Appendix A

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from a normal distribution with mean μ and variance σ^2 measuring the characteristic under investigation. The natural estimator $\hat{C}_{pk}$ is obtained by replacing the process mean μ and the process standard deviation σ by their conventional estimators $\overline{X}$ and S_{n-1} $(= [\sum_{i=1}^{n}(X_i - \overline{X})^2/(n-1)]^{½})$, respectively. Then we have the following expression:

$$\hat{C}_{pk} = \frac{d - \left|\overline{X} - M\right|}{3S_{n-1}}. \tag{A.1}$$

For the sake of deriving the cumulative distribution function of $\hat{C}_{pk}$, the following notations are introduced:

1. $K = (n-1)S_{n-1}^2/\sigma^2$, which is distributed as χ_{n-1}^2,
2. $Z' = \sqrt{n}(\overline{X} - M)/\sigma$, which is distributed as $N(\xi\sqrt{n}, 1)$ with $\xi = (\mu - M)/\sigma$,
3. $H = |Z'|$, which is distributed as a folded-normal distribution with probability density function $f_H(h) = \phi(h + \xi\sqrt{n}) + \phi(h - \xi\sqrt{n})$ for $h \geq 0$, where $\phi(\cdot)$ is the probability density function of the standard normal distribution.

For $x > 0$, the cumulative distribution function of $\hat{C}_{pk}$ can be derived as:

$$F_{\hat{C}_{pk}}(x) = P(\hat{C}_{pk} \leq x) = P\left(\frac{\sqrt{n-1}(b\sqrt{n} - H)}{3\sqrt{nK}} \leq x\right)$$

$$= 1 - P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - H)}{3x}\right)$$

$$= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - H)}{3x} \,\middle|\, H = h\right) f_H(h)dh$$

$$= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - h)}{3x}\right) f_H(h)dh, \tag{A.2}$$

where $b = d/\sigma$. Since K is distributed as χ_{n-1}^2, we have

$$P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - h)}{3x}\right) = 0 \text{ for } h > b\sqrt{n} \text{ and } x > 0. \tag{A.3}$$

Therefore, $F_{\hat{C}_{pk}}(x) = 1 - \int_0^{b\sqrt{n}} P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - h)}{3x}\right) f_H(h)dh$

$$= 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-h)^2}{9nx^2}\right) f_H(h)dh \text{ , for } x > 0, \qquad \text{(A.4)}$$

where $G(\cdot)$ is the cumulative distribution function of χ^2_{n-1}. Substituting $f_H(t)$ leads to the result:

$$F_{\hat{C}_{pk}}(x) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9nx^2}\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right] dt \text{ , for } x > 0 \, . \qquad \text{(A.5)}$$