hat{L}_e'') - L_e''$, and the mean squared error, which is more relevant to the analysis of process quality, is $MSE(\hat{L}_e'') = Var(\hat{L}_e'') + [Bias(\hat{L}_e'')]^2$. To explore the behavior of the estimator $\hat{L}_e''$, the bias and the mean squared error were calculated using computer software for various values of $a = (\mu-T)/\sigma, b = \sigma/d^*$,

d_u, d_l, and sample size n. For example, Table 26 displays the bias and the MSE of $\hat{L}_e''$ for a = –1.0(0.5)1.0, b = 1, d_u = 5/4, d_l = 5/6, and n = 10(10)100.

The results in Table 26 indicate that as $|a|$ increases, the bias and the mean squared error also increase. Further, as the sample size increases, the bias and the mean squared error decrease. The bias of $\hat{L}_e''$ versus n are plotted in Figure 55 with a = –1.0, 0, 1.0 (from bottom to top in the plot). And Figure 56 plot the MSE of $\hat{L}_e''$ versus n with a = 0, –1.0, 1.0 (from bottom to top in the plot).

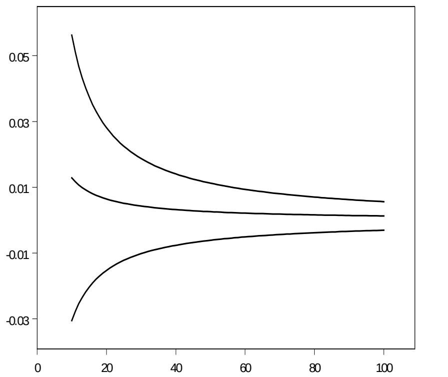

Figure 55. Plots of bias of $\hat{L}_e''$ versus n with a = –1.0, 0, 1.0 (bottom to top in plot).

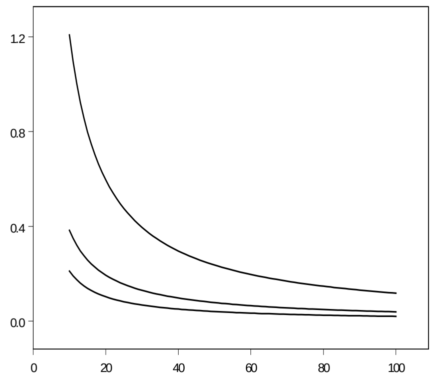

Figure 56. Plots of MSE of $\hat{L}_e''$ versus n with a = 0, –1.0, 1.0 (bottom to top in plot).

Table 27 display the relative error and relative bias of $\hat{L}_e''$, defined as $[MSE_R(\hat{L}_e'')]^{½} = \{E[(\hat{L}_e'' - L_e'')/L_e'']^2\}^{½}$, $Bias_R(\hat{L}_e'') = [E(\hat{L}_e'') - L_e'']/L_e''$, respectively, for a = –1.0(0.5)1.0, b = 1, d_u = 5/4, d_l = 5/6, and n = 10(10)100. The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true L_e''. For example, with n = 100, a = 0.5 we has $[MSE_R(\hat{L}_e'')]^{½} = 0.1521$. Thus, for n = 100, a = 0.5 we expect that the average error of $\hat{L}_e''$ would be no greater than 15.21% of the true L_e''. On the other hand, the relative bias $Bias_R(\hat{L}_e'')$ is investigated to analyze the accuracy of the natural estimator $\hat{L}_e''$. For example, with n = 100, a = 0.5 we has $Bias_R(\hat{L}_e'')$ = 0.0040, that is, 0.4% relative bias for the true L_e''.

From the case where the production tolerance is symmetric, since $d_u = d_l = 1$, $\hat{L}_e''$ is an unbiased estimator of L_e'', or equivalently, $Bias(\hat{L}_e'') = 0$. The unbiased estimator depends only on the complete, sufficient statistic $(\bar{X}, S_n^2)$ for (μ, σ^2), by the Lehmann-Scheffé Theorem we know that $\hat{L}_e''$ is an uniformly minimum variance unbiased estimator (UMVUE) of L_e''. In addition, we have the rth moment of $\hat{L}_e''$ for symmetric tolerance as

$$E(\hat{L}_e'')^r = E(\hat{L}_e)^r = \left(\frac{\sigma^2}{nd^2}\right)^r \sum_{j=0}^{\infty}\left(\frac{e^{-\lambda/2}(\lambda/2)^j}{j!}\right)\cdot\left(\frac{2^r\,\Gamma((n/2)+j+r)}{\Gamma((n/2)+j)}\right). \qquad (6.15)$$

6.3.2 *Estimation of* L''_{ot}

To estimate the new off-target loss index $L''_{ot} = (A)^2/(d^*)^2$, we consider the natural estimator $\hat{L}''_{ot} = (\hat{A})^2/(d^*)^2$. The rth moment about zero for $\hat{L}''_{ot}$ is:

$$E(\hat{L}''_{ot})^r = \left(\frac{\sigma^2}{nd^{*2}}\right)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}(P_j)\cdot 2^r \cdot \Gamma\left(\frac{1+j}{2}+r\right)\cdot\left[d_u^{2r}+(-1)^j d_l^{2r}\right]. \tag{6.16}$$

In particular, the expected value and the variance of $\hat{L}''_{ot}$ can be obtained as follows:

$$E(\hat{L}''_{ot}) = \left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}(P_j)\cdot \Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot\left[d_u^{2}+(-1)^j d_l^{2}\right], \tag{6.17}$$

$$\begin{aligned} Var(\hat{L}''_{ot}) = {} & \left(\frac{\sigma^4}{n^2 d^{*4}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}(P_j)\cdot \Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot(3+j)\cdot\left[d_u^{4}+(-1)^j d_l^{4}\right] \\ & -\left\{\left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}(P_j)\cdot \Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot\left[d_u^{2}+(-1)^j d_l^{2}\right]\right\}^2. \end{aligned} \tag{6.18}$$

We note that the estimator $\hat{L}''_{ot}$ is biased. The bias of $\hat{L}''_{ot}$ may be computed as $Bias(\hat{L}''_{ot}) = E(\hat{L}''_{ot}) - L''_{ot}$, and the mean squared error, which is more relevant to the analysis of process quality, is $MSE(\hat{L}''_{ot}) = Var(\hat{L}''_{ot}) + [Bias(\hat{L}''_{ot})]^2$. Table 28 displays the bias and the MSE of $\hat{L}''_{ot}$ for a = –1.0(0.5)1.0, b = 1, d_u = 5/4, d_l = 5/6, and n = 10(10)50. The results in Table 28 indicate that as $|a|$ increases, the mean squared error also increases. Further, as the sample size increases, the bias and the mean squared error decrease. The bias of $\hat{L}''_{ot}$ versus n are plotted in Figure 57 with a = –1.0, 0, 1.0 (from bottom to top in the plot). And Figure 58 plots the MSE of $\hat{L}''_{ot}$ versus n with a = 0, –1.0, 1.0 (from bottom to top in the plot).

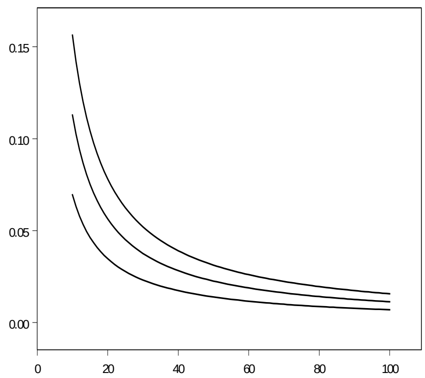

Figure 57. Plots of bias of $\hat{L}''_{ot}$ versus n with a = –1.0, 0, 1.0 (bottom to top in plot).

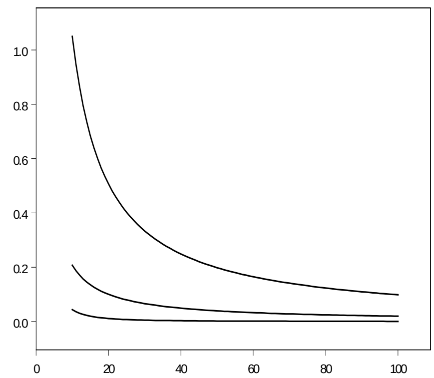

Figure 58. Plots of MSE of $\hat{L}''_{ot}$ versus n with a = 0, –1.0, 1.0 (bottom to top in plot).

Table 29 display the relative error and relative bias of $\hat{L}''_{ot}$, defined as $[MSE_R(\hat{L}''_{ot})]^{1/2} = \{E[(\hat{L}''_{ot} - L''_{ot})/L''_{ot}]^2\}^{1/2}$ and $Bias_R(\hat{L}''_{ot}) = [E(\hat{L}''_{ot}) - L''_{ot}]/L''_{ot}$, respectively, for a = –1.0(0.5)1.0, b = 1, d_u = 5/4, d_l = 5/6, and n = 10(10)100. The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true L''_{ot}. For example, with n = 100, a = 0.5 we has $[MSE_R(\hat{L}''_{ot})]^{1/2} =$ 0.4060. Thus, for n = 100, a = 0.5 we expect that the average error of L''_{ot} would be no greater than 40.6% of the true L''_{ot}. On the other hand, the relative bias $Bias_R(\hat{L}''_{ot})$ is investigated to analyze the accuracy of the natural estimator L''_{ot}. For example, with n = 100, a = 0.5 we has $Bias_R(\hat{L}''_{ot}) =$ 0.0400, that is, 4% relative bias for the true L''_{ot}.

For the case when the production tolerance is symmetric, $\hat{A}$ may be simplified as $|\bar{X} - T|$ and the estimator $\hat{L}''_{ot}$ reduces to $\hat{L}_{ot} = (\bar{X} - T)^2/(d)^2$, which is the maximum likelihood estimator (MLE) of L_{ot}. This is because that $\bar{X}$ is the MLE of μ, then by the invariance property of MLE the result follows. Thus, we have the rth moment of $\hat{L}''_{ot}$ for symmetric tolerance as

$$E(\hat{L}''_{ot})^r = E(\hat{L}_{ot})^r = \left(\frac{\sigma^2}{nd^2}\right)^r \sum_{j=0}^{\infty}\left(\frac{e^{-\lambda/2}(\lambda/2)^j}{j!}\right)\cdot\left(\frac{2^r\,\Gamma((1/2)+j+r)}{\Gamma((1/2)+j)}\right). \tag{6.19}$$

6.3.3 Estimation of L''_{pe}

The index L''_{pe} reflects the process inconsistency loss, and its natural estimator can be defined as $\hat{L}''_{pe} = S^2_{n-1}/d^{*2}$, where $S^2_{n-1} = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. This estimator is unbiased and depends only on the complete, sufficient statistic S^2_{n-1} for σ^2, by the Lehmann-Scheffé Theorem we know that $\hat{L}''_{pe}$ is an uniformly minimum variance unbiased estimator (UMVUE) of L''_{pe}. On the assumption of normality, $\hat{L}''_{pe}$ is distributed as $\sigma^2/[(n-1)d^{*2}]$ times a chi-square variable with $(n-1)$ degrees of freedom. The rth moment about zero for $\hat{L}''_{pe}$ is:

$$E(\hat{L}''_{pe})^r = \left(\frac{\sigma^2}{(n-1)d^{*2}}\right)^r\cdot\left(\frac{2^r\,\Gamma((n-1)/2+r)}{\Gamma((n-1)/2)}\right). \tag{6.20}$$

In particular, the expected value and the variance of $\hat{L}''_{pe}$ can be obtained as follows:

$$E(\hat{L}''_{pe}) = L''_{pe}, \tag{6.21}$$

and

$$Var(\hat{L}''_{pe}) = \left(\frac{2\sigma^4}{(n-1)d^{*4}}\right). \tag{6.22}$$

For the case when the production tolerance is symmetric, d^* may be simplified as d and the estimator $\hat{L}''_{pe}$ reduces to $\hat{L}_{pe} = S^2_{n-1}/d^2$, which is an uniformly minimum variance unbiased estimator (UMVUE) of L_{pe}. The rth moment of $\hat{L}''_{ot}$ for symmetric tolerance becomes

$$E(\hat{L}''_{pe})^r = E(\hat{L}_{pe})^r = \left(\frac{\sigma^2}{(n-1)d^2}\right)^r \cdot \left(\frac{2^r \Gamma((n-1)/2+r)}{\Gamma((n-1)/2)}\right). \qquad (6.23)$$

6.4 An Application Example

We consider a case study for illustration purpose. Consider the following example involving a factory manufacturing high density Light Emitting Diode (LED). Application of LEDs is expanding rapidly since high intensity LEDs of wide range of colors have been recently developed and become available, which enabled application of LEDs in a wide variety of areas such as instrument cluster lighting, color displays, traffic signals, roadway signs (barricade lights), airport signaling and lighting, automotive backlighting in dashboards and switches, telecommunication indicator and backlighting in telephone and fax backlighting for audio and video equipment, backlighting in office equipment, indoor and outdoor message boards, flat backlight for LCDs, switches and symbols, illumination purposes, alternative to incandescent lamps, etc.

LEDs are peculiar light sources much different from lamps in terms of physical size, flux level, spectrum, and spatial intensity distribution. And LED technology provides a number of benefits over incandescent bulbs. With a focus on the critical characteristic, the luminous intensity of LED sources, we examine a particular LED product model. The upper and the lower specification limits of luminous intensity are set to USL = 100 mcd, LSL = 50 mcd, and the target value is set to T = 80 mcd. We note that it's an asymmetric tolerance case.

Table 30. The 30 consecutive days $\hat{L}''_e$.

0.644	0.817	0.942	0.691	0.754	0.458
0.485	0.610	0.707	0.577	0.732	0.512
0.683	0.764	0.870	0.653	0.574	0.623
0.551	0.690	0.582	0.744	0.658	0.491
0.725	0.673	0.455	0.649	0.971	0.521

Now we consider a particular type of LED manufacturing process. Historical data based on routine process monitoring shows that the process is under statistical control and the process distribution is justified and is shown to be fairly close to the normal distribution. A sample data collection procedure is implemented in the factory on a daily basis to monitor/control process quality. The factory production resource and schedule allows the data collection plan be implemented with a sample size $n \leq 40$. A simple approach to determine the true value (rather than a upper confidence bound) of L''_e is to perform the sampling on a routine basis consecutively for a number of, say, 30 days. The calculated values of single-day $\hat{L}''_e$ for 30 consecutive days are displayed in Table 30. The average $\hat{L}''_e$ value for the 30 days is obtained as $E(\hat{L}''_e) = 0.660$. Checking Table 27, the values of $Bias_R(\hat{L}''_e)$ is between –0.0065 and 0.0101. Therefore, the true value of L''_e can be determined as 0.66/(1–0.65%) = 0.6643. The error of the approximation becomes negligibly small over time.